Generalized sampling and the stable and accurate reconstruction of piecewise analytic functions from their Fourier coefficients

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Abstract

Suppose that the first m Fourier coefficients of a piecewise analytic function are given. Direct expansion in a Fourier series suffers from the Gibbs phenomenon and lacks uniform convergence. Nonetheless, in this paper we show that, under very broad conditions, it is always possible to recover an n-term expansion in a different system of functions using only these coefficients. Such an expansion can be made arbitrarily close to the best possible n-term expansion in the given system. Thus, if a piecewise polynomial basis is employed, for example, exponential convergence can be restored. The resulting method is linear, numerically stable and can be implemented efficiently in only $\mathcal{O}(nm)$ operations.

A key issue is how the parameter m must scale in comparison to n to ensure recovery. We derive analytical estimates for this scaling for large classes of polynomial and piecewise polynomial bases. In particular, we show that in many important cases, including the case of piecewise Chebyshev polynomials, this scaling is quadratic: $m = \mathcal{O}(n^2)$. Therefore, using a system of polynomials that the user is essentially free to choose, one can restore exponential accuracy in n and root exponential accuracy in m. This generalizes a result proved recently for piecewise Legendre polynomials.

The method developed in this paper is part of new numerical framework for sampling and reconstruction in abstract Hilbert spaces, known as generalized sampling. This paper extends previous work by the authors by introducing a substantially more flexible methodology which allows for sampling and reconstruction with respect to different inner products. In the final part of this paper we illustrate the application of generalized sampling to a related family of problems.

1 Introduction

The Fourier expansion of an analytic and periodic function f converges exponentially fast in the truncation parameter m. For this reason – as well as the fact that the expansion can be computed rapidly via the Fast Fourier Transform (FFT) – such approximations are extremely widely used in all areas of computational mathematics, with one important instance being the spectral solution of partial differential equations (PDEs) [15, 19]. However, rapid convergence is lost when the function is no longer analytic. Indeed, when f is only piecewise smooth, one encounters the familiar Gibbs phenomenon. This not only destroys local accuracy – characteristic $\mathcal{O}(1)$ oscillations are witnessed near each jump in the function – it also inhibits global approximation: the truncated expansion converges only very slowly in the L^2 norm, and not at all when the error is measured in the uniform norm. Notably, this issue arises when computing spectral approximations to PDEs that develop shocks – hyperbolic conservation laws, for example [31, 41].

This naturally leads to the following question: given the first m Fourier coefficients of a piecewise analytic function f, how can one recover high orders of approximation? The problem is not new, and there have been many different approaches developed for its solution (see, for example, [2, 11, 12, 17, 18, 22, 23, 24, 25, 52] and references therein). Of this large collection, perhaps the most successful and commonly used is the method of spectral reprojection [30, 33, 34, 35]. Introduced by D. Gottlieb et al [35], in this approach the Fourier coefficients of f are used to approximate the first $n = \mathcal{O}(m)$ coefficients in a new basis of piecewise polynomials (the *reconstruction* basis). Provided this basis is chosen appropriately, one obtains exponential accuracy in m. To date, spectral reprojection has been successfully used in a range of

applications, including image reconstruction [9, 10], and the postprocessing of spectral discretizations of hyperbolic PDEs [28, 31, 41].

Whilst spectral reprojection has been widely successful, there are a number of drawbacks. In particular, spectral reprojection achieves rapid convergence by employing a particular choice of reconstruction basis. Herein lies a problem. Only very few bases (known as *Gibbs complementary bases* [30, 34]) have this property, with the two most commonly used being ultraspherical (Gegenbauer) and Freud polynomials. In both cases, however, the parameter α defining the polynomials must scale linearly with n. This is not only computationally inconvenient – changing n requires recomputation of the whole reprojection basis – in addition, a rather careful selection of parameters must be employed to ensure such convergence and avoid potentially a Runge-type phenomenon [16]. Because of the stringent requirements placed on the reconstruction basis, spectral reprojection consequently affords little flexibility to the user. At the very least, Gegenbauer and Freud polynomials are no where near as easy to use and manipulate as Chebyshev polynomials, for example, with the latter being amenable to the FFT. Another issue, as shown empirically in [7], is that, even if exponential convergence occurs, the rate of this convergence in the Gibbs complementary basis may be quite slow in practice, meaning that many Fourier coefficients may be required to achieve high accuracy.

With this in mind, the purpose of this paper is to consider a different approach based on an alternative idea. Unlike spectral reprojection, where the reconstruction basis must be suitably chosen to ensure rapid convergence, the method we develop in this paper allows the user to employ an arbitrary basis. In other words, we address the following problem:

Problem 1.1. *Given the first m Fourier coefficients of a piecewise analytic function f, where m is sufficiently large, recover an approximation $f_{n,m}$ to the n-term expansion $\mathcal{Q}_n f$ in an arbitrary basis of piecewise polynomials (the reconstruction basis) orthogonal to with respect the inner product $\langle \cdot, \cdot \rangle$ satisfying $\|f - f_{n,m}\| \asymp \|f - \mathcal{Q}_n f\|$ as $n, m \to \infty$, where $\|\cdot\|$ is the corresponding norm corresponding.*

Here and elsewhere we use the notation $a_n \asymp b_n$ for nonnegative sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ to mean that, for all sufficiently large n there exist $c_1, c_2 > 0$ such that $c_1 a_n \leq b_n \leq c_2 a_n$. Observe that $\|f - f_{n,m}\| \asymp \|f - \mathcal{Q}_n f\|$ implies that the approximation $f_{n,m}$ is *quasi-optimal* to f from the set $\mathbb{P}_n$: as $n \to \infty$, $f_{n,m}$ converges to f at precisely the same rate as the best approximation $\mathcal{Q}_n f$.

As we show, by letting m vary independently of n, the restriction of having to employ a Gibbs complementary basis is avoided. This allows one to use far more convenient polynomial bases in reconstruction, such as the aforementioned Chebyshev basis. Note that the case of (piecewise) Legendre polynomial bases was developed in [7] (therein several examples demonstrating improved performance over spectral reprojection were also given). The main contribution of this paper (see §1.4 for a more detailed description) is to generalize this work to arbitrary bases of piecewise polynomials.

1.1 Reconstructions in Hilbert spaces

Our solution to Problem 1.1 is based on recasting it in terms of sampling and reconstruction in abstract Hilbert spaces. To this end, let H^S and H^R be subspaces of a vector space V that form Hilbert spaces with respect to the bilinear forms $\langle \cdot, \cdot \rangle_S$ and $\langle \cdot, \cdot \rangle_R$. Suppose that $\{\psi_j\}_{j=1}^{\infty}, \{\phi_j\}_{j=1}^{\infty}$ are orthonormal bases for $(\mathrm{H}^S, \langle \cdot, \cdot \rangle_S)$ and $(\mathrm{H}^R, \langle \cdot, \cdot \rangle_R)$ respectively (the *sampling* and *reconstruction* bases), and let U $\subseteq \mathrm{H}^R$ be a subspace (U will consist of those functions f we wish to reconstruct). For $f \in$ U, let

$$\hat{f}_j = \langle f, \psi_j \rangle_S, \quad j = 1, 2, \ldots,$$

be the *samples* of f. The method we develop in this paper is designed to solve the problem:

Problem 1.2. *Given the first m samples $\hat{f}_1, \ldots, \hat{f}_m$ of $f \in$ U, recover an approximation $f_{n,m}$ to $\mathcal{Q}_n f$ satisfying $\|f - f_{n,m}\|_R \asymp \|f - \mathcal{Q}_n f\|_R$, where $\mathcal{Q}_n : (\mathrm{H}^R, \langle \cdot, \cdot \rangle_R) \to \mathrm{T}_n := \mathrm{span}\{\phi_1, \ldots, \phi_n\}$ is the orthogonal projection.*

It transpires that, under certain assumptions, this problem is solvable, and the resulting framework is linear, and completely numerically stable. The key is that m must scale appropriately to n. The required scaling can be derived numerically, and, in the cases we consider in this paper, analytically.

It is straightforward to see that this framework can be applied to Problem 1.1. In particular, consider the simple scenario where f is analytic, but nonperiodic (equivalently, $f : \mathbb{T} \to \mathbb{R}$ has a jump at $x = -1$,

where $\mathbb{T} = [-1, 1)$ is the unit torus). Given the Fourier coefficients of f, we would like to reconstruct in the orthonormal basis of Chebyshev polynomials $\phi_j(x) = c_j T_j(x)$ of the first kind, where

$$T_j(x) = \cos(j \arccos x), \quad j = 0, 1, 2, \ldots,$$

and $c_0 = \frac{1}{\sqrt{\pi}}$, $c_j = \sqrt{\frac{2}{\pi}}$, $j \neq 1$. Given that the first kind Chebyshev polynomials T_j are orthogonal with respect to the weight function $w(x) = (1 - x^2)^{-\frac{1}{2}}$, we thus let $\mathrm{H}^S = \mathrm{L}^2(-1, 1)$ and $\mathrm{H}^R = \mathrm{L}^2_w(-1, 1)$ (the spaces of square-integrable functions and weight square-integrable functions with respect to the weight function w respectively), and define $\psi_j(x) = \frac{1}{\sqrt{2}} \exp(\mathrm{i}j\pi x)$ to be the standard Fourier basis.

As is no doubt apparent to the reader, Problem 1.2 is actually far more general than Problem 1.1. Indeed, the abstract framework introduced to solve Problem 1.2 can be applied far more widely than the problem of reconstructing from Fourier data. We shall discuss several other important applications of this framework at the end of this paper. In particular, we shall illustrate how this framework can be used to solve another common problem of this type: namely, the reconstruction of a piecewise analytic function from its coefficients with respect to an orthogonal polynomial basis. Much like the Fourier case, the problem also notably occurs in spectral discretizations of hyperbolic PDEs [31, 34, 41].

1.2 Orthogonal polynomial systems

It is useful at this moment to introduce some notation. We shall be principally concerned in this paper with Jacobi polynomials, with corresponding weight function

$$w_{\alpha,\beta}(x) = (1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1.$$

We write $\mathrm{L}^2_{\alpha,\beta}(-1, 1)$ for the space of weighted square-integrable functions with respect to $w_{\alpha,\beta}$, and $\|\cdot\|_{\alpha,\beta}$ for the corresponding norm. Our main examples herein will be ultraspherical (or Gegenbauer) polynomials ($\alpha = \beta = \gamma$), and in particular, Legendre polynomials ($\gamma = 0$) and Chebyshev polynomials of the first ($\gamma = -\frac{1}{2}$) and second ($\gamma = \frac{1}{2}$) kinds. Whenever $\alpha = \beta = \gamma$ we shall use the slightly more succinct notation $\mathrm{L}^2_\gamma(-1, 1)$, $\|\cdot\|_\gamma$ and w_γ.

Legendre and Chebyshev polynomials are the most common in applications – in particular, spectral methods for PDEs [15, 19]. More specifically, Chebyshev polynomials are chosen for reconstruction because of their aforementioned computational flexibility, whereas Legendre polynomials are desirable because of the simplicity of the Legendre weight function $w(x) \equiv 1$. However, it is of both theoretical and practical interest to maintain the generality of Jacobi polynomials. Indeed, whilst Legendre and Chebyshev polynomials are most frequently used in practice, there are a number of applications which employ Jacobi polynomial systems [15, 19, 36, 37].

