Analytical and Numerical Aspects of a Generalization of the Complementary Error Function

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Abstract

In this paper we discuss analytical and numerical properties of the function $V_{\nu,\mu}(\alpha,\beta,z) = \int_0^\infty e^{-zt}(t+\alpha)^\nu(t+\beta)^\mu\,dt$, with $\alpha,\beta,\Re z > 0$, which can be viewed as a generalization of the complementary error function, and in fact also as a generalization of the Kummer $U-$function. The function $V_{\nu,\mu}(\alpha,\beta,z)$ is used for certain values of the parameters as an approximant in a singular perturbation problem. We consider the relation with other special functions and give asymptotic expansions as well as recurrence relations. Several methods for its numerical evaluation and examples are given.

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1 Introduction

The complementary error function is defined by

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt, \tag{1.1}$$

where the path of integration goes to infinity in such a way that the integral is well defined. Another form is given by (see [1, p. 297])

$$w(z) = \frac{1}{\pi i} \int_{-\infty}^\infty \frac{e^{-t^2}}{t-z}\, dt, \quad \Im z > 0, \tag{1.2}$$

with analytic continuation through the relation

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz), \quad z \in \mathbb{C}. \tag{1.3}$$

The complementary error function plays a role in asymptotic problems of integrals when a saddle point and a pole are close together or even coalesce. It also occurs when a saddle point is near an endpoint of the interval of integration. See §4.3 and [13, pp. 356–366]. For its role in a singularly perturbed convection-diffusion problem in a rectangle with corner singularities we refer the reader to [8].

Generalizations of the complementary error function can be made in several ways, to cover integrals with more complicated structure. It is a special case of the incomplete gamma function, which is given by

$$\begin{aligned} \Gamma(a,z) &= \int_z^\infty t^{a-1} e^{-t}\, dt, \quad |\operatorname{ph} z| < \pi, \\ &= z^a e^{-z} \int_0^\infty (t+1)^{a-1} e^{-zt}\, dt, \quad \Re z > 0, \\ &= \frac{e^{-z}}{\Gamma(1-a)} \int_0^\infty t^{-a} e^{-zt} \frac{dt}{t+1}, \quad \Re a < 0, \quad \Re z > 0. \end{aligned} \tag{1.4}$$

We have

$$\operatorname{erfc} z = \frac{1}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}, z^2\right). \tag{1.5}$$

The next step is given by the Kummer $U-$function, defined by

$$U(a,c,z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{c-a-1} e^{-zt}\, dt, \quad \Re a > 0, \quad \Re z > 0. \tag{1.6}$$

We have

$$\Gamma(a,z) = z^a e^{-z} U(1, a+1, z) = e^{-z} U(1-a, 1-a, z), \tag{1.7}$$

where the last expression is obtained by applying the functional relation

$$U(a,c,z) = z^{1-c} U(a-c+1, 2-c, z). \tag{1.8}$$

As particular cases we have

$$\operatorname{erfc} z = \frac{1}{\sqrt{\pi}} e^{-z^2} U(\tfrac{1}{2}, \tfrac{1}{2}, z^2) = \frac{z}{\sqrt{\pi}} e^{-z^2} U(1, \tfrac{3}{2}, z^2). \tag{1.9}$$

In this paper we consider generalizations of the complementary error function motivated by singular perturbation problems. More precisely, we discuss properties of the function

$$V_{\nu,\mu}(\alpha,\beta,z) = \int_0^\infty e^{-zt}(t+\alpha)^\nu (t+\beta)^\mu \, dt, \quad \alpha, \beta, \Re z > 0. \tag{1.10}$$

This function can be viewed as a generalization of the complementary error function, in the sense that the integrand shows poles (or algebraic singularities) at two different points $t = -\alpha$ and $t = -\beta$, and therefore it allows us to treat more general problems in uniform asymptotic expansions, when more than one singularity of the integrand coalesce with a saddle point or an endpoint.

A special case of this function occurs in [9] in the form

$$V(u,v) = \int_0^\infty \frac{te^{-t^2}}{\sqrt{t^2+u^2}(t^2+v^2)} \, dt = \tfrac{1}{2} V_{-\frac{1}{2},-1}(u^2, v^2, 1) \tag{1.11}$$

as a first order approximation of an elliptic 3D singular perturbation problem in the half-space $z \geq 0$ with discontinuous boundary values at $z = 0$. The function $V(u,v)$ defined in (1.11) has been used in the following singular perturbation convection-diffusion problem with a "square shaped source of contamination" located at the plane $z = 0$:

$$-\varepsilon \Delta U + U_z = 0, \quad \text{if } (x,y,z) \in \Omega, \tag{1.12}$$

where Δ is the 3D-Laplacian, Ω is the 3D half-space $z > 0$, and ε is a small positive parameter. The boundary values at $z = 0$ are $U(x,y,0) = 1$ inside the unit square $|x| < 1, |y| < 1$ and $U(x,y,0) = 0$ outside this square. Observe that these Dirichlet data at $z = 0$ are discontinuous at the boundary of the unit square in the plane $z = 0$.

The solution of this problem has been obtained in the form

$$U(x,y,z) = \frac{e^{\omega z}}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(\omega t)}{t} \frac{\sin(\omega s)}{s} e^{i\omega x t + i\omega y s - z\omega\sqrt{1+t^2+s^2}}\, dt\, ds, \tag{1.13}$$

where $\omega = \frac{1}{2\varepsilon}$. To apply the saddle point method the sine functions are replaced by exponential functions, which introduces poles that can be avoided by deforming the contours in the complex s and t planes.

The special interest in (1.11) is large ω and small values of u and v, and in that case the behaviour of the solution of the 3D singular perturbation problem at $x \sim 1, y \sim 1$ (and at other corner points of the unit square) can be described in terms of $V(u,v)$ of (1.11). Along the sides of the unit square the saddle points of the integrand in (1.13) are close to the origin (where poles arise when writing the sine functions in terms of exponential functions), and the transition from 0 to 1 can be described by using the complementary error function. Near the corners of the square this description fails, and a "two-dimensional" generalization of the error function (in fact, the function $V(u,v)$ of (1.11)) can be used. For further details we refer to [9] and [8].

We observe that $V_{\nu,\mu}(\alpha,\beta,z)$ of (1.10) can also be seen as a generalization of the Kummer $U-$function given in (1.6). Indeed, by differentiating with respect to z, we see that we can relate derivatives of the $V-$function with the $V-$function itself with shifted parameters. For instance:

$$\begin{aligned} V'_{\nu,\mu}(\alpha,\beta,z) &= -V_{\nu+1,\mu}(\alpha,\beta,z) + \alpha V_{\nu,\mu}(\alpha,\beta,z) \\ &= -V_{\nu,\mu+1}(\alpha,\beta,z) + \beta V_{\nu,\mu}(\alpha,\beta,z) \end{aligned} \tag{1.14}$$

Using similar manipulations, it is not difficult to show that the $V-$function satisfies the following inhomogeneous second order differential equation:

$$\begin{aligned} zf''(z) &+ [\nu+\mu+2-(\alpha+\beta)z]f'(z) \\ &- [\beta(\nu+1)+\alpha(\mu+1)-\alpha\beta z]\, f(z) = \alpha^{\nu+1}\beta^{\mu+1} \end{aligned} \tag{1.15}$$

The solutions of the homogeneous differential equation can be expressed in terms of confluent hypergeometric functions $U(a,c,z)$ and ${}_1F_1(a;c;z)$. Upon examination of the solutions of the homogeneous equation it seems reasonable to identify the parameters as $a = \nu+1$ and $c = \nu+\mu+2$.