If f is piecewise smooth with jumps at $-1 < x_1 < \ldots < x_l < 1$ we shall seek to reconstruct in bases of piecewise polynomials. Thus, with $\alpha = \{\alpha_0, \ldots, \alpha_l\}$ and $\beta = \{\beta_0, \ldots, \beta_l\}$ we define the piecewise Jacobi weight function $w_{\alpha,\beta}(x)$ by

$$w_{\alpha,\beta}(x) = (x_{r+1} - x)^{\alpha_r} (x - x_r)^{\beta_r}, \quad x \in I_r := (x_r, x_{r+1}), \qquad r = 0, \ldots, l.$$

Here $x_0 = -1$ and $x_{l+1} = 1$. The corresponding orthonormal system of piecewise polynomials can be obtained by appropriately scaling from the standard interval $[-1, 1]$ to each subinterval of smoothness I_r (see §3.1). When considering a finite expansion in such functions, we shall write $n = (n_0, \ldots, n_l) \in \mathbb{N}^{l+1}$ for the vector of polynomial degrees in each subinterval I_r, and define

$$\mathrm{T}_n = \{\phi : \phi|_{I_r} \in \mathbb{P}_{n_r}, \ r = 0, \ldots, l\}. \tag{1.1}$$

The operator $\mathcal{Q}_n : \mathrm{L}^2_{\alpha,\beta}(-1, 1) \to \mathrm{T}_n$ will be the orthogonal projection with respect to the weight function $w_{\alpha,\beta}$. As in the no-jump (analytic and nonperiodic) setting, the main examples we consider in this paper involve reconstructions in piecewise Chebyshev or Legendre polynomials, i.e. $\alpha = \beta = \gamma$, where $\gamma = \{-\frac{1}{2}, \ldots, -\frac{1}{2}\}$ (first kind), $\gamma = \{\frac{1}{2}, \ldots, \frac{1}{2}\}$ (second kind) or $\gamma = \{0, \ldots, 0\}$ respectively.

Remark 1.3 Many of the results we prove in this paper extend quite trivially to the modified Jacobi weight

$$w_{\alpha,\beta}(x) = g(x)(1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1, \tag{1.2}$$

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where $g(x)$ is analytic and positive on $[-1, 1]$. Likewise, one may define the piecewise modified Jacobi weight

$$w_{\alpha,\beta}(x) = g_r(x)(x_{r+1} - x)^{\alpha_r}(x - x_r)^{\beta_r}, \quad x \in I_r := (x_r, x_{r+1}), \tag{1.3}$$

where g_r is analytic and positive in $\overline{I_r}$. Although $w_{\alpha,\beta}$, as given by (1.3), is only unique (for given α and β) up to multiplication by a collection $g_0, \ldots, g_l$ of positive, analytic functions, we shall continue to write $L^2_{\alpha,\beta}(-1, 1)$ for the corresponding space of square-integrable functions with respect to (1.3), and shall not make the dependence on the functions $g_0, \ldots, g_l$ explicit.

1.3 Key results

We devote the first part of this paper to the solution of Problem 1.2 and the properties of the resulting framework, including numerical stability. In the second half, we consider its application to Problem 1.1. In this regard, the key result we prove is as follows:

Theorem 1.4. *Let $\alpha = \{\alpha_0, \ldots, \alpha_l\}$ and $\beta = \{\beta_0, \ldots, \beta_l\}$, where $\alpha_r, \beta_r > -1$, $r = 0, \ldots, l$, be parameters corresponding to the (piecewise) modified Jacobi weight $w_{\alpha,\beta}$, and suppose that $\mathcal{Q}_n$ is the orthogonal projection onto T_n, as given by (1.1), with respect to $w_{\alpha,\beta}$. Define*

$$\tilde{\alpha}_r = \begin{cases} \alpha_r & -1 < \alpha_r < 1 \\ 1 - \epsilon_r & \alpha_r \geq 1 \end{cases}, \qquad \tilde{\beta}_r = \begin{cases} \beta_r & -1 < \beta_r < 1 \\ 1 - \epsilon'_r & \beta_r \geq 1 \end{cases},$$

where $0 < \epsilon_r, \epsilon'_r < 2$ are arbitrary, and let $\mathrm{U} := L^2_{\tilde{\alpha},\tilde{\beta}}(-1, 1)$. Suppose that the first m Fourier coefficients of $f \in \mathrm{U}$ are given. Then there exists a $c > 0$ independent of m, $n = (n_0, \ldots, n_l)$ and f such that, whenever $m \geq c\bar{n}^r$, where $\bar{n} = \max\{n_0, \ldots, n_l\}$ and

$$r = \max\left\{2, p+1, 2 + 2\frac{p-q}{q+1}\right\},$$

$$p = \max\{\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_l\}, \quad q = \min\{\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_l\},$$

one may compute, in a completely numerically stable manner, a reconstruction $f_{n,m} \in \mathrm{T}_n$ satisfying

$$\|f - \mathcal{Q}_n f\|_{\alpha,\beta} \leq \|f - f_{n,m}\|_{\alpha,\beta} \leq c'\|f - \mathcal{Q}_n f\|_{\tilde{\alpha},\tilde{\beta}},$$

for some c' depending only on c. Moreover, excluding preprocessing step of at most $\mathcal{O}\left(m^2\right)$ cost, the number of operations required to compute $f_{n,m}$ is $\mathcal{O}\left(nm\right)$. If $w_{\alpha,\beta}$ is a piecewise ultraspherical weight, then such preprocessing involves only $\mathcal{O}\left(nm\right)$ operations.

This theorem guarantees solution of Problem 1.1 for a very large class of functions. In fact, if $-1 < \alpha_r, \beta_r < 1$, $r = 0, \ldots, l$, then we can recover any $f \in \mathrm{U} \equiv \mathrm{H}^R$, where $\mathrm{H}^R = L^2_{\alpha,\beta}(-1, 1)$. In the general case, note that U, although not equal to H^R, is still a very large space, and is therefore certainly sufficient for practical purposes (recall that the principal concern of this paper is piecewise analytic functions). Of course, higher smoothness of f ensures fast convergence of $\mathcal{Q}_n f$, and therefore $f_{n,m}$. However, this theorem demonstrates that quasi-optimal, stable recovery with GS is always possible regardless of smoothness.

In practice, it is important to understand the constant c of Theorem 1.4. Fortunately, c can always be computed numerically (see §2.3). Also, empirical evidence suggests that a small value of c is usually suitable. In the numerical experiments in this paper we use $c = \frac{1}{4}$, which gives perfectly acceptable results (see §4). One could also use a smaller value, with the only difference being a slightly larger, but still $\mathcal{O}\left(1\right)$, condition number.

The proof of Theorem 1.4 relies on careful analysis of the behaviour of Fourier expansions in certain weighted spaces. Herein, a connection is made with the Helson–Szegö Theorem on positive measures [26]. Another feature of our analysis is that, in many cases, the best constants corresponding to the quasi-optimality $\|f - f_{n,m}\|_R \asymp \|f - \mathcal{Q}_n f\|_R$ are known, and can be explicitly computed. Specifically, one has the sharp bounds

$$\|f - f_{n,m}\|_R \leq \frac{1}{C_{n,m}}\|f - \mathcal{Q}_n f\|_R,$$

where the constant $C_{n,m}$ is bounded (provided $m \geq c\bar{n}^r$) and has a geometrical interpretation in terms of the angle between two particular finite-dimensional subspaces.

Note also the great flexibility guaranteed by this theorem. In many important cases, $m = \mathcal{O}\left(n^2\right)$ samples of a piecewise analytic function allow one to recover close to the best n-term approximation in a given piecewise polynomial system. This includes (piecewise) Legendre ($p = q = 0$) and Chebyshev polynomials ($p = q = \pm\frac{1}{2}$), both of which are commonly used in applications. In particular,

Corollary 1.5. *Given the first m Fourier coefficients of a piecewise analytic function f, one can compute, in $\mathcal{O}(m^{\frac{3}{2}})$ operations, a root-exponentially convergent piecewise polynomial approximation of f that consists of arbitrary piecewise ultraspherical polynomials with parameter $-1 < \gamma < 1$.*

In summary, the method we present in this paper improves on the more conventional approach of spectral reprojection by allowing reconstructions in arbitrary polynomial bases. In addition, spectral reprojection incurs a relatively large cost of $\mathcal{O}\left(m^2\right)$ operations. Here we also obtain an improvement, by a factor of $\mathcal{O}(m^{\frac{1}{2}})$. Having said this, spectral reprojection is formally exponentially convergent, whereas this approach obtains only root-exponential convergence. However, as we shall demonstrate by numerical example (see §4.2), this rate of exponential convergence may be quite slow in practice, meaning that the approach of this paper, whilst formally less rapidly convergent, actually gives a better numerical approximation.

Remark 1.6 At first sight it may appear that the quadratic scaling $m = \mathcal{O}\left(n^2\right)$, resulting in root exponential convergence in m, is less than optimal, and therefore could be improved. As discussed in [6] (see also [50] for a closely related result), root-exponential convergence is the best that can be attained by a stable method for this problem, and thus the framework developed in this paper is *optimally stable*.

1.4 Relation to previous work

The special case $\mathrm{H}^S = \mathrm{H}^R = \mathrm{H}$, $\langle\cdot,\cdot\rangle_R = \langle\cdot,\cdot\rangle_S = \langle\cdot,\cdot\rangle$ was first considered in [4, 7]. Therein an abstract framework, known as *generalized sampling (GS)*, was developed to solve Problem 1.2 in this more simple setting. The framework introduced in this paper to address the significantly more general case $\mathrm{H}^S \neq \mathrm{H}^R$, $\langle\cdot,\cdot\rangle_S \neq \langle\cdot,\cdot\rangle_R$ is a direct extension of this work, and, for this reason, we shall continue to refer to the resulting framework as generalized sampling.

The original GS framework of [4, 7] can be applied successfully to Problem 1.1 whenever $\mathcal{Q}_n f$ is the Legendre polynomial expansion of f (this corresponds to $\mathrm{H}^R = \mathrm{H}^S = \mathrm{L}^2(-1, 1)$), leading to a stable numerical method. This particular instance of GS was also is described [42]. Unfortunately this framework is not as flexible as one may hope. Although one can also reconstruct using arbitrary bases of polynomials, the resulting method is only numerically stable when Legendre polynomials are employed. Moreover, in the ultraspherical case, for example, this method also incurs an increased cost (over the value of $\mathcal{O}(m^{\frac{3}{2}})$ for the Legendre case) proportional to $\mathcal{O}(m^{\frac{3+|\gamma|}{2}})$, where γ is the corresponding polynomial parameter.