This procedure allows us to think of the V function as an inhomogeneous analogue of the Kummer functions, and obtain an analytic continuation of the V function with less restrictions on the parameters α, β and z than the ones imposed by (1.11). We will not explore this idea any further, however, since this integral representation is general enough for our purposes.

We work with the general form as given in (1.10), because the parameters ν and μ do not play an essential role in the analytic relations to be given, and because they also provide other special cases that are relevant in other problems.

The plan of the paper is as follows. In Section 2 we present the basic properties of the function $V_{\nu,\mu}(\alpha,\beta,z)$, as well as the relation with other special functions and integrals. Section 3 is devoted to inhomogeneous three-term recurrence relations satisfied by the function $V_{\nu,\mu}(\alpha,\beta,z)$ and their computational aspects. In Section 4 we give asymptotic expansions both for large values of z and for large values of the parameters ν and μ, and finally in Section 5 we give several numerical examples using the methods analysed in the previous sections.

2 Properties of the function $V_{\nu,\mu}(\alpha,\beta,z)$

We observe first that one of the parameters α,β,z is redundant because it can be removed by a simple transformation of the variable of integration. For example:

$$V_{\nu,\mu}(\alpha,\beta,z) = z^{-1-\nu-\mu}V_{\nu,\mu}(\alpha z,\beta z,1). \tag{2.1}$$

Similarly, a straightforward calculation gives:

$$V_{\nu,\mu}(\alpha,\beta,z) = \alpha^{\nu+\mu+1}V_{\nu,\mu}(1,\beta/\alpha,\alpha z) = \beta^{\nu+\mu+1}V_{\nu,\mu}(\alpha/\beta,1,\beta z). \tag{2.2}$$

Secondly, it is clear from the definition of the function that we can interchange α and β together with ν and μ, i.e.

$$V_{\nu,\mu}(\alpha,\beta,z) = V_{\mu,\nu}(\beta,\alpha,z), \tag{2.3}$$

and for this reason, in the sequel we will assume that $\beta > \alpha$, without loss of generality.

2.1 Particular cases

The function in (1.10) cannot simply be expressed in terms of the Kummer $U-$function. However, when $\alpha \to 0$ we obtain

$$V_{\nu,\mu}(0,\beta,z) = \beta^{\mu+\nu+1}\Gamma(\nu+1)U(\nu+1,\mu+\nu+2,\beta z), \quad \Re\nu > -1. \tag{2.4}$$

A similar result holds when $\beta \to 0$.

As particular cases of the Kummer $U-$function we find several known special functions, or combinations of them. For instance, when $\mu = m = 0, 1, 2, \ldots$ and $\Re\alpha > 0$, we can expand the binomial term and we have

$$V_{\nu,m}(\alpha,\beta,z) = \alpha^{\nu+1}\beta^m \sum_{k=0}^{m} \binom{m}{k} (\alpha/\beta)^k k!\, U(k+1, \nu+k+2, \alpha z), \quad (2.5)$$

with a similar result when ν is a nonnegative integer. For $m = 0$ formula (2.5) gives an incomplete gamma function, see (1.7):

$$V_{\nu,0}(\alpha,\beta,z) = \alpha^{\nu+1} U(1, \nu+2, \alpha z) = z^{-\nu-1} e^{\alpha z} \Gamma(\nu+1, \alpha z). \quad (2.6)$$

For $m > 0$ the $U-$functions in (2.5) are derivatives of incomplete gamma functions. We have (see [1, Eq. 13.4.22] and (1.7))

$$U(k+1, \nu+k+2, z) = \frac{(-1)^k}{k!} \frac{d^k}{dz^k} \left(z^{-\nu-1} e^z \Gamma(\nu+1, z) \right), \quad (2.7)$$

for $k = 0, 1, 2, \ldots$ The incomplete gamma function appears in two other important cases: firstly, when $\alpha = 0$ or $\beta = 0$ and additionally $\mu = -1$, since in that case:

$$V_{\nu,-1}(0,\beta,z) = \beta^\nu e^{\beta z} \Gamma(\nu+1) \Gamma(-\nu, \beta z). \quad (2.8)$$

Secondly, when $\alpha = \beta$ we obtain

$$V_{\nu,\mu}(\alpha,\alpha,z) = \alpha^{\nu+\mu+1} U(1, \nu+\mu+1, \alpha z) = z^{-\mu-\nu-1} e^{\alpha z} \Gamma(\nu+\mu+1, \alpha z). \quad (2.9)$$

Since the complementary error function can be obtained as a special case of the incomplete gamma function, see (1.5), then we have

$$V_{-\frac{1}{2},-1}(0,1,z) = \pi e^z \operatorname{erfc}\sqrt{z}, \quad V_{0,-\frac{1}{2}}(0,1,z) = \sqrt{\frac{\pi}{z}} e^z \operatorname{erfc}\sqrt{z}. \quad (2.10)$$

2.2 Related and generalized functions and integrals

The following integrals are other generalizations of the complementary error function, and some of them play a role in uniform asymptotic expansions.

1. Goodwin-Staton integral, see [11, pp. 44, 115],

$$G(z) = \int_0^\infty \frac{e^{-t^2}}{z+t}\, dt, \quad |\operatorname{ph} z| < \pi. \quad (2.11)$$

   This is a "half-time" complementary error function, and it is "elementary" because it can be expressed in terms of the error function and the exponential integral.

2. A generalized Goodwin-Staton, see [7],

$$I(\mu, z) = \int_0^\infty \frac{t^\mu e^{-t^2}}{t+z}\, dt, \quad 0 < |\mathrm{ph}\, z| < \pi, \quad \Re\mu > -1. \tag{2.12}$$

Observe that Jones considers $I(\mu, -z)$ and writes this function in [7, Eq. (2.4)] as a sum of two incomplete gamma functions. For a numerical approach based on series expansion and integration term by term see [10].

3. Incomplete Goodwin-Staton:

$$G(u, v) = \int_u^\infty \frac{e^{-t^2}}{v+t}\, dt, \quad |\mathrm{ph}\, u| < \pi, \quad |\mathrm{ph}\, v| < \pi. \tag{2.13}$$

See also [2] for an application with a more general phase function. In this case, we have a saddle point at the origin that can coalesce with the pole at $t = -v$ if v is small and with the endpoint $t = u$ if u is small. These two situations can be analysed separately using standard methods in uniform asymptotics (in particular with the complementary error function, see [2]). An alternative form for this function is

$$G(u, v) = e^{-u^2} \int_0^\infty \frac{e^{-t^2-2ut}}{u+v+t}\, dt, \quad |\mathrm{ph}(u+v)| < \pi. \tag{2.14}$$

4. Ciarkowski [3]:

$$H(u, v) = \int_0^\infty \frac{e^{-t^2}}{\sqrt{u+t}(v+t)}\, dt, \quad |\mathrm{ph}\, u| < \pi, \quad |\mathrm{ph}\, v| < \pi. \tag{2.15}$$

In the mentioned reference, a more general algebraic singularity is considered, in the form $(u+t)^r$ with $-1 < r < 1$. When only two critical points are allowed to coalesce (saddle and pole or saddle and algebraic singularity) an asymptotic approximation can again be obtained using classical methods. This integral is of interest because it has a saddle point at the endpoint $t = 0$, which is not present in the corresponding integral in (1.10), with $\nu = -1$ and $\mu = -\frac{1}{2}$.

3 Recurrence relations

The function $V_{\nu,\mu}(\alpha,\beta,z)$ defined in (1.10) satisfies a inhomogeneous three-term recurrence relation, which is similar to the one for the related Kummer $U-$function.

Integration by parts gives

$$\begin{aligned}(\nu+1)V_{\nu,\mu} &= \int_0^\infty e^{-zt}(t+\beta)^\mu \, d(t+\alpha)^{\nu+1} \\ &= -\beta^\mu\alpha^{\nu+1} + zV_{\nu+1,\mu} - \mu V_{\nu+1,\mu-1} \\ &= -\beta^\mu\alpha^{\nu+1} + zV_{\nu,\mu+1} + z(\alpha-\beta)V_{\nu,\mu} - \mu V_{\nu+1,\mu-1}\end{aligned} \tag{3.1}$$

The last step follows from writing $(t+\alpha) = (t+\beta)+(\alpha-\beta)$, and thereby

$$V_{\nu+1,\mu} = V_{\nu,\mu+1} + (\alpha-\beta)V_{\nu,\mu}. \tag{3.2}$$

Another integration by parts shows that

$$zV_{\nu,\mu+1} = \alpha^\nu\beta^{\mu+1} + \nu V_{\nu-1,\mu+1} + (\mu+1)V_{\nu,\mu}. \tag{3.3}$$

Substituting this in (3.1) and rearranging terms we obtain the following inhomogeneous recurrence relation:

$$\mu V_{\nu+1,\mu-1} + [(\beta-\alpha)z + \nu - \mu]V_{\nu,\mu} - \nu V_{\nu-1,\mu+1} = \alpha^\nu\beta^\mu(\beta-\alpha). \tag{3.4}$$

If we write

$$\nu = a + \epsilon_1 n, \quad \mu = c + \epsilon_2 n, \tag{3.5}$$

with integer n and $\epsilon_j = \pm 1$ (not both zero), we can find recursions with respect to n in different directions using (3.4). In the sequel, we will designate the different cases using the notation (ϵ_1, ϵ_2).

3.1 Computational aspects. The $(1,-1)$ recursion

The general solution of the inhomogeneous recursion can be written as

$$y_n(z) = Af_n(z) + Bg_n(z) + h_n(z), \tag{3.6}$$

where $f_n(z)$ and $g_n(z)$ are independent solutions of the homogeneous recursion (which in this case can be written in terms of Kummer functions), $h_n(z)$ is a particular solution of the inhomogeneous one and A and B are constants.

From a computational perspective it is crucial to know the asymptotic behaviour of the different solutions for large n (including the ones of the homogeneous recurrence relation, which in our case are Kummer functions). The asymptotic behaviour of the Kummer functions is well known in the different directions of recursion, see for example [4], and an analysis of the $V-$function for large n is given later.

3.1.1 Minimal and dominant solutions

If we set $\nu = a+n$, $\mu = c-n$ in the recursion (3.4), we obtain

$$(c-n)y_{n+1}(z)+[(\beta-\alpha)z+a-c+2n]y_n(z)-(a+n)y_{n-1}(z) = \alpha^{a+n}\beta^{c-n}(\beta-\alpha). \tag{3.7}$$

Solutions of the homogeneous recurrence are

$$\begin{aligned} y_n^{(1)}(z) &= \frac{\Gamma(a+n+1)}{\Gamma(a+2-c+n)}M(a+n+1,c,(\beta-\alpha)z), \\ y_n^{(2)}(z) &= \Gamma(a+n+1)U(a+n+1,a+c+2,(\beta-\alpha)z), \\ y_n^{(3\pm)}(z) &= \frac{(-1)^n}{\Gamma(a+2-c+n)}U(c-a-n-1,c,(\beta-\alpha)ze^{\pm\pi i}). \end{aligned} \tag{3.8}$$

In this case the solution $y_n^{(2)}(z)$ is minimal for increasing n and $-\pi < \mathrm{ph}(\beta-\alpha)z < \pi$, see [4, Sec. 4].

These results are relevant if one intends to use the recurrence relations for computational purposes, since it is possible to combine backward recursion with a normalising series to obtain a method of evaluation of a minimal solution of the recursion that does not require any initial values, see [6, Sec. 4.6]. As we show below, the $(1,-1)$ case provides a useful example, since we obtain a normalization series which is convergent and the V function has the right asymptotic behaviour for large n, see section §4.3.1.

3.1.2 Normalization relations

If we sum the functions V in n we obtain a convergent series:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}V_{a+n,c-n}(\alpha,\beta,z) = (\beta-\alpha)^{-\lambda}V_{a,c+\lambda}(\alpha,\beta,z), \tag{3.9}$$

which is convergent for fixed values of z. For $\lambda=-c$ this reduces to

$$\sum_{n=0}^{\infty}\frac{(-c)_n}{n!}V_{a+n,c-n}(\alpha,\beta,z) = (\beta-\alpha)^{c}z^{-a-1}e^{\alpha z}\Gamma(a+1,\alpha z), \tag{3.10}$$

and more generally, when $\lambda = m - c$ $(m = 0, 1, 2, \ldots)$ we have

$$\sum_{n=0}^{\infty} \frac{(m-c)_n}{n!} V_{a+n,c-n}(\alpha,\beta,z) = (\beta-\alpha)^{c-m} V_{a,m}(\alpha,\beta,z), \tag{3.11}$$

which can be expressed as a finite sum of $U-$functions (see (2.5)), and these in turn in terms of incomplete gamma functions.

Corresponding series for the minimal solution of the homogeneous recursion (that is, the function $y_n^{(2)}(z)$ given in (3.9)) are

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Gamma(a+n+1) U(a+n+1, a+c+2, (\beta-\alpha)z) = \\ &\qquad \Gamma(a+1) U(a+1, a+c+\lambda+2, (\beta-\alpha)z), \end{aligned} \tag{3.12}$$

with for $\lambda = m - c$ $(m = 0, 1, 2, \ldots)$ results in a simple expression:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(m-c)_n}{n!} \Gamma(a+n+1) U(a+n+1, a+c+2, (\beta-\alpha)z) = \\ &\qquad \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(a+k+1)}{((\beta-\alpha)z)^{a+k+1}}. \end{aligned} \tag{3.13}$$

3.1.3 A Miller algorithm

We can use the recurrence relation in the homogeneous form by eliminating the right-hand side in (3.6). This gives the recursion

$$\begin{aligned} \rho(a+n) y_{n-1} = (n-c) y_{n+2} - (c-n) \left(\frac{\zeta + a - c + 2n + 2}{c-n-1} - \rho \right) y_{n+1} \\ + \left((a+n+1) \frac{c-n}{c-n-1} + \rho(\zeta + a - c + 2n) \right) y_n, \end{aligned} \tag{3.14}$$

where

$$\zeta = (\beta - \alpha) z, \quad \rho = \frac{\alpha}{\beta}. \tag{3.15}$$

Three solutions are $y_n^{(1)}(z)$ and $y_n^{(2)}(z)$ defined in (3.9) and the function $V_{a+n,c-n}(\alpha,\beta,z)$ defined in (1.10). Let us denote

$$u_n = y_n^{(2)}(z), \quad v_n = V_{a+n,c-n}(\alpha,\beta,z), \tag{3.16}$$

and we want to compute v_0.