Reconstructions in arbitrary orthogonal polynomial systems from Fourier coefficients have also been considered in [44, 45]. However, the approach introduced therein is exponentially unstable, and liable to suffer from a Runge-type phenomenon (i.e. divergence for insufficiently analytic functions). The work of [42] can be viewed as a regularization of this approach, yet only the case of Legendre polynomials was considered. Fortunately, the extended GS framework we develop in this paper, which exploits the orthogonality of both the sampling and reconstruction bases, allows for completely stable reconstructions in only $\mathcal{O}(m^{\frac{3}{2}})$ operations using significantly more general families of polynomials. Moreover, as we explain in due course, the extended GS framework can be reinterpreted succinctly as a preconditioning of the original GS framework. This preconditioner acts to stabilize the particular linear system, and thereby reduces the overall computational cost.

1.5 Outline

The outline of the remainder of this paper is as follows. In §2 we introduce and analyze the extended version of the GS framework to solve Problem 1.2. §3 is devoted to (piecewise) polynomial reconstructions from Fourier samples, and in particular, the solution of Problem 1.1 and proof of Theorem 1.4. In §4 we give numerical examples and in §5 we discuss applications of this extended GS framework to several related reconstruction problems.

2 An extended generalized sampling framework

Let H^S and H^R be subspaces of a vector space V that form Hilbert spaces in their own right with respect to the bilinear forms $\langle \cdot, \cdot \rangle_S$ and $\langle \cdot, \cdot \rangle_R$ respectively. Let U $\subseteq H^R$ be a subspace (not necessarily closed) and assume that $f \in$ U and that $\{\phi_j\}_{j=1}^\infty \subseteq$ U, where $\{\phi_j\}$ is some orthonormal basis for $(H^R, \langle \cdot, \cdot \rangle_R)$. The subspace U consists of those functions f that we wish to recover and the subspace $T_n := \mathrm{span}\{\phi_1, \ldots, \phi_n\}$ is the space in which we seek to reconstruct. If $\{\psi_j\}_{j=1}^\infty \subseteq$ U is an orthonormal basis of $(H^S, \langle \cdot, \cdot \rangle_S)$, we assume that we have access to the first m samples of f

$$\hat{f}_j = \langle f, \psi_j \rangle_S, \quad j = 1, \ldots, m, \tag{2.1}$$

with respect to this basis (we shall assume that V is such that these values exist and are finite). In practice, the basis $\{\phi_j\}_{j=1}^\infty$ is chosen with some *a priori* knowledge about f. For the applications considered in this paper, where f is a piecewise analytic function on $[-1, 1]$, $\{\phi_j\}_{j=1}^\infty$ will consist of orthonormal piecewise polynomials with respect to some weight function and ψ_j will correspond to the complex exponential $\frac{1}{\sqrt{2}} e^{ij\pi x}$ (in this case we shall enumerate ψ_j over $\mathbb{Z}$ as opposed to $\mathbb{N}$).

Remark 2.1 Note that the assumption of orthonormality in the sampling and reconstruction vectors $\{\psi_j\}_{j=1}^\infty$, $\{\phi_j\}_{j=1}^\infty$ is not necessary. It is reasonably straightforward to relax this condition to that of a Riesz basis [7] (and also further to a frame [8]). However, since all sampling and reconstruction vectors of interest in this paper will be orthonormal, we shall accept this slight loss of generality.

We seek to solve Problem 1.2: namely, compute an approximation $f_{n,m} \in T_n$ from the samples (2.1) that is close to $\mathcal{Q}_n f$, where $\mathcal{Q}_n : (H^R, \langle \cdot, \cdot \rangle_R) \to T_n$ is the orthogonal projection. Note that $\mathcal{Q}_n f$ is defined by the equations

$$\langle \mathcal{Q}_n f, \phi_j \rangle_R = \langle f, \phi_j \rangle_R, \quad j = 1, \ldots, n, \quad \mathcal{Q}_n f \in T_n. \tag{2.2}$$

Had we had access to the values $\langle f, \phi_j \rangle_R$, i.e. the coefficients of f in a particular polynomial basis, for example, we could have computed $\mathcal{Q}_n f$ immediately. However, in general we do not have this information. Specifically, all we know is $\{\hat{f}_j\}_{j=1}^m$. Since the sampling basis $\{\psi_j\}_{j \in \mathbb{N}}$ is fixed, and cannot be altered, we are unable to form $\mathcal{Q}_n f$ from the given data.

Nonetheless, let us write $\mathcal{P}_m : H^S \to S_m = \mathrm{span}\{\psi_1, \ldots, \psi_m\}$ for orthogonal projection onto $S_m \subseteq$ U with respect to $\langle \cdot, \cdot \rangle_S$, i.e.

$$\mathcal{P}_m g = \sum_{j=1}^m \langle g, \psi_j \rangle_S \psi_j, \quad g \in H^S.$$

We now define $f_{n,m} \in T_n$ as the solution to

$$\langle \mathcal{P}_m f_{n,m}, \mathcal{P}_m \phi \rangle_R = \langle \mathcal{P}_m f, \mathcal{P}_m \phi \rangle_R, \quad \forall \phi \in T_n, \qquad f_{n,m} \in T_n. \tag{2.3}$$

Suppose for a moment that $(H^S, \langle \cdot, \cdot \rangle_S)$ and $(H^R, \langle \cdot, \cdot \rangle_R)$ coincide. This is precisely the setting of the original GS framework of [7]. It is now quite simple to give an intuitive explanation as to how $f_{n,m}$ solves Problem 1.2. Indeed, since $\{\psi_j\}_{j=1}^\infty$ is an orthonormal basis for $(H^S, \langle \cdot, \cdot \rangle_S)$, we have that $\mathcal{P}_m \to \mathcal{I}$ in the strong operator topology on $(H^S, \langle \cdot, \cdot \rangle_S)$ [20], where $\mathcal{I} : H^S \to H^S$ is the identity operator. Thus, for fixed $n \in \mathbb{N}$, if $m \to \infty$ the equations (2.3) resemble those equations (2.2) defining $\mathcal{Q}_n f$. Therefore, for sufficiently large m, we expect $f_{n,m}$ to inherit the features of $\mathcal{Q}_n f$ (in particular, $f_{n,m}$ should exist uniquely), and critically its good approximation properties.

In the general case $(H^S, \langle \cdot, \cdot \rangle_S) \neq (H^R, \langle \cdot, \cdot \rangle_R)$, we need the additional assumption that $\mathcal{P}_m \to \mathcal{I}$ strongly on $(U, \langle \cdot, \cdot \rangle_R)$:

$$\|g - \mathcal{P}_m g\|_R \to 0, \quad m \to \infty, \quad \forall g \in U. \tag{A1}$$

Note that this assumption is not guaranteed in general, and must be verified for each particular problem. However, as we shall see in §3, it does hold in all cases of interest in this paper. We will also require one further assumption: namely, a uniform boundedness condition for $\{\mathcal{P}_m\}_{m \in \mathbb{N}}$ in terms of some norm $\|\cdot\|$ defined on U:

$$\|\mathcal{P}_m g\|_R \leq c \|g\|, \quad \forall g \in U, \ m \in \mathbb{N}, \tag{A2}$$

where $c > 0$ is independent of m and g.

In some circumstances, but not all, it transpires that the following stronger version of (A2) holds:

$$\mathrm{U} = \mathrm{H}^R, \qquad \mathcal{P}_m : \left(\mathrm{H}^R, \langle\cdot,\cdot\rangle_R\right) \to \left(\mathrm{H}^R, \langle\cdot,\cdot\rangle_R\right) \text{ bounded.} \tag{A2$'$}$$

To see that this is stronger than (A2), we note that

Lemma 2.2. *Suppose that (A1) and (A2$'$) hold. Then the family $\{\mathcal{P}_m\}_{m\in\mathbb{N}}$ is uniformly bounded. In other words, (A2) holds with $\|\cdot\| = \|\cdot\|_R$.*

Proof. (A1) and (A2$'$) imply that $\sup_{m\in\mathbb{N}} \|\mathcal{P}_m g\| < \infty$ for any $g \in \mathrm{U} = \mathrm{H}^R$. The uniform boundedness principle now gives the result. $\qquad\square$

Whenever (A2$'$) holds instead of (A2) we may give a more precise analysis of the approximation $f_{n,m}$ (see §2.2). In §3 we detail the situations in which (A2$'$) holds as opposed to (A2) for the Fourier reconstruction problem.

2.1 Recovery from Fourier coefficients

Before analysing this framework, let us first discuss the main example of this paper. Suppose that $f : [-1,1] \to \mathbb{R}$ is piecewise analytic with jumps at $-1 < x_1 < \ldots < x_l < 1$. Let $\psi_j(x) = \frac{1}{\sqrt{2}} \exp(\mathrm{i} j\pi x)$, $j \in \mathbb{Z}$, and assume that the coefficients

$$\hat{f}_j = \int_{-1}^1 f(x)\overline{\psi_j}(x)\,\mathrm{d}x, \quad j = -\left\lfloor \frac{m}{2} \right\rfloor, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, \tag{2.4}$$

are given. In particular, $\mathcal{P}_m f$ is the truncated Fourier series of f (we henceforth refer to $\mathcal{P}_m$ as the Fourier projection operator in this case). We wish to recover f in a basis of orthonormal piecewise polynomials corresponding to the piecewise (modified) Jacobi weight $w_{\alpha,\beta}$ given by (1.3), where $\alpha = \{\alpha_0, \ldots, \alpha_l\}$ and $\beta = \{\beta_0, \ldots, \beta_l\}$. To this end, let $x_0 = -1$, $x_{l+1} = 1$ and write

$$\mathrm{T}_n = \left\{\phi : \phi|_{I_r} \in \mathbb{P}_{n_r} : r = 0, \ldots, l\right\}, \quad n = (n_0, \ldots, n_l), \tag{2.5}$$

where $I_r = (x_r, x_{r+1})$. We now compute $f_{n,m} \in \mathrm{T}_n$ using (2.3). Thus, we let $\mathrm{V} = \mathrm{L}^1(-1,1)$ (the space of absolutely integrable functions), $\mathrm{H}^R = \mathrm{L}^2_{\alpha,\beta}(-1,1)$, $\|\cdot\|_R = \|\cdot\|_{\alpha,\beta}$, and set $\mathrm{H}^S = \mathrm{L}^2(-1,1)$, $\|\cdot\|_S = \|\cdot\|$.

There are now three key questions:

  (i) How do we choose the subspace U and the norm $\|\cdot\|$ so that (A1) and (A2) hold?
 (ii) Does such a choice of U include all functions of interest? Specifically, U should at the very least contain all piecewise analytic functions.
(iii) In what circumstances does (A2$'$) hold instead of (A2)?

In §3 we provide answers to these questions.

Remark 2.3 Throughout we assume that the discontinuity locations $x_1, \ldots, x_l$ are known. Indeed, this is necessary prior knowledge for constructing an appropriate reconstruction basis of piecewise polynomials. In practice, a fully-automated algorithm must also incorporate a scheme for locating such singularities. There are numerous methods for this problem, and we shall not discuss this issue any further. We refer the reader to [29, 52].

2.2 Analysis of the extended framework

We present two types of analysis of this framework. The first, for which the corresponding error estimates (i.e. bounds for $\|f - f_{n,m}\|_R$) are not sharp, is valid whenever (A1) and (A2) hold. The second, which assumes the stronger condition (A2$'$), leads to sharp bounds.

2.2.1 Version I

The definition (2.3) of $f_{n,m}$ motivates the introduction of the quantity

$$E_{n,m} = \sup\{\|\phi - \mathcal{P}_m\phi\|_R : \phi \in \mathrm{T}_n,\ \|\phi\|_R = 1\}. \tag{2.6}$$

This quantity measures how close the restriction $\mathcal{P}_m|_{\mathrm{T}_n}$ is to the identity operator $\mathcal{I}|_{\mathrm{T}_n}$. Since (A1) holds and T_n is finite-dimensional, the following lemma is unsurprising:

Lemma 2.4. *Suppose that $E_{n,m}$ is defined by (2.6). Then $E_{n,m} \leq 1 + cd_n$ for all $m,n \in \mathbb{N}$, where c is as in (A2) and*

$$d_n = \sup\{\|\phi\| : \phi \in \mathrm{T}_n,\ \|\phi\|_R = 1\}. \tag{2.7}$$

Moreover, $E_{n,m} \to 0$ as $m \to \infty$ for fixed n.

Proof. Note that $E_{n,m} \leq 1 + \sup\{\|\mathcal{P}_m\phi\|_R : \phi \in \mathrm{T}_n,\ \|\phi\|_R = 1\}$. (A2) now gives the first result. For the second, observe that the set $\mathrm{B}_n := \{\phi \in \mathrm{T}_n : \|\phi\|_R = 1\}$ is compact. Hence, since $\mathcal{P}_m \to \mathcal{I}$ strongly on $\mathrm{U} \supseteq \mathrm{B}_n$, this convergence is uniform on B_n. $\qquad\square$

With this to hand, we are now able to prove the main result of this section:

Theorem 2.5. *For every $n \in \mathbb{N}$ there exists an m_0 such that the approximation $f_{n,m}$, defined by (2.3), exists and is unique and satisfies the estimates*

$$\|f_{n,m}\|_R \leq \frac{c}{1 - E_{n,m}}\|f\|, \tag{2.8}$$

and

$$\|f - f_{n,m}\|_R \leq \|f - \mathcal{Q}_nf\|_R + \frac{cE_{n,m}}{(1 - E_{n,m})^2}\|f - \mathcal{Q}_nf\| + \frac{1}{(1 - E_{n,m})^2}\|(\mathcal{I} - \mathcal{P}_m)(f - \mathcal{Q}_nf)\|_R, \tag{2.9}$$

where $\mathcal{Q}_n : \mathrm{H}^R \to \mathrm{T}_n$ is the orthogonal projection with respect to $\langle\cdot,\cdot\rangle_R$ and $E_{n,m}$ is as in (2.6). Specifically, m_0 is the least m such that $E_{n,m} < 1$.

Note that (2.8) is a continuous stabilty estimate for $f_{n,m}$. In particular, this implies that the coefficients of $f_{n,m}$ cannot grow large.

Proof. Let $\mathcal{U} : \mathrm{T}_n \to \mathbb{C}^n$ be the linear operator $g \mapsto \{\langle\mathcal{P}_mg, \mathcal{P}_m\phi_j\rangle_R\}_{j=1}^n$. It suffices to show that $\mathcal{U}$ is invertible; in other words, $\ker(\mathcal{U}) = \{0\}$. Suppose that $g \in \mathrm{T}_n$ with $\mathcal{U}g = 0$. Then, by linearity, $\langle\mathcal{P}_mg, \mathcal{P}_m\phi\rangle_R = 0$, $\forall\phi \in \mathrm{T}_n$. In particular, $0 = \|\mathcal{P}_mg\|_R \geq (1 - E_{n,m})\|g\|_R$. Thus, if $m \geq m_0$, where m_0 is the least m such that $E_{n,m} < 1$, then we must have that $g = 0$. Hence, $\mathcal{U}$ is nonsingular, and $f_{n,m}$ exists uniquely.

Now consider (2.8). Setting $\phi = f_{n,m}$ in (2.3) gives

$$\|\mathcal{P}_mf_{n,m}\|_R^2 = \langle\mathcal{P}_mf, \mathcal{P}_mf_{n,m}\rangle_R \leq \|\mathcal{P}_mf\|_R\|\mathcal{P}_mf_{n,m}\|_R \leq c\|f\|\|\mathcal{P}_mf_{n,m}\|_R.$$

The inequality (2.8) now follows directly from the definition of $E_{n,m}$ and (A2).

For (2.9) we first write $e = f_{n,m} - \mathcal{Q}_nf \in \mathrm{T}_n$ and notice that

$$\|\mathcal{P}_me\|_R^2 = \langle\mathcal{P}_m(f - \mathcal{Q}_nf), \mathcal{P}_me\rangle_R.$$

Since $e \in \mathrm{T}_n$ and $f - \mathcal{Q}_nf \perp \mathrm{T}_n$ is the orthogonal projection, the right hand side may be written as

$$\langle\mathcal{P}_m(f - \mathcal{Q}_nf), \mathcal{P}_me\rangle_R - \langle f - \mathcal{Q}_nf, e\rangle_R$$
$$= -\langle(\mathcal{I} - \mathcal{P}_m)(f - \mathcal{Q}_nf), e\rangle_R - \langle\mathcal{P}_m(f - \mathcal{Q}_nf), (\mathcal{I} - \mathcal{P}_m)e\rangle_R.$$

This gives

$$(1 - E_{n,m})^2\|e\|_R^2 \leq \|\mathcal{P}_me\|_R^2 \leq \|(\mathcal{I} - \mathcal{P}_m)(f - \mathcal{Q}_nf)\|_R\|e\|_R + \|\mathcal{P}_m(f - \mathcal{Q}_nf)\|_R\|(\mathcal{I} - \mathcal{P}_m)e\|_R.$$

Hence, by (A2) and the definition of $E_{n,m}$, we obtain

$$(1 - E_{n,m})^2\|e\|_R \leq \|(\mathcal{I} - \mathcal{P}_m)(f - \mathcal{Q}_nf)\|_R + cE_{n,m}\|f - \mathcal{Q}_nf\|.$$

The full result follows from the inequality $\|f - f_{n,m}\|_R \leq \|e\|_R + \|f - \mathcal{Q}_nf\|_R$. $\qquad\square$

This theorem confirms solution of Problem 1.2 using (2.3) whenever (A1) and (A2) hold, and provided m is sufficiently large in comparison to n. The question of how large will be discussed and quantified in §2.3. Observe that

$$\|g - \mathcal{P}_m g\|_R \leq \|g\|_R + c\|g\|, \quad \forall g \in U, \quad \forall m \in \mathbb{N},$$

hence the term $\|(\mathcal{I} - \mathcal{P}_m)(f - \mathcal{Q}_n f)\|_R$ in right-hand side of (2.9) can be easily bounded independently of m to give the more convenient bound

$$\|f - f_{n,m}\|_R \leq \left[1 + \frac{1}{(1 - E_{n,m})^2}\right]\|f - \mathcal{Q}_n f\|_R + \frac{c(1 + E_{n,m})}{(1 - E_{n,m})^2}\|f - \mathcal{Q}_n f\|. \tag{2.10}$$

However, this bound obscures one interesting facet of $f_{n,m}$: namely, its *asymptotic optimality*. That is, by increasing the number of samples m, $f_{n,m}$ can be made arbitrarily close to the best approximation $\mathcal{Q}_n f$. Indeed,

Corollary 2.6. *Suppose that $f_{n,m}$ is defined by (2.3). Then, for fixed n, $f_{n,m} \to \mathcal{Q}_n f$ as $m \to \infty$.*

Proof. Since $E_{n,m} \to 0$ as $m \to \infty$ (Lemma 2.4), we have

$$\|e\|_R \leq \frac{cE_{n,m}}{(1 - E_{n,m})^2}\|f - \mathcal{Q}_n f\| + \frac{1}{(1 - E_{n,m})^2}\|(\mathcal{I} - \mathcal{P}_m)(f - \mathcal{Q}_n f)\|_R \to 0,$$

as $m \to \infty$, where $e = f_{n,m} - \mathcal{Q}_n f$. Thus, $f_{n,m} \to \mathcal{Q}_n f$ as required. $\qquad\square$

The analysis of this section demonstrates quasi-optimality of $f_{n,m}$ (as well as asymptotic optimality). However, it is clearly less than ideal to have an error estimate of the form (2.9) involving $\|f - \mathcal{Q}_n f\|$. In general, there is no guarantee that this term decays as $n \to \infty$ (although this is always true in the applications of this paper). Thus, one may ask: can a better analysis give an error bound involving only $\|f - \mathcal{Q}_n f\|_R$ (which must tend to zero since $\mathcal{Q}_n : H^R \to T_n$ is the orthogonal projection)? As it transpires, whenever (A2′) holds instead of (A2), this is indeed the case. This is described in the next section. The resulting analysis also provides bounds for $\|f_{n,m}\|_R$ and $\|f - f_{n,m}\|_R$ which, unlike those given in Theorem 2.5, are sharp.

2.2.2 Version II

Suppose now that (A2′) holds instead of (A2). To derive an improved analysis, it is useful to introduce the notion of oblique projections in Hilbert spaces:

Definition 2.7. *Let U and V be closed subspaces of a Hilbert space H. A mapping $\mathcal{W} := \mathcal{W}_{UV} : H \to U$ is an oblique projection onto U along V if $\mathcal{W}^2 = \mathcal{W}$ and $\mathcal{W}(v) = 0$, $\forall v \in V$.*

We shall also require the following:

Definition 2.8. *Let U and V be closed subspaces of a Hilbert space H and $\mathcal{Q}_V : H \to V$ the orthogonal projection onto V. The subspace angle $\theta = \theta_{UV}$ between U and V is given by*

$$\cos(\theta_{UV}) = \inf\left\{\|\mathcal{Q}_V u\| : u \in U, \|u\| = 1\right\}.$$

Note that, $U \oplus V = H$ if and only if there exists a unique oblique projection onto U along V [51]. Equivalently, the cosine of the subspace angle $\cos\theta_{UV^\perp} \neq 0$. In the particular case that $V = U^\perp$ (i.e. $\cos\theta_{UV} = 1$), $\mathcal{W}_{UV} = \mathcal{Q}_U$ is the orthogonal projection onto U.

We now require the following theorem:

Theorem 2.9 ([6]). *Suppose that U and V are closed subspaces of H satisfying $U \oplus V^\perp = H$, and let $\mathcal{W} : H \to U$ be the oblique projection onto U along $V^\perp$. Then we have the sharp bounds*

$$\|\mathcal{W}f\| \leq \frac{1}{\cos\theta_{UV}}\|f\|, \quad \forall f \in H, \tag{2.11}$$

and

$$\|f - \mathcal{Q}f\| \leq \|f - \mathcal{W}f\| \leq \frac{1}{\cos\theta_{UV}}\|f - \mathcal{Q}f\|, \quad \forall f \in H, \tag{2.12}$$

where $\mathcal{Q} : H \to U$ is the orthogonal projection. In particular, if $f \in U$ then $\mathcal{W}f = f$.