We assume that the condition holds that $y_n^{(1)}(z)$ is a dominant solution and u_n and v_n are minimal solutions with the same exponential behaviour, as explained in §4.3.1. We choose a large number N and start the recursion with the values

$$y_N = 1, \quad y_{N+1} = 0, \quad y_{N+2} = 0. \tag{3.17}$$

Then, as in the classical Miller algorithm [6, §4.6], we expect that

$$y_n \doteq Au_n + Bv_n, \quad n = 0, 1, \ldots, N_0 \ll N, \tag{3.18}$$

with A and B not depending on n. The A and B can be obtained by using the normalisation series given in (3.10)–(3.13). During recursion we compute for $m = 0$ and a value of m larger then zero the finite sums

$$S_m = \sum_{n=0}^{N} \frac{(m-c)_n}{n!} y_n, \tag{3.19}$$

and we can compute from (3.11) and (3.13) the values U_m and V_m given by

$$U_m = \sum_{n=0}^{N_0} \frac{(m-c)_n}{n!} u_n, \quad V_m = \sum_{n=0}^{N_0} \frac{(m-c)_n}{n!} v_n, \tag{3.20}$$

for the same values of m. This gives

$$A \doteq \frac{S_m V_0 - S_0 V_m}{U_m V_0 - U_0 V_m}, \quad B \doteq \frac{U_m S_0 - U_0 S_m}{U_m V_0 - U_0 V_m}. \tag{3.21}$$

From (3.18) we can compute v_0, assuming that the value of u_0 is known. This quantity can be computed in the same algorithm with the standard Miller algorithm by using the homogeneous part of the recursion in (3.6) and the normalisation series of u_n in (3.20).

4 Power series and asymptotic behaviour

In this section we study the behaviour of the function $V_{\nu,\mu}(\alpha, \beta, z)$, firstly for small values of z and then both for large values of the variable z and for large values of the parameters ν and μ.

4.1 Power series expansion

For certain parameters it is convenient to use the representation in terms of the Kummer $U-$ function together with a fast converging series. If we split the integral, we can write for $\beta \geq \alpha$

$$V_{\nu,\mu}(\alpha,\beta,z) = e^{\alpha z} \int_{\alpha}^{\infty} e^{-zt} t^{\nu} (t+\beta-\alpha)^{\mu}\, dt, \tag{4.1}$$

which gives, together with (1.6)

$$\begin{aligned} &G_{\nu,\mu}(\alpha,\beta,z) + V_{\nu,\mu}(\alpha,\beta,z) = \\ &\qquad (\beta-\alpha)^{\nu+\mu+1} e^{\alpha z} \Gamma(\nu+1) U(\nu+1, \nu+\mu+2, (\beta-\alpha)z), \end{aligned} \tag{4.2}$$

where

$$G_{\nu,\mu}(\alpha,\beta,z) = \int_{0}^{\alpha} e^{(\alpha-t)z} t^{\nu} (t+\beta-\alpha)^{\mu}\, dt. \tag{4.3}$$

By expanding the exponential function we obtain

$$G_{\nu,\mu}(\alpha,\beta,z) = (\beta-\alpha)^{\nu+\mu+1} \sum_{k=0}^{\infty} \frac{[(\beta-\alpha)z]^k}{k!} G_k(\nu,\mu,\gamma), \tag{4.4}$$

where

$$G_k(\nu,\mu,\gamma) = \int_{0}^{\gamma} (\gamma-t)^k t^{\nu} (t+1)^{\mu}\, dt, \quad \gamma = \frac{\alpha}{\beta-\alpha}, \quad k = 0,1,2,\ldots \tag{4.5}$$

In fact, the function $G_k(\nu,\mu,\gamma)$ can be identified as a Gauss hypergeometric function:

$$G_k(\nu,\mu,\gamma) = \gamma^{\nu+k+1} \frac{\Gamma(k+1)\Gamma(\nu+1)}{\Gamma(k+\nu+2)}\, {}_2F_1\left(\begin{matrix} -\mu, \nu+1 \\ \nu+k+2 \end{matrix} ; -\gamma \right), \tag{4.6}$$

for $k = 0,1,2,\ldots$. Therefore

$$G_{\nu,\mu}(\alpha,\beta,z) = (\beta-\alpha)^{\nu+\mu+1} \gamma^{\nu+1} \Gamma(\nu+1) \sum_{k=0}^{\infty} d_k H_k(\mu,\nu,\gamma), \tag{4.7}$$

where $d_k = (\alpha z)^k$ and

$$H_k(\mu,\nu,\gamma) = \frac{1}{\Gamma(\nu+k+2)}\, {}_2F_1\left(\begin{matrix} -\mu, \nu+1 \\ \nu+k+2 \end{matrix} ; -\gamma \right), \quad k = 0,1,2,\ldots. \tag{4.8}$$

The functions H_k are minimal solutions of the corresponding recursion for increasing k, and therefore we should not compute them using forward recursion. However, a backward recursion can be used efficiently. We postpone the computational details until Section §5.2.

4.2 Asymptotic expansions for large z

For values of α and β bounded away from zero we use (1.10) and expand

$$(t+\alpha)^\nu (t+\beta)^\mu = \alpha^\nu \beta^\mu \sum_{k=0}^{\infty} c_k t^k \tag{4.9}$$

We apply Watson's lemma (see [13, pp. 19–29]) to obtain

$$V_{\nu,\mu}(\alpha,\beta,z) \sim \alpha^\nu \beta^\mu \sum_{k=0}^{\infty} c_k \frac{k!}{z^{k+1}}, \quad z \to \infty. \tag{4.10}$$

When α and β are positive, this expansion is valid for $|\mathrm{ph}\, z| < \frac{3}{2}\pi$. The first few coefficients are

$$c_0 = 1, \quad c_1 = \frac{\mu\alpha + \nu\beta}{\alpha\beta}, \quad c_2 = \frac{2\nu\mu\alpha\beta + \alpha^2\mu(\mu-1) + \beta^2\nu(\nu-1)}{2\alpha^2\beta^2}. \tag{4.11}$$

Higher coefficients follow from the recursion

$$\alpha\beta(k+1)c_{k+1} = (\mu\alpha+\nu\beta-k(\alpha+\beta))c_k + (\nu+\mu-k+1)c_{k-1}, \quad k = 1,2,3,\ldots. \tag{4.12}$$

A modified asymptotic expansion can be obtained following the ideas exposed in [5]:

$$(t+\alpha)^\nu = (\alpha/\beta)^\nu (t+\beta)^\nu \sum_{k=0}^{\infty} \binom{\nu}{k} \left(\frac{\beta-\alpha}{\alpha}\right)^k \frac{t^k}{(t+\beta)^k}, \tag{4.13}$$

and substituting this in (1.10). This gives the expansion

$$V_{\nu,\mu}(\alpha,\beta,z) = \alpha^\nu \beta^{\mu+1} \sum_{k=0}^{\infty} \binom{\nu}{k} \left(\frac{\beta-\alpha}{\alpha}\right)^k k!\, U(k+1, \nu+\mu+2, \beta z), \tag{4.14}$$

which will be convergent provided that $0 < \beta < 2\alpha$. This result follows from observing that (4.13) is convergent for $|t/(t+\beta)| < \alpha/|\beta-\alpha|$. This bound holds for all positive t if $\alpha/|\beta-\alpha| > 1$, which is true when $0 < \beta < 2\alpha$.