This aside, note that, since (A2′) holds, the operator $\mathcal{P}_m$ is bounded on H^R. Hence it has a bounded adjoint $\mathcal{P}_m^* : \left(\mathrm{H}^R, \langle \cdot, \cdot \rangle\right) \to \left(\mathrm{H}^R, \langle \cdot, \cdot \rangle\right)$. Write $\mathcal{W}_m := \mathcal{P}_m^* \circ \mathcal{P}_m$. We require the following lemma:

Lemma 2.10. *The operators $\mathcal{W}_m$ converge strongly to $\mathcal{I}$ on $\left(\mathrm{H}^R, \langle \cdot, \cdot \rangle\right)$.*

Proof. For $g \in \mathrm{H}^R$, we have

$$\begin{aligned}
\|g - \mathcal{W}_m g\|^2 &= \langle g - \mathcal{P}_m^* \mathcal{P}_m g, g - \mathcal{P}_m^* \mathcal{P}_m g \rangle \\
&\leq |\langle \mathcal{P}_m^*(g - \mathcal{P}_m g), g - \mathcal{P}_m^* \mathcal{P}_m g \rangle| + |\langle g - \mathcal{P}_m^* g, g - \mathcal{P}_m^* \mathcal{P}_m g \rangle|.
\end{aligned}$$

Since $\{\mathcal{P}_m\}_{m \in \mathbb{N}}$ is uniformly bounded, so is $\{\mathcal{P}_m^*\}_{m \in \mathbb{N}}$. Moreover, the adjoint operation is strongly continuous on bounded sets. Hence $\mathcal{P}_m^* \to \mathcal{I}$ strongly on H^R and we obtain, for some $c > 0$ independent of g and m,

$$\|g - \mathcal{W}_m g\|^2 \leq c \left(\|g - \mathcal{P}_m g\| + \|g - \mathcal{P}_m^* g\| \right) \|g\| \to 0, \quad m \to \infty,$$

as required. $\qquad\square$

We are now ready to analyze $f_{n,m}$. First, note that we may rewrite (2.3) as

$$\langle f_{n,m}, \mathcal{W}_m \phi \rangle_R = \langle f, \mathcal{W}_m \phi \rangle_R, \quad \forall \phi \in \mathrm{T}_n.$$

In other words, $f_{n,m}$ (whenever it exists) is precisely the oblique projection of f onto T_n along the subspace $[\mathcal{W}_m(\mathrm{T}_n)]^\perp$ (here $\perp$ is taken with respect to $(\mathrm{H}^R, \langle \cdot, \cdot \rangle_R)$). Letting

$$C_{n,m} = \cos \theta_{\mathrm{T}_n, \mathcal{W}_m(\mathrm{T}_n)}, \tag{2.13}$$

we immediately deduce the following:

Theorem 2.11. *Suppose that $n, m \in \mathbb{N}$ are such that $C_{n,m} > 0$, where $C_{n,m}$ is given by (2.13). Then $f_{n,m}$, as defined by (2.3), exists uniquely and satisfies the sharp bounds*

$$\|f_{n,m}\|_R \leq \frac{1}{C_{n,m}} \|f\|_R, \tag{2.14}$$

and

$$\|f - f_{n,m}\|_R \leq \frac{1}{C_{n,m}} \|f - \mathcal{Q}_n f\|_R. \tag{2.15}$$

Naturally, to use this theorem, we need to understand the quantity $C_{n,m}$. We have

Lemma 2.12. *The quantity $C_{n,m}$ defined by (2.13) satisfies*

$$\frac{1}{c}(1 - E_{n,m}) \leq C_{n,m} \leq 1, \quad \forall n, m \in \mathbb{N}, \tag{2.16}$$

where $c = \sup_{m \in \mathbb{N}} \|\mathcal{P}_m\|$ and $\|\cdot\|$ is the operator norm on the space of bounded linear maps $(\mathrm{H}^R, \langle \cdot, \cdot \rangle_R) \to (\mathrm{H}^R, \langle \cdot, \cdot \rangle_R)$. Moreover, $C_{n,m} \to 1$ as $m \to \infty$ for fixed n.

Proof. That $C_{n,m} \leq 1$ follows immediately from the definition (2.13). Moreover, by the definition of subspace angles

$$C_{n,m} = \inf_{\substack{\phi \in \mathrm{T}_n \\ \|\phi\|_R = 1}} \|\mathcal{Q}_{\mathcal{W}_m(\mathrm{T}_n)} \phi\|_R.$$

Consider the quantity $\|\mathcal{Q}_{\mathcal{W}_m(\mathrm{T}_n)} \phi\|_R$. By the standard duality pairing,

$$\begin{aligned}
\|\mathcal{Q}_{\mathcal{W}_m(\mathrm{T}_n)} \phi\|_R &= \sup \left\{ \frac{\langle \mathcal{Q}_{\mathcal{W}_m(\mathrm{T}_n)} \phi, \mathcal{W}_m \phi' \rangle_R}{\|\mathcal{W}_m \phi'\|_R} : \phi' \in \mathrm{T}_n, \ \phi' \neq 0 \right\} \\
&= \sup \left\{ \frac{\langle \phi, \mathcal{W}_m \phi' \rangle_R}{\|\mathcal{W}_m \phi'\|_R} : \phi' \in \mathrm{T}_n, \ \phi' \neq 0 \right\}. \tag{2.17}
\end{aligned}$$

Setting $\phi' = \phi$, we obtain

$$\|\mathcal{Q}_{\mathcal{W}_m(\mathrm{T}_n)} \phi\|_R \geq \frac{\langle \phi, \mathcal{W}_m \phi \rangle_R}{\|\mathcal{W}_m \phi\|_R} = \frac{\|\mathcal{P}_m \phi\|_R^2}{\|\mathcal{W}_m \phi\|_R}.$$

Since $\mathcal{W}_m = \mathcal{P}_m^* \circ \mathcal{P}_m$ we obtain

$$\|\mathcal{Q}_{\mathcal{W}_m(\mathrm{T}_n)}\phi\|_R \geq \frac{1}{c}\|\mathcal{P}_m\phi\|_R \geq \frac{1}{c}\left(1 - E_{n,m}\right)\|\phi\|,$$

which gives (2.16). The proof that $C_{n,m} \to 1$ as $m \to \infty$ is analogous to the proof of Lemma 2.4, and follows directly from the fact that $\mathcal{W}_m \to \mathcal{I}$ (Lemma 2.10). $\qquad\square$

The importance of (2.17) is that it allows $C_{n,m}$ to be estimated in terms of $E_{n,m}$; a somewhat easier task. Thus, estimates obtained later for the asymptotic behaviour of $E_{n,m}$ automatically hold for $C_{n,m}$, and thus guarantee Theorem 2.11. We shall also exploit this fact in §3. Note that this lemma also implies asymptotic optimality of $f_{n,m}$: since

$$\|f_{n,m} - \mathcal{Q}_n f\|^2 = \|f - f_{n,m}\|^2 - \|f_{n,m} - \mathcal{Q}_n f\|^2 \leq K_{n,m}\|f - \mathcal{Q}_n f\|^2 \leq K_{n,m}\|f\|^2,$$

where $K_{n,m} = \frac{1}{C_{n,m}^2} - 1$, we have $f_{n,m} \to \mathcal{Q}_n f$ as $m \to \infty$.

It is instructive to relate this general framework to the example of §2.1. Suppose that we sample an analytic, nonperiodic f via its Fourier coefficients (so that $\mathcal{P}_m$ is the standard Fourier projection operator) and wish to reconstruct in Chebyshev polynomials. We thus let $V = L^1(-1,1)$, $H^S = L^2(-1,1)$ and $H^R = L_w^2(-1,1)$, where $w(x) = (1 - x^2)^{-\frac{1}{2}}$ is the Chebyshev weight on $[-1,1]$. It transpires that both (A1) and (A2$'$) hold in this case (see §3). Therefore Theorem 2.11 and Lemma 2.12 apply to this example. Note that in this setting the operators $\mathcal{P}_m^*$ and $\mathcal{W}_m$ are given explicitly by

$$\mathcal{P}_m^* g = \frac{1}{w}\mathcal{P}_m(wg), \quad \mathcal{W}_m g = \frac{1}{w}\mathcal{P}_m\left(w\mathcal{P}_m g\right), \quad g \in L_w^2(-1,1). \tag{2.18}$$

2.3 Numerical implementation

Let us now consider the computation of $f_{n,m}$. Writing $f_{n,m} = \sum_{j=1}^n \alpha_j \phi_j$ for unknowns $\alpha_1, \ldots, \alpha_n$, it is easily seen that the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ is defined by

$$U^* C U \alpha = U^* C \hat{f},$$

where $\hat{f} = \{\langle f, \psi_1 \rangle_S, \ldots, \langle f, \psi_m \rangle_S\}$, and $U \in \mathbb{C}^{m \times n}$ and $C \in \mathbb{C}^{m \times m}$ have $(j, k)^{\text{th}}$ entries $\langle \phi_k, \psi_j \rangle_S$ and $\langle \psi_j, \psi_k \rangle_R$ respectively. Thus the coefficients of $f_{n,m}$ are defined by an $n \times n$ linear system of equations, with self-adjoint matrix $A = U^* C U$.

The condition number $\kappa(A)$ of the matrix A is critical from a numerical perspective. It determines both the stability of the numerical method – its susceptibility to noise and round-off errors, in particular – as well as the cost of any iterative solver (conjugate gradients, for example) for computing the vector of unknowns α. Specifically, the number of iterations required to compute α is proportional to $\sqrt{\kappa(A)}$. Thus, if $\kappa(A) = \mathcal{O}(1)$ for all n and sufficiently large m (which turns out to be the case), then the cost of computing $f_{n,m}$ is determined solely by the cost of performing matrix-vector multiplications involving A. Since the (typically dense) matrices U and C are of size $m \times n$ and $m \times m$ respectively, straightforward implementation of this method, that is, without any fast transforms, requires $\mathcal{O}\left(m^2\right)$ operations. However, as we shall explain in §4, C always has a Toeplitz structure when one considers Fourier samples. Thus, matrix-vector multiplications involving C require only $\mathcal{O}\left(m \log m\right)$ operations, giving a total figure of $\mathcal{O}\left(mn\right)$ for this framework.

We now give an estimate for $\kappa(A)$:

Lemma 2.13. *Let $E_{n,m}$ be given by (2.6). Then the condition number of $A = U^* C U$ satisfies*

$$\kappa(A) \leq \left(\frac{1 + E_{n,m}}{1 - E_{n,m}}\right)^2.$$

In particular, if n is fixed then $A \to I$ as $m \to \infty$, where $I \in \mathbb{C}^{n \times n}$ is the identity.

Proof. Let $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{C}^n$ be an normalized eigenvector of A with corresponding eigenvalue λ. If $\phi = \sum_{j=1}^n \alpha_j \phi_j$ it follows that

$$\langle \mathcal{P}_m \phi, \mathcal{P}_m \phi_j \rangle_R = \lambda \alpha_j, \quad j = 1, \ldots, n,$$

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and, in particular, $\|\mathcal{P}_m \phi\|_R^2 = \lambda$. Hence,

$$\lambda_{\min}(A) = \inf\left\{\|\mathcal{P}_m\phi\|_R^2 : \phi \in \mathrm{H}^R,\ \|\phi\|_R = 1\right\}, \quad \lambda_{\max}(A) = \sup\left\{\|\mathcal{P}_m\phi\|_R^2 : \phi \in \mathrm{H}^R,\ \|\phi\|_R = 1\right\}.$$

Using the definition $E_{n,m}$ we deduce that

$$(1 - E_{n,m})^2 \le \lambda_{\min}(A) \le \lambda_{\max}(A) \le (1 + E_{n,m})^2,$$

which gives the first result. For the second, we merely note that $A \to I$ componentwise as $m \to \infty$ (since $E_{n,m} \to 0$ by Lemma 2.4), and therefore in norm. $\qquad\square$

This lemma confirms good conditioning of A whenever m is sufficiently large in comparison to n. In fact, the condition number of A can be made arbitrarily close to unity by increasing m sufficiently.

Remark 2.14 As mentioned in §1.4, the extended GS framework introduced in this section can actually be reinterpreted as a preconditioning of the original GS framework of [7]. In the original framework, one solves $U^*U\alpha = U^*\hat{f}$. However, unless $(\mathrm{H}^R, \langle\cdot,\cdot\rangle_R) = (\mathrm{H}^S, \langle\cdot,\cdot\rangle_S)$ the matrix U^*U is not well-conditioned, leading to a higher cost in computing α. The extended framework merely replaces this with the preconditioned equations $U^*CU\alpha = U^*C\hat{f}$, where C is as above. This judicious choice of C preconditions the original equations (by exploiting the orthogonality of both the sampling and reconstruction vectors) and therefore gives the aforementioned improvement.

2.4   Scaling of m with n: the stable sampling rate

Observe that the same quantity $E_{n,m}$ which determines the magnitude of the error committed by $f_{n,m}$ also occurs in the estimate for the condition number. Indeed, provided m is chosen so that $1 - E_{n,m}$ is bounded away from zero, we can ensure both quasi-optimality of $f_{n,m}$ and numerical stability. This motivates the following definition. We let

$$\Theta(n;\theta) = \min\left\{m \in \mathbb{N} : E_{n,m} < 1 - \theta\right\}, \quad \theta \in (0,1). \tag{2.19}$$

For a given n, choosing $m \ge \Theta(n;\theta)$ ensures that

$$\|f - f_{n,m}\|_R \le \left(1 + \theta^{-2}\right)\|f - \mathcal{Q}_n f\|_R + c(2-\theta)\theta^{-2}\|f - \mathcal{Q}_n f\|, \tag{2.20}$$

and that

$$\kappa(A) \le \left(2\theta^{-1} - 1\right)^2.$$

In other words, $f_{n,m}$ is quasi-optimal to f from T_n, and its computation is numerically stable, uniformly in $n \in \mathbb{N}$.

The quantity $\Theta(n;\theta)$, originally introduced in [7] and named the *stable sampling rate* in [6], measures how large m (the number of samples) must be for a given n for stable, quasi-optimal reconstruction. Importantly, $\Theta(n;\theta)$ is computable. Indeed,

Lemma 2.15. *Let $E_{n,m}$ be given by (2.6). Then $E_{n,m} = \sqrt{\|B\|}$, where $B \in \mathbb{C}^{n \times n}$ is the Hermitian matrix with $(j,k)^{\mathrm{th}}$ entry $\langle\phi_j - \mathcal{P}_m\phi_j, \phi_k - \mathcal{P}_m\phi_k\rangle_R$.*

Proof. For $\phi \in \mathrm{T}_n$, write $\phi = \sum_{j=1}^n \alpha_j \phi_j$. Then

$$E_{n,m}^2 = \sup\left\{\frac{\sum_{j,k=1}^n \alpha_j\overline{\alpha_k}\langle\phi_j - \mathcal{P}_m\phi_j, \phi_k - \mathcal{P}_m\phi_k\rangle_R}{\sum_{j,k=1}^n \alpha_j\overline{\alpha_k}\langle\phi_j, \phi_k\rangle_R} : \alpha \in \mathbb{C}^n,\ \alpha \ne 0\right\}$$

$$= \sup\left\{\frac{\alpha^* B\alpha}{\alpha^*\alpha} : \alpha \in \mathbb{C}^n,\ \alpha \ne 0\right\}.$$

This is precisely $\|B\|$. $\qquad\square$

Note that the matrix B can be expressed as $B = I - V^*U - U^*V + A$, where $A = U^*CU$ and $V \in \mathbb{C}^{m \times n}$ has $(j,k)^{\mathrm{th}}$ entry $\langle\phi_k, \psi_j\rangle_R$. Hence, given U, C and V, one can readily compute $E_{n,m}$ and therefore $\Theta(n;\theta)$.

The estimate (2.20) for $\|f - f_{n,m}\|$ holds with the stronger assumption (A2′). In the case that this does hold, then it is preferable to replace (2.20) by the sharp bound (2.15). Using (2.16) we obtain the improved estimate

$$\|f - f_{n,m}\|_R \leq \frac{c}{\theta}\|f - \mathcal{Q}_n f\|_R. \tag{2.21}$$

In fact, one could, in theory, also use (2.15) directly, by numerically computing $C_{n,m}$ (recall that $C_{n,m}$ is the angle between two finite-dimensional subspaces, and hence computable), thereby giving a sharp (and consequently slightly better) bound.

The significance of $\Theta(n; \theta)$ is that it allows one to control *a priori* both the stability and accuracy of the numerical method. Whilst it is convenient that $\Theta(n; \theta)$ can always be computed numerically, it is also important to have analytical estimates for each particular application. Naturally, such estimates are completely dependent on both the sampling and reconstruction spaces, and thus must be obtained on a case-by-case basis. We shall devote much of the second half of this paper to the derivation of such estimates for the case of Fourier sampling and (piecewise) polynomial reconstructions.

3 Recovery from Fourier coefficients

We now consider the specific application of recovering a piecewise analytic function from its Fourier coefficients using a piecewise polynomial basis. As we show, it is possible to reconstruct in a completely stable manner in an arbitrary basis of piecewise polynomials, orthogonal with respect to the (piecewise) modified Jacobi weight. Thus, the individual polynomials employed can be specified by the user, making the method extremely flexible in this sense. In §5 we consider a number of other related reconstruction problems which can be addressed with this framework.

3.1 Preliminaries

Recall the setup of §2.1. Suppose that $f : [-1, 1] \to \mathbb{R}$ is piecewise analytic with jumps at the known locations $-1 < x_1 < \ldots < x_l < 1$. Set $x_0 = -1$ and $x_{l+1} = 1$. We shall assume that the first m Fourier coefficients (2.4) of f are given. Thus, we let $\mathrm{H}^S = \mathrm{L}^2(-1, 1)$ with its standard inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and define $\psi_j(x) = \frac{1}{\sqrt{2}} \exp(\mathrm{i}j\pi x)$ (recall that in this case we enumerate $\{\psi_j\}$ over $\mathbb{Z}$ as opposed to $\mathbb{N}$).

We shall reconstruct in the space T_n defined by (2.5), with a basis of this space consisting piecewise polynomials orthogonal with respect to the piecewise (modified) Jacobi weight (1.3) with parameters $\alpha = \{\alpha_0, \ldots, \alpha_l\}$, $\beta = \{\beta_0, \ldots, \beta_l\}$, where $\alpha_r, \beta_r > -1$, $r = 0, \ldots, l$. We let $\mathrm{H}^R = \mathrm{L}^2_{\alpha,\beta}(-1, 1)$ with corresponding inner product $\langle \cdot, \cdot \rangle_{\alpha,\beta}$ and norm $\|\cdot\|_{\alpha,\beta}$.

We enumerate the orthonormal basis for T_n as follows. First, let $c_r = \frac{1}{2}(x_{r+1} - x_r)$ and define $\Lambda_r(x) = \frac{x - x_r}{c_r} - 1$, so that $\Lambda(I_r) = [-1, 1]$, where $I_r = [x_r, x_{r+1}]$. For given α_r, β_r, let ϕ_j be the corresponding orthonormal modified Jacobi polynomial of degree j on $[-1, 1]$. Assume that $\phi_j \equiv 0$ outside $[-1, 1]$ and define the local (modified) Jacobi polynomial on I_r by $\phi_{r,j} = \frac{1}{\sqrt{c_r}}\phi_j \circ \Lambda_r$. It follows that the set

$$\{\phi_{r,j} : j = 0, \ldots, n_r - 1, \ r = 0, \ldots, l\}, \tag{3.1}$$

forms an orthonormal basis for T_n, and hence we may write $f_{n,m}$ as

$$f_{n,m} = \sum_{r=0}^{l} \sum_{j=0}^{n_r-1} \alpha_{r,j}\phi_{r,j},$$

with unknowns $\alpha_{r,j} \in \mathbb{C}$. We now compute $f_{n,m}$ via (2.3).

We first wish to analyze $f_{n,m}$ (implementation will be discussed in §4). This means applying the analysis of §2, and for this we need to verify (A1) and (A2). Additionally, we also wish to determine when we may replace (A2) by the stronger condition (A2′), and hence when the sharp bounds of Theorem 2.11 apply. Verifying these assumptions requires understanding the convergence of Fourier series in the weighted spaces $\mathrm{L}^2_{\alpha,\beta}(-1, 1)$. This is the content of the next section.

3.2 Convergence of Fourier series in weighted norms

Somewhat surprisingly, this question has been considered before. Convergence of Fourier series with respect to piecewise modified Jacobi weights are simple consequences of this more general theory, which we now recap. The reader is referred to [40, 43] or more recently [13], for further details.

Suppose that $w(x)$ is a weight function on $[-1, 1]$ with $w(x) \geq 0$ almost everywhere and such that $0 < w(x) < \infty$ almost everywhere in $[-1, 1]$. We shall denote the space of weighted square integrable functions by $\mathrm{L}_w^2(-1, 1)$, with corresponding norm $\|\cdot\|_w$. We say that $w(x)$ is a *Helson–Szegö* weight function if and only if there exists a constant $c > 0$ such that

$$\left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I \frac{1}{w} \right) \leq c,$$

for all subsets $I \subseteq [-1, 1]$ that are either subintervals or complements of subintervals in $[-1, 1]$ (here $|I|$ denotes the length of I).