This, together with the estimates to be given in §4.3.1 for the $U-$functions, proves the convergence of the expansion (4.14).

In the same manner, we have

$$V_{\nu,\mu}(\alpha,\beta,z) = \alpha^{\nu+1}\beta^{\mu}\sum_{k=0}^{\infty}\binom{\mu}{k}\left(\frac{\alpha-\beta}{\beta}\right)^{k} k!\,U(k+1,\nu+\mu+2,\alpha z), \quad (4.15)$$

with convergence if $0 < \alpha < 2\beta$. Both convergence criteria in (4.14) and (4.15) give overlapping domains in the quarter plane $\alpha > 0, \beta > 0$.

As explained in [5] the expansions in (4.14) and (4.15) have an asymptotic character for large values of βz and αz, respectively. However, convergence of these approximations becomes slow when βz in (4.2) or αz in (4.15) is small, since we have the estimate:

$$y_n^{(2)}(z) \sim \sqrt{\pi}[(\beta-\alpha)z]^{\frac{-3-2a-2c}{4}} e^{\frac{(\beta-\alpha)z}{2}} n^{\frac{1+2a+2c}{4}} e^{-2\sqrt{n(\beta-\alpha)z}}. \quad (4.16)$$

The $U-$functions in these series can be computed by using a backward recursion scheme; for details we refer to §3.1.3. When $\alpha = \beta$ the expansions reduce to the relation in (2.9), and in fact the first term in both series is an incomplete gamma function.

4.3 Asymptotic behaviour for large n

In this section we study the asymptotic properties of the function

$$V_{a+\epsilon_1 n, c+\epsilon_2 n}(\alpha,\beta,nz) \quad (4.17)$$

for large values of n, where $\epsilon_{1,2} = 0, \pm 1$ (not both equal to 0). More general cases can be considered, but we will restrict ourselves to these ones in the present discussion. It is important to note that the symmetry property (2.3) allows as to take $\beta > \alpha$ without loss of generality, interchanging α and β and ν and μ if necessary.

The asymptotic behaviour in the different directions can be obtained by using saddle point analysis of the integral representation. It is immediate that we can write

$$V_{a+\epsilon_1 n, c+\epsilon_2 n}(\alpha,\beta,nz) = \int_0^{\infty} (t+\alpha)^a (t+\beta)^c e^{-n\phi(t)}\,dt, \quad (4.18)$$

where

$$\phi(t) = zt - \epsilon_1 \log(t+\alpha) - \epsilon_2 \log(t+\beta). \quad (4.19)$$

The zeros of $\phi'(t)$ are given by

$$t_\pm = -\frac{(\alpha+\beta)z - \epsilon_1 - \epsilon_2}{2z} \pm \frac{\sqrt{[(\alpha+\beta)z - \epsilon_1 - \epsilon_2]^2 - 4z\Delta}}{2z}, \tag{4.20}$$

where

$$\Delta = \alpha\beta z - \epsilon_1\beta - \epsilon_2\alpha. \tag{4.21}$$

When one of the roots or both of them are positive for positive values of α, β and z, we can apply Watson's lemma, with the change of variable $\phi(t) - \phi(t_\pm) = u$. If both roots are negative or complex then a similar result holds, taking $t = 0$ as the relevant point to construct the asymptotic expansion.

The case where $\Delta = 0$ is more interesting, since the root t_+ coalesces with the endpoint of the interval of integration $t = 0$, and the complementary error function has to be used in order to give a valid expansion. It is straightforward to check that $\Delta = 0$ when $z = z_0$, where

$$z_0 = \frac{\epsilon_1}{\alpha} + \frac{\epsilon_2}{\beta}. \tag{4.22}$$

This value is positive in the $(1,0)$, $(0,1)$, $(-1,1)$ and $(1,1)$ cases.

As an example, we will give more details on the $(1,-1)$ case in the following section. Similar expansions can be obtained in the rest of the cases, using the appropriate values of z_0.

4.3.1 Asymptotic behaviour of $V_{a+n,c-n}(\alpha,\beta,nz)$ for large n

In this case we substitute $\epsilon_1 = 1$ and $\epsilon_2 = -1$ in the previous expression

$$V_{a+n,c-n}(\alpha,\beta,nz) = \int_0^\infty e^{-nzt}(t+\alpha)^{a+n}(t+\beta)^{c-n}\,dt = \int_0^\infty f(t)e^{-n\phi(t)}\,dt, \tag{4.23}$$

and we get

$$f(t) = (t+\alpha)^a(t+\beta)^c, \qquad \phi(t) = zt - \log(t+\alpha) + \log(t+\beta). \tag{4.24}$$

and $\alpha,\beta > 0$. We have

$$\phi'(t) = \frac{zt^2 + z(\alpha+\beta)t + z\alpha\beta + \alpha - \beta}{(\alpha+t)(\beta+t)}. \tag{4.25}$$

We look for saddle points. Solving $\phi'(t) = 0$, we obtain two roots:

$$t_\pm = -\frac{\alpha+\beta}{2} \pm \frac{\sqrt{z(\beta-\alpha)(z(\beta-\alpha)+4)}}{2z}. \tag{4.26}$$

Assuming also $z > 0$ we note that both roots are real and $t_- < 0$. The factor $\Delta = \alpha\beta z - \beta + \alpha$ vanishes at

$$z_0 = \frac{1}{\alpha} - \frac{1}{\beta}, \tag{4.27}$$

which is indeed positive since we take $\beta > \alpha$.

If $z > z_0$ then both t_+ and t_- are negative, the function $\phi(t)$ is increasing on $[0, \infty)$ and the relevant point to construct the asymptotic approximation is $t = 0$. We have

$$\int_0^\infty f(t)e^{-n\phi(t)}\,dt = e^{-n\phi(0)} \int_0^\infty f(u)e^{-nu}\frac{dt}{du}\,du, \tag{4.28}$$

with the change of variable

$$\phi(t) - \phi(0) = u, \tag{4.29}$$

which is implicit since

$$\phi(t) - \phi(0) = zt + \log\frac{t+\beta}{t+\alpha} - \log\frac{\beta}{\alpha}. \tag{4.30}$$

Using Lagrange's inversion theorem, one can find that:

$$t = \frac{1}{z_0}u + \frac{\alpha^2 - \beta^2}{2z_0^3}\,u^2 + \mathcal{O}(u^3), \tag{4.31}$$

and the first order approximation is

$$V_{a+n,c-n}(\alpha,\beta,nz) \sim \frac{a^{\alpha+n}b^{\beta-n}}{z_0}\frac{1}{n}, \qquad n \to \infty, \tag{4.32}$$

for α, β bounded away from 0.