There is a fundamental relationship between Helson–Szegö weight functions and Fourier series. This is summarized in the following theorem:

Theorem 3.1 (see [43] or [13])**.** *The following four statements are equivalent:*

  (i) *w is a Helson–Szegö weight function,*
 (ii) *There exist real functions u, v on $[-1, 1]$ with $\|u\|_\infty, \|v\|_\infty < \infty$ such that $w(x) = \mathrm{e}^{u(x) + \tilde{v}(x)}$ almost everywhere,*
(iii) *There exists a finite constant $c > 0$ such that, for all $f \in \mathrm{L}_w^2(-1, 1)$, we have $f \in \mathrm{L}^1(-1, 1)$ and $\|\mathcal{P}_m f\|_w \leq c\|f\|_w$, $\forall m \in \mathbb{N}$, where $\mathcal{P}_m f$ is the truncated Fourier expansion of f,*
 (iv) *For every $f \in \mathrm{L}_w^2(-1, 1)$, we have $f \in \mathrm{L}^1(-1, 1)$ and $\|f - \mathcal{P}_m f\|_w \to 0$ as $m \to \infty$.*

Using this theorem (specifically conditions (iii) and (iv)), we immediately obtain the following:

Corollary 3.2. *Let $\mathcal{P}_m$ be the Fourier projection operator. Then (A1) and (A2′) hold for $\mathrm{H}^R = \mathrm{L}_w^2(-1, 1)$ if and only if the weight function w is of Helson–Szegö type.*

Remark 3.3 Recall that in §2.2.2 we introduce the operator $\mathcal{W}_m = \mathcal{P}_m^* \circ \mathcal{P}_m$, and Lemma 2.10 gives that $\mathcal{W}_m \to \mathcal{I}$ strongly on H^R. This can actually be proved quite easily in the case considered in this section. If w is a Helson–Szegö weight function then, recalling the explicit form (2.18) for $\mathcal{W}_m$,

$$\|g - \mathcal{W}_m g\|_w \leq \|wg - \mathcal{P}_m(wg)\|_{w^{-1}} + \|\mathcal{P}_m(wg - w\mathcal{P}_m g)\|_{w^{-1}}.$$

If w is a Helson–Szegö weight function then so is $w^{-1}(x) = \frac{1}{w(x)}$ (wherever defined). Also, $g \in \mathrm{L}^2(-1, 1)$ if and only if $wg \in \mathrm{L}_{w^{-1}}^2(-1, 1)$. Therefore Theorem 3.1 gives

$$\|g - \mathcal{W}_m g\|_w \leq \|wg - \mathcal{P}_m(wg)\|_{w^{-1}} + c\|g - \mathcal{P}_m g\|_w \to 0, \quad m \to \infty,$$

as required.

We are interested in the case where (piecewise) modified Jacobi weight functions are of Helson–Szegö type. We have

Lemma 3.4. *Suppose that $w_{\alpha,\beta}$ is a piecewise modified Jacobi weight function with parameters $\alpha = \{\alpha_0, \dots, \alpha_l\}$ and $\beta = \{\beta_0, \dots, \beta_l\}$. Then $w_{\alpha,\beta}$ is a Helson–Szegö weight function if and only if $-1 < \alpha_r, \beta_r < 1$ for all $r = 0, \dots, l$.*

Proof. Note that if $\tilde{w}$ is a Helson–Szegö weight function and $w \asymp \tilde{w}$ uniformly in x, then so is w. We now recall a result proved in [13]. Let $w(t) = \varphi(\mathrm{e}^{\mathrm{i}\pi t})$, where φ is finite, positive and continuous on the unit circle except at a finite set of points $z_1, \dots, z_l$. Moreover, suppose that

$$\varphi(z) = \mathcal{O}\left(|z - z_r|^{\gamma_r}\right), \quad \frac{1}{\varphi(z)} = \mathcal{O}\left(|z - z_r|^{-\gamma_r}\right), \quad z \to z_r.$$

Then w is a Helson–Szegö weight function if and only if $-1 < \gamma_r < 1$, $r = 1, \dots, l$. Noting that $1 - t^2 \asymp |1 + \mathrm{e}^{\mathrm{i}\pi t}|$ now gives the full result. $\qquad\square$

This lemma, in combination with Theorem 3.1, gives the following:

Corollary 3.5. *Let $\mathcal{P}_m$ be the Fourier projection operator and $\mathrm{H}^R = \mathrm{L}^2_{\alpha,\beta}(-1,1)$. Then assumptions (A1) and (A2′) hold if and only if $-1 < \alpha_r, \beta_r < 1$ for all $r = 0, \ldots, l$.*

Note that this corollary includes the cases $\alpha = \beta = \{-\frac{1}{2}, \ldots, -\frac{1}{2}, \}, \{\frac{1}{2}, \ldots, \frac{1}{2}\}$ corresponding to Chebyshev weights (of the first and second kinds) that form the principal interest of this paper. With this corollary in hand, the analysis of §2.2.2 immediately applies to this problem. In particular, we deduce stability, convergence, asymptotic optimality and the sharp bounds of Theorem 2.11.

Let us now address the other case, where at least one α_j or β_j exceeds 1. As shown by Lemma 3.4, $w_{\alpha,\beta}$ is no longer a Helson–Szegö weight function. However, one can still verify (A1) and (A2) in this case, for a suitable choice of the subspace U:

Corollary 3.6. *Let $\mathcal{P}_m$ be the Fourier projection operator and $w_{\alpha,\beta}$ a piecewise Jacobi weight function with parameters $\alpha = \{\alpha_0, \ldots, \alpha_l\}$ and $\beta = \{\beta_0, \ldots, \beta_l\}$. For $r = 0, \ldots, l$ define*

$$
\tilde{\alpha}_r = \left\{ \begin{array}{ll} \alpha_r & -1 < \alpha_r < 1 \\ 1 - \epsilon_r & \alpha_r \geq 1 \end{array} \right. , \qquad \tilde{\beta}_r = \left\{ \begin{array}{ll} \beta_r & -1 < \beta_r < 1 \\ 1 - \epsilon'_r & \beta_r \geq 1 \end{array} \right. ,
$$

where $0 < \epsilon_r, \epsilon'_r < 2$ are arbitrary. Then (A1) and (A2) hold with $\mathrm{U} = \mathrm{L}^2_{\tilde{\alpha},\tilde{\beta}}(-1,1)$ and $\|\cdot\| = \|\cdot\|_{\tilde{\alpha},\tilde{\beta}}$.

Proof. By construction, $w_{\tilde{\alpha},\tilde{\beta}}$ is a piecewise modified Jacobi weight function of Helson–Szegö type. Moreover, we have the continuous embedding $\mathrm{U} \hookrightarrow \mathrm{H}^R$. In other words, there exists $c > 0$ such that $\|g\|_{\alpha,\beta} \leq c\|g\|_{\tilde{\alpha},\tilde{\beta}}, \forall g \in \mathrm{U}$. Therefore, by Theorem 3.1,

$$
\|g - \mathcal{P}_m g\|_{\alpha,\beta} \leq \|g - \mathcal{P}_m g\|_{\tilde{\alpha},\tilde{\beta}} \to 0, \quad \forall g \in \mathrm{U},
$$

and

$$
\|\mathcal{P}_m g\|_{\alpha,\beta} \leq \|\mathcal{P}_m g\|_{\tilde{\alpha},\tilde{\beta}} \leq c\|g\|_{\tilde{\alpha},\tilde{\beta}}, \quad \forall g \in \mathrm{U}.
$$

This gives the result. $\square$

As a result of this corollary, the analysis of §2.2.1 applies in this setting. Hence one verifies stability and convergence, albeit without the sharp bounds of §2.2.2. As commented, the space U of functions we can reconstruct with this approach is very large, and certainly includes all functions of interest in practice.

Observe that Corollaries 3.5 and 3.6, in combination with the analysis of §2, confirm the first part of the key theorem of this paper: namely, Theorem 1.4. For the second part, we need to estimate the stable sampling rate $\Theta(n; \theta)$, as defined by (2.19). This is the content of the next section. The final part of Theorem 1.4, i.e. the computational cost estimates, will be addressed in §4.

3.3 Estimates for the stable sampling rate $\Theta(n; \theta)$

The key theorem we prove in this section is the following:

Theorem 3.7. *Let $\mathcal{P}_m$ be the Fourier projection operator, T_n be given by (2.5) for $n = (n_0, \ldots, n_l)$, and suppose that $\mathrm{H}^R = \mathrm{L}^2_{\alpha,\beta}(-1,1)$, where $\alpha = \{\alpha_0, \ldots, \alpha_l\}$, $\beta = \{\beta_0, \ldots, \beta_l\}$. Then, for $0 < \theta < 1$ there exists $c > 0$ depending only on θ such that*

$$
\Theta(n; \theta) \leq c\bar{n}^r, \quad \forall n_0, \ldots, n_l \in \mathbb{N},
$$

where $\Theta(n; \theta)$ is given by (2.19), $\bar{n} = \max\{n_0, \ldots, n_l\}$, and

$$
r = \max\left\{ 2, p+1, 2 + 2\frac{p-q}{q+1} \right\},
$$

$$
p = \max\{\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_l\}, \quad q = \min\{\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_l\}.
$$

A proof of this theorem is given later in this section. Recall that the most important case in this paper is when $\alpha_0 = \ldots = \alpha_l = \beta_0 = \ldots = \beta_l = \gamma$. In other words, a piecewise ultraspherical weight with the same parameter values in each of the intervals $I_0, \ldots, I_l$. In this case, $p = q = \gamma$, and Theorem 3.7 gives that

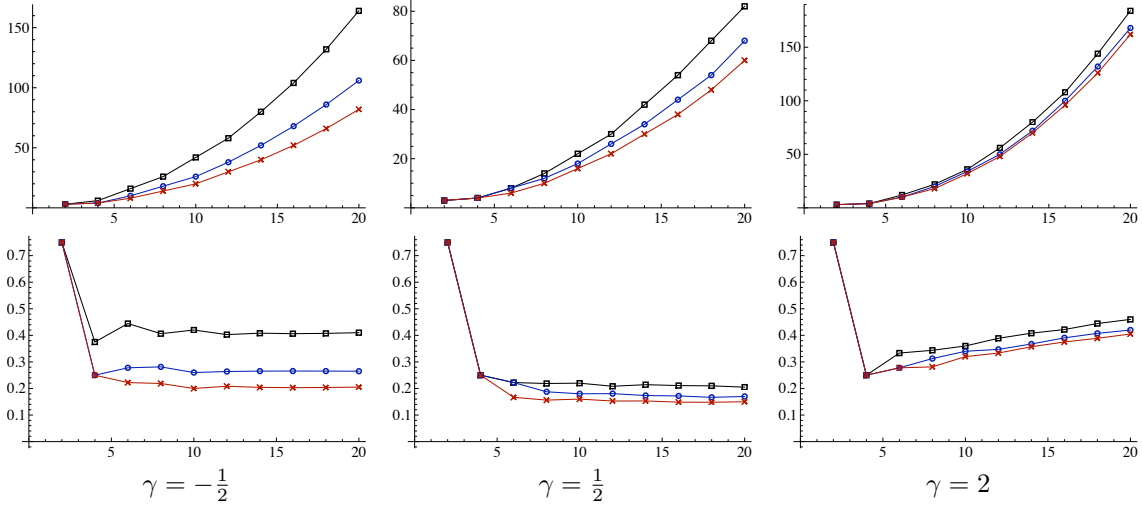$$
\Theta(n; \theta) \leq c\bar{n}^{\max\{2, \gamma+1\}}.
$$

Figure 1: The functions $\Theta(n;\theta)$ (top) and $n^{-2}\Theta(n;\theta)$ (bottom) against $n = 1,\ldots,20$ for $\theta = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ (squares, circles and crosses respectively).

For example, whenever $-1 < \gamma < 1$, such as in the Legendre or Chebyshev cases, we obtain an $\mathcal{O}\left(n^2\right)$ scaling for $\Theta(n;\theta)$. Note that this result was first proved in [42] (see also [7]) for the Legendre case. In fact, quadratic scaling of $\Theta(n;\theta)$ with n is the best possible predicted by this theorem (indeed, one cannot obtain a better scaling whilst retaining stability with *any* method – see Remark 1.6 and [6]). One negative consequence of this result is that if one chooses α and β so that either $p \neq q$ or $p > 1$, then the growth of $\Theta(n;\theta)$ is more severe. Thus, more Fourier coefficients are required to reconstruct in terms of the corresponding piecewise polynomial basis. This is clearly less than ideal. Nevertheless, whenever α and β are chosen so that $p = q$ and $p \leq 1$, we retain quadratic growth.

The reason for the increased scaling whenever $p > 1$ is that the weighted norm $\|\cdot\|_{\alpha,\beta}$ is too weak in comparison to the standard L^2 norm to adequately control the error of the Fourier projection. Conversely, when $p \neq q$, the local convergence of the Fourier series of an arbitrary function ϕ at any point $x \in (-1,1)$ is determined by its global behaviour at *every* jump $x_0, x_1, \ldots, x_l$. Thus, convergence in the local norm over each I_r subinterval must balance this global behaviour, which leads to the higher scaling when $p \neq q$.

In Figure 1 we plot $\Theta(n;\theta)$ and $n^{-2}\Theta(n;\theta)$ for case where $l = 0$ (i.e. analytic and nonperiodic functions f) and $\alpha = \beta = \gamma$ (i.e. reconstructions in ultraspherical polynomials). As is evident, when $\gamma = \pm\frac{1}{2}$, $\Theta(n;\theta)$ scales quadratically with n. Conversely, this scaling is cubic when the value $\gamma = 2$ is used. Thus, these results confirm Theorem 3.7.

The remainder of this section is devoted to a proof of Theorem 3.7. For this, we first require some general results about Fourier series, and in particular, Fourier series of arbitrary piecewise algebraic polynomials. For this, we require some additional notation. We shall write $A \lesssim B$ to mean that there exists a constant $c > 0$ independent of all relevant parameters such that $A \leq cB$. Second, if $I \subseteq [-1,1]$ is an interval and $\alpha, \beta > -1$, we write $\|\cdot\|_{\infty,I}$ and $\|\cdot\|_{\alpha,\beta,I}$ for the norms corresponding to the spaces $\mathrm{L}^2_\infty(I)$ and $\mathrm{L}^2_{w_{\alpha,\beta}}(I)$ respectively, where $w_{\alpha,\beta}$ is a modified Jacobi weight, appropriately scaled to I. Finally, we let $[g](x) = g(x^+) - g(x^-)$ be value of the jump in a function $g : [-1,1] \to \mathbb{R}$ at the point $x \in [-1,1]$, with the understanding that $[g](-1) = [g](1) = g(-1) - g(1)$.

Our first lemma is as follows:

Lemma 3.8. *Suppose that $f : [-1,1] \to \mathbb{R}$ is piecewise smooth with singularities at $-1 < x_1 < \ldots < x_l < 1$. Then we have the asymptotic expansion*

$$\hat{f}_j \sim \frac{1}{\sqrt{2}} \sum_{r=0}^{l} \sum_{s=0}^{\infty} \frac{(-1)^{s+1}\mathrm{e}^{-\mathrm{i}\pi j x_r}}{(-\mathrm{i}j\pi)^{s+1}} \left[f^{(s)}\right](x_r), \quad |j| \gg 1, \tag{3.2}$$

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In particular, if $f \in T_n$, where T_n is as in (2.5), then

$$\hat{f}_j = \frac{1}{\sqrt{2}} \sum_{r=0}^{l} \sum_{s=0}^{n_r-1} \frac{(-1)^{s+1}e^{-i\pi j x_r}}{(-ij\pi)^{s+1}} \left[f^{(s)}\right](x_r), \quad \forall j \in \mathbb{Z}\backslash\{0\}. \tag{3.3}$$

Proof. This follows from repeated integration by parts applied to the integral $\int_{-1}^{1} f(x)e^{-ij\pi x}\,dx$. $\qquad\square$

Note that the symbol $\sim$ in (3.2) denotes an asymptotic expansion (in the usual Poincaré sense) in the parameter $j \to \pm\infty$. As is typical, the right-hand side of (3.2) does usually not converge for any fixed j. On the other hand, (3.3) holds for any finite $j \neq 0$ and arbitrary piecewise polynomials f.

A convenient means to describe the Fourier series of an arbitrary function piecewise smooth function f is in terms of certain cardinal functions [3, 49]. For $r \in \mathbb{N}$, let $\tilde{C}_r(x)$ be the periodic extension of $-\frac{2^r}{(r+1)!}B_{r+1}(x)$ to the real line, where B_{r+1} is the $(r+1)^{\text{th}}$ Bernoulli polynomial [1, chpt. 23], and define $C_r(x) = \tilde{C}_r(\frac{1}{2}x+1)$. It follows that C_r is a piecewise polynomial of degree $r+1$ with singularity at $x = 0$. Moreover, since the Bernoulli polynomials form an Appell sequence, we have that

$$\left[C_r^{(s)}\right](0) = \delta_{r-s}, \quad r, s \in \mathbb{N}.$$

Define the piecewise polynomials

$$q_{r,s}(x) = C_s(x - x_r), \quad r = 0, \ldots, l, \quad s \in \mathbb{N},$$

via translation. Substituting $f = q_{r,s}$ in (3.3), we note that

$$\widehat{q_{r,s}}_j = \frac{1}{\sqrt{2}} \frac{(-1)^{s+1}e^{-ij\pi x_r}}{(-ij\pi)^{s+1}}, \quad j \in \mathbb{Z}\backslash\{0\}. \tag{3.4}$$

Thus (3.2) can be expressed more succinctly as

$$\hat{f}_j \sim \sum_{r=0}^{l} \sum_{s=0}^{\infty} \widehat{q_{r,s}}_j \left[f^{(r)}\right](x_r).$$

Note also that if $\phi \in T_n$, where T_n is defined by (2.5), then

$$\phi(x) - \mathcal{P}_m\phi(x) = \sum_{r=0}^{l} \sum_{s=0}^{n_r-1} \left[\phi^{(s)}\right](x_r)\{q_{r,s}(x) - \mathcal{P}_m q_{r,s}(x)\}, \tag{3.5}$$

Thus, to study the convergence of the Fourier series of an arbitrary $\phi \in T_n$, it suffices to consider the Fourier series of the cardinal functions $q_{r,s}$. We use this approach when proving the next two results.

Lemma 3.9. *Suppose that $\phi \in T_n$. Then, there exists $c > 0$ independent of n and ϕ such that $\|\phi - \mathcal{P}_m\phi\|_\infty \lesssim \|\phi\|_\infty$ whenever $m \geq c\bar{n}^2$.*

Proof. By (3.5) and Markov's inequality [14],

$$\|\phi'\| \lesssim n^2\|\phi\|, \quad \forall\phi \in \mathbb{P}_{n-1}, \tag{3.6}$$

we have

$$\|\phi - \mathcal{P}_m\phi\|_\infty \leq \sum_{r=0}^{l} \sum_{s=0}^{n_r-1} \left|\left[\phi^{(s)}\right](x_r)\right| \|q_{r,s} - \mathcal{P}_m q_{r,s}\|_\infty$$

$$\lesssim \sum_{r=0}^{l} \sum_{s=0}^{n_r-1} \left\{n_r^{2s}\|\phi\|_{\infty,I_r} + n_{r-1}^{2s}\|\phi\|_{\infty,I_{r-1}}\right\} \|q_{r,s} - \mathcal{P}_m q_{r,s}\|_\infty, \tag{3.7}$$

where, for convenience we let $I_{-1} = I_l$ and $n_{-1} = n_l$. Consider $q_{r,s}$. For $s = 1, 2, \ldots$, the function $q_{r,s} \in C^1(\mathbb{T})$ and the derivative $q_{r,s}''$ is of bounded variation. Hence $\mathcal{P}_m q_{r,s}$ converges uniformly to $q_{r,s}$, the coefficients $\widehat{q_{r,s}}_j$ are absolutely summable and we may write $q_{r,s}(x) - \mathcal{P}_m q_{r,s}(x)$ as the infinite sum

$$q_{r,s}(x) - \mathcal{P}_m q_{r,s}(x) = \sum_{|j| \geq \frac{m}{2}} \widehat{q_{r,s}}_j \psi_j(x), \quad \forall x \in [-1, 1].$$

Using the expression (3.4) we deduce that

$$\|q_{r,s} - \mathcal{P}_m q_{r,s}\|_\infty \lesssim \sum_{|j| \geq \frac{m}{2}} |\widehat{q_{r,s_j}}| \lesssim \sum_{j \geq \frac{m}{2}} j^{-s-1} \lesssim m^{-s}.$$

Now consider $q_{r,0}$. By translation, it suffices to consider $q_{0,0}(x) = \frac{1}{4} - \frac{1}{2}x$. This function clearly has uniformly bounded Fourier series. Hence, substituting this and the previous result into (3.7), we obtain

$$\|\phi - \mathcal{P}_m \phi\|_\infty \lesssim \|\phi\|_\infty \sum_{r=0}^{l} \sum_{s=0}^{n_r - 1} \left(\frac{\tilde{c} n_r^2}{m} \right)^s,$$

for some $\tilde{c} > 0$. Therefore, provided $m \geq c\bar{n}^2$ with $c > \tilde{c}$, this sum is bounded independently of n, m and the result follows. $\square$

Lemma 3.10. *Suppose that $\phi \in \mathrm{T}_n$ and let $x \in (x_r, x_{r+1})$ for some $r = 0, \ldots, l$. Then, there exists $c > 0$ independent of n, r, ϕ and x such that*

$$|\phi(x) - \mathcal{P}_m \phi(x)| \lesssim m^{-1}\|\phi\|_\infty \sum_{r=0}^{l} \left(1 + |\sin \tfrac{\pi}{2}(x - x_r)|^{-1}\right),$$

whenever $m \geq c\bar{n}^2$.

Proof. Suppose that we can show that

$$|q_{r,s}(x) - \mathcal{P}_m q_{r,s}(x)| \lesssim m^{-s-1} \left(1 + |\sin \tfrac{\pi}{2}(x - x_r)|^{-1}\right), \quad s \in \mathbb{N}, \ r = 0, \ldots, l, \quad x \neq x_r. \quad (3.8)$$

Then the result follows directly from (3.5) and Markov's inequality (3.6):

$$|\phi(x) - \mathcal{P}_m \phi(x)| \lesssim \sum_{r=0}^{l} \left(1 + |\sin \tfrac{\pi}{2}(x - x_r)|^{-1}\right) \sum_{s=0}^{n_r - 1} m^{-s-1} \left(n_r^{2s} + n_{r-1}^{2s}\right) \|\phi\|_\infty