When $z < z_0$ then the root t_+ is positive and becomes relevant in the asymptotic analysis for large n. In this case, we apply the classical Laplace method and obtain

$$V_{a+n,c-n}(\alpha,\beta,nz) \sim \sqrt{\frac{2\pi}{n\phi''(t_+)}}f(t_+)e^{-n\phi(t_+)}, \qquad n \to \infty, \tag{4.33}$$

where

$$\phi(t_+) = zt_+ + \log(1 + \beta z + zt_+) \tag{4.34}$$

and

$$\phi''(t_+) = \frac{z}{\beta - \alpha}\sqrt{z(\beta-\alpha)(4 + z(\beta-\alpha))}. \tag{4.35}$$

When $z = z_0$ we have t_+ coalescing with the endpoint $t = 0$, and when $z \sim z_0$ we can use an error function to describe the transition from $t_+ < 0$ to $t_+ > 0$. Instead of the transformation in (4.29) we take a quadratic function in the right-hand side, because we want to introduce a movable saddle point in the transformation. We put

$$\phi(t) - \phi(0) = \tfrac{1}{2}v^2 - \eta v, \tag{4.36}$$

where η is not depending on t or v, and is chosen such that the saddle point t_+ corresponds with the saddle point of the quadratic function, that is, with $v = \eta$. Notice also the corresponding points $t = 0$ and $v = 0$, and corresponding relations at $+\infty$. For η we find

$$\phi(t_+) - \phi(0) = -\tfrac{1}{2}\eta^2, \tag{4.37}$$

with the agreement $\operatorname{sign}\eta = \operatorname{sign} t_+$. After using the transformation (4.36) in (4.23) we obtain

$$V_{a+n,c-n}(\alpha, \beta, nz) = \left(\frac{\alpha}{\beta}\right)^n \int_0^\infty e^{-n(\frac{1}{2}v^2 - \eta v)} g(v)\, dv, \tag{4.38}$$

where

$$g(v) = f(t)\frac{dt}{dv}, \quad \frac{dt}{dv} = \frac{v-\eta}{\phi'(t)}. \tag{4.39}$$

When we replace $g(v)$ with a constant g_c (that is, independent of v), we obtain an approximation in terms of the complementary error function; see (1.1). The optimal choice for this constant is as follows:

$$\begin{aligned} g_c &= g(0) = -\alpha^a \beta^c \frac{z_0}{\eta}, && \text{if} \quad t_+ \le 0, \\ g_c &= g(\eta) = (t_+ + \alpha)^a (t_+ + \beta)^c \frac{1}{\sqrt{\phi''(t_+)}}, && \text{if} \quad t_+ \ge 0, \end{aligned} \tag{4.40}$$

where $\phi''(t_+)$ is given in (4.35). In this way we find

$$V_{a+n,c-n}(\alpha, \beta, nz) \sim \sqrt{\frac{\pi}{2n}}\, g_c e^{-n\phi(t_+)} \operatorname{erfc}\left(-\eta\sqrt{n/2}\right). \tag{4.41}$$

See [13] for further details on this method.

Remark 1 The approximation in (4.41) is in particular useful when $t_+ \sim 0$ (in which case $\eta \sim 0$). We verify how the approximation behaves when

$\alpha \sim \beta$, a degenerate case, as remarked in the second case after (4.26). Applying Watson's lemma to (4.23) we obtain

$$V_{a+n,c-n}(\alpha, \alpha, nz) \sim \frac{\alpha^{a+c}}{zn}, \tag{4.42}$$

and using (4.41) we find the same estimate, because

$$t_+ = -\alpha, \quad \eta = -\sqrt{2\alpha z}, \quad \phi(t_+) = -\alpha z, \quad g_c = \alpha^{a+c}\sqrt{2\alpha/z}, \tag{4.43}$$

and

$$\operatorname{erfc}\left(\sqrt{\alpha z n}\right) \sim \frac{e^{-\alpha z n}}{\sqrt{\alpha z n \pi}}. \tag{4.44}$$

Remark 2 The approximation in (4.33) is most relevant for deciding if the backward recursion scheme can be used for the numerical evaluation of $V_{\nu,\mu}(\alpha, \beta, z)$. To see this, observe that we have derived this estimate with the scaled value nz. So, for considering the computation of $V_{\nu,\mu}(\alpha, \beta, z)$ for moderate values of z, the asymptotic estimate should also be used for moderate values of nz, that is, for $z = o(1)$, when n is large. This shows that the condition $z\alpha\beta < \beta - \alpha$ is fulfilled, unless $\beta \sim \alpha$, the degenerate case.

When we expand $\phi(t_+)$ given in (4.34) for small values of z, we obtain

$$\phi(t_+) = 2\sqrt{z(\beta - \alpha)} - \tfrac{1}{2}(\alpha + \beta)z + \mathcal{O}\left(z^{3/2}\right), \tag{4.45}$$

and we see that this gives the same relevant factor $e^{-2\sqrt{n(\beta-\alpha)z}}$ as given in the asymptotic estimate (4.16) for the minimal solution of the homogeneous part of the recurrence relation (3.8).

With this in mind we consider both $V_{\nu,\mu}(\alpha, \beta, z)$ and $y_n^{(2)}(z)$ given in (3.9) as minimal solutions of the recurrence relation in (3.6) when (4.45) holds.

5 Numerical results

In this section we provide some examples of numerical evaluation of the $V-$function and give some details on the computational use of the methods explained in the previous sections. We will restrict ourselves to the case $\nu = -1/2$ and $\mu = -1$, which is relevant for the singular perturbation problem explained in the introduction. It appears that the recursion to compute the $U-$functions in the expansion (4.14) is more efficient and reliable than the one described in §3.1.3, but further numerical tests are needed for comparing both methods.

5.1 Modified asymptotic series

We recall the modified asymptotic series given in (4.14):

$$V_{\nu,\mu}(\alpha,\beta,z) = \alpha^{\nu}\beta^{\mu+1}\sum_{k=0}^{\infty} d_k\Phi_k, \tag{5.1}$$

where

$$d_k = \left(\frac{\beta-\alpha}{\alpha}\right)^k \binom{\nu}{k}, \qquad \Phi_k = k!\; U(k+1,\nu+\mu+2,\beta z), \tag{5.2}$$

and the series is convergent provided that $0 < \beta < 2\alpha$. We write the series as follows:

$$V_{\nu,\mu}(\alpha,\beta,z) = \alpha^{\nu}\beta^{\mu+1}\sum_{k=0}^{K-1} d_k\Phi_k + R_K, \tag{5.3}$$

and will give an estimation for R_K. Now

$$R_K = \alpha^{\nu}\beta^{\mu+1}\sum_{k=K}^{\infty} d_k\Phi_k = \alpha^{\nu}\beta^{\mu+1}\sum_{k=0}^{\infty} d_{K+k}\Phi_{K+k}. \tag{5.4}$$

Since

$$\binom{\nu}{k} = (-1)^k\frac{(-\nu)_k}{k!}, \tag{5.5}$$

and $(a)_{K+k} = (a)_K(a+K)_k$, we can write

$$\sum_{k=0}^{\infty} d_{K+k}\Phi_{K+k} = C_K\sum_{k=0}^{\infty}\frac{(-\nu+K)_k}{(K+1)_k}\times \int_0^{\infty} e^{-\beta zt}t^K(1+t)^{\nu+\mu-K}\left(\frac{(\alpha-\beta)t}{\alpha(1+t)}\right)^k dt, \tag{5.6}$$

where

$$C_K = \left(\frac{\alpha-\beta}{\alpha}\right)^K\frac{(-\nu)_K}{(1)_K}. \tag{5.7}$$

Interchanging summation and integration, we get

$$\sum_{k=0}^{\infty} \frac{(-\nu+K)_k}{(K+1)_k} \int_0^{\infty} e^{-\beta z t} t^K (1+t)^{\nu+\mu-K} \left(\frac{(\alpha-\beta)t}{\alpha(1+t)} \right)^k dt$$

$$= \int_0^{\infty} e^{-\beta z t} t^K (1+t)^{\nu+\mu-K} \ {}_2F_1 \left(\begin{matrix} -\nu+K, 1 \\ K+1 \end{matrix} ; \frac{(\alpha-\beta)t}{\alpha(1+t)} \right) dt$$

$$= \alpha \int_0^{\infty} e^{-\beta z t} t^K (1+t)^{\nu+\mu-K+1} (\alpha+\beta t)^{-1} \ {}_2F_1 \left(\begin{matrix} \nu+1, 1 \\ K+1 \end{matrix} ; \frac{(\beta-\alpha)t}{\beta t+\alpha} \right) dt$$

$$\approx \alpha \int_0^{\infty} e^{-\beta z t} t^K (1+t)^{\nu+\mu-K+1} (\alpha+\beta t)^{-1} \ dt \leq K! \ U(K+1, \nu+\mu+3, \beta z).$$

Here we have estimated the Gauss hypergeometric function by 1 for large enough K, and used the well known identity [12]:

$$ {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = (1-z)^{-b} \ {}_2F_1 \left(\begin{matrix} c-a, b \\ c \end{matrix} ; \frac{z}{z-1} \right). \tag{5.8}$$

For the Kummer $U-$function we have the following asymptotic estimate for large values of K (see also (4.16)):

$$\Gamma(K+1) U(K+1, \nu+\mu+3, \beta z) \sim \lambda K^{\frac{2\nu+2\mu+3}{4}} e^{-2\sqrt{K\beta z}}, \tag{5.9}$$

where

$$\lambda = \sqrt{\pi} (\beta z)^{-\frac{5+2\nu+2\mu}{4}} e^{\frac{\beta z}{2}}. \tag{5.10}$$

If we set $\nu = -1/2$ and $\mu = -1$ the expression simplifies to

$$\Gamma(K+1) U(K+1, \nu+\mu+3, \beta z) \sim \sqrt{\frac{\pi}{\beta z}} \ e^{\frac{\beta z}{2}} e^{-2\sqrt{K\beta z}}, \tag{5.11}$$

This last expression can be easily checked for increasing values of K until a given threshold is reached. It is clear that large values of z and β will make this method more effective.

Once we have estimated the truncation index K, it is convenient to evaluate the series in the following nested form:

$$\sum_{k=0}^{K} d_k \Phi_k = d_0 \Phi_0 \left(1 + \frac{d_1}{d_0} \frac{\Phi_1}{\Phi_0} \left(1 + \frac{d_2}{d_1} \frac{\Phi_2}{\Phi_1} \left(\ldots + \left(1 + \frac{d_K}{d_{K-1}} \frac{\Phi_K}{\Phi_{K-1}} \right) \right) \right) \right), \tag{5.12}$$

since in this way the ratios can be obtained via continued fractions and we avoid the numerical computation of the U-functions. More precisely, we evaluate $r_K = \Phi_K/\Phi_{K-1}$ using the modified Lentz-Thompson method [6], and once we have that ratio we update it applying the backward recursion:

$$r_k = \frac{-\alpha_k}{\beta_k + r_{k+1}}, \qquad j = K-1, K-2, \ldots, 1. \tag{5.13}$$

Finally, the first function of the series is $\Phi_0 = U(1, \nu + \mu + 2)$, which in general is not directly available in Matlab. However, in the case $\nu = -1/2$, $\mu = -1$, which is our main interest for applications, this function reduces to the complementary error function.

In the example shown in Figures 1 and 2 these estimations have been used, with ϵ given by the machine epsilon in Matlab. The figures show a comparison between the result given by the modified asymptotic series in Matlab and direct integration using extended precision in Maple, using $\nu = -1/2$, $\mu = -1$ and 1000 random points in the region $0 < \beta < 2\alpha$, $0 < \alpha < 20$.

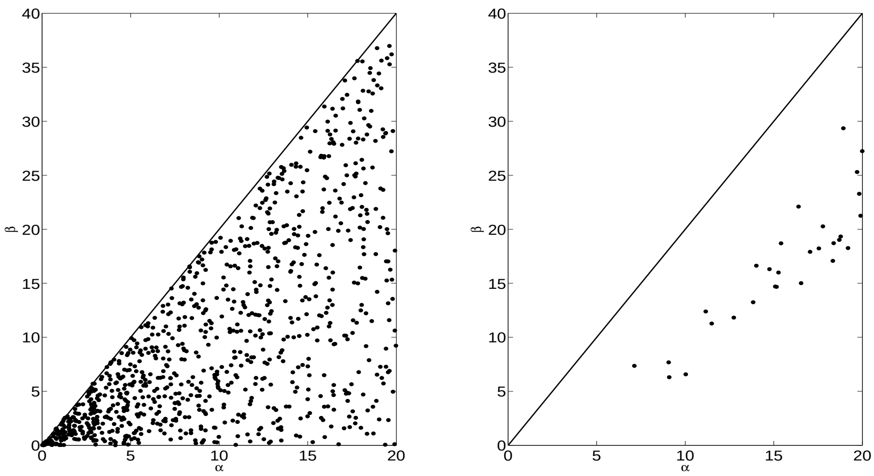

Figure 1: Absolute error in the computation of the $V_{-1/2,-1}(\alpha, \beta, z)$ function using modified asymptotic series. On the left, black dots indicate values of α and β for which the error is less than 10^{-14}, on the right black dots show where the error is larger than 10^{-14}. Here $z = 10.45$. The maximum error in this example is 1.78×10^{-14}.

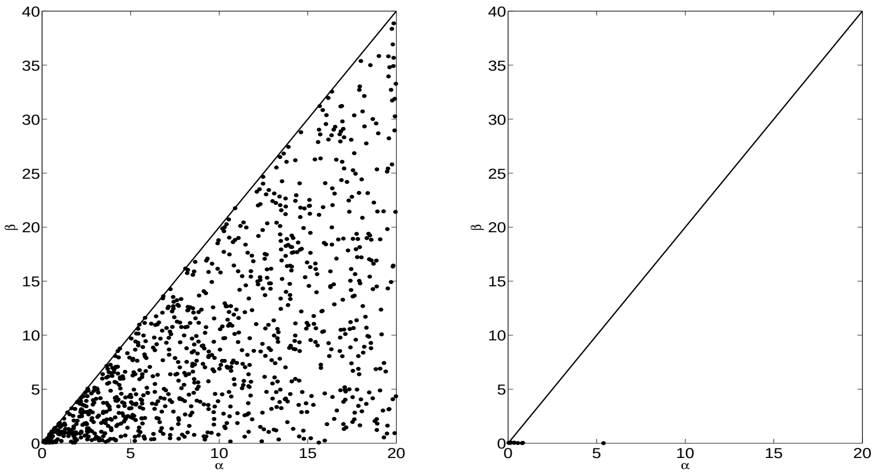

Figure 2: Same as in Figure 1 but with $z = 0.45$. The maximum error in this example is 1.05×10^{-13}, for values of β very close to 0.

5.2 Power series expansion

Another interesting method for numerical evaluation of the V function is given by formula (4.2) and (4.7). Recall that

$$G_{\nu,\mu}(\alpha,\beta,z) + V_{\nu,\mu}(\alpha,\beta,z) = (\beta-\alpha)^{\nu+\mu+1} e^{\alpha z}\Gamma(\nu+1) U(\nu+1, \nu+\mu+2, (\beta-\alpha)z), \tag{5.14}$$

where

$$G_{\nu,\mu}(\alpha,\beta,z) = \alpha^{\nu+1}(\beta-\alpha)^{\mu}\Gamma(\nu+1)\sum_{k=0}^{\infty} d_k H_k(\mu,\nu,\gamma), \tag{5.15}$$

the coefficients are $d_k = (\alpha z)^k$ and $H_k(\mu,\nu,\gamma)$ are Gauss hypergeometric functions, see formula (4.8). Now

$$\sum_{k=0}^{\infty} d_k H_k(\nu,\mu,\gamma) = \sum_{k=0}^{K-1} d_k H_k(\nu,\mu,\gamma) + R_K. \tag{5.16}$$

Using the integral representation of the H_k functions and interchanging summation and integration (which is permissible when $\beta > 2\alpha$, since in that case $\gamma < 1$), as we did in the previous subsection, we obtain

$$R_K \approx \frac{(\alpha z)^K}{\Gamma(\nu + K + 2)}, \tag{5.17}$$

which we used to estimate an adequate truncation index K. Then the finite series should be evaluated in nested form, as in (5.12), and using the corresponding continued fraction for the Gauss functions, which can be deduced from [1, Eq. 15.2.12]. We also need the function $H_0(\nu, \mu, \gamma)$, which is general is not elementary. However, in the case $\nu = -1/2$, $\mu = -1$, we have:

$$H_0(-1/2, -1, \gamma) = \frac{\arctan \sqrt{\gamma}}{\sqrt{\gamma}}. \tag{5.18}$$

Observe also that when $\nu = -1/2$ and $\mu = -1$, the Kummer $U-$function in (4.2) reduces to a complementary error function.

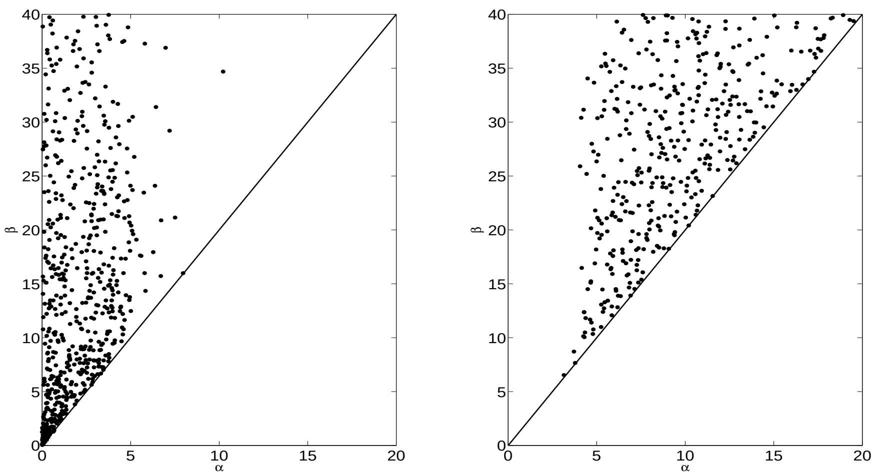

Figure 3: Absolute error in the computation of the $V_{-1/2,-1}(\alpha, \beta, z)$ function using power series. On the left, black dots indicate values of α and β for which the error is less than 10^{-14}, on the right black dots indicate where the error is larger than 10^{-14}. Here $z = 0.87$, and the maximum error is 2.7×10^{-9}.

Figures 3 and 4 show another example with 1000 points in the region $\beta > 2\alpha$, $0 < \alpha < 20$. As can be seen, the results get considerably worse when α and z grow. This behaviour can be expected, since large values of

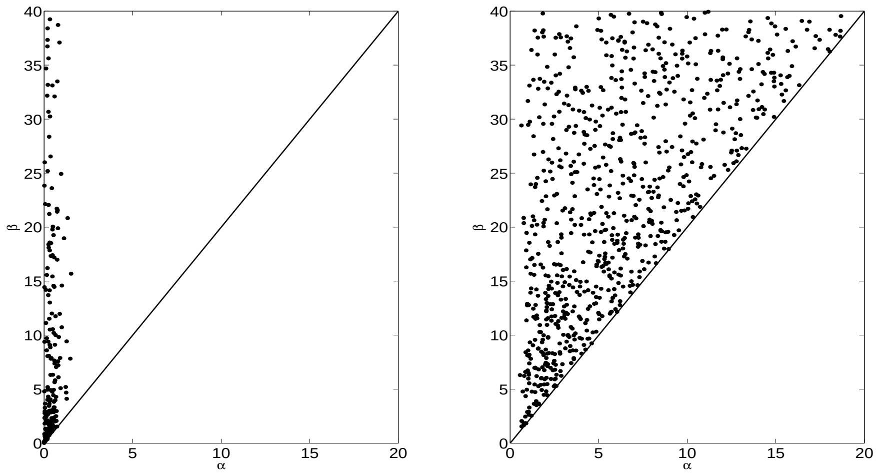

Figure 4: Same as in Figure 3 but with $z = 4.31$. The maximum error in this example is 2.82×10^{19}.

α are likely to produce severe cancellations when subtracting $G_{\nu,\mu}(\alpha,\beta,z)$ from the Kummer $U-$function. Similarly, large values of z will be harmful, since from (4.10) we have that

$$V_{\nu,\mu}(\alpha,\beta,z) \sim \frac{\alpha^\nu \beta^\mu}{z}, \qquad z \to \infty, \tag{5.19}$$

whereas both the first and last terms in (4.2) are exponentially large when z is large.

On the other hand, for small values of the parameters the results are very satisfactory, which is quite relevant since other methods such as modified asymptotic expansions or quadrature become less attractive when z is small.

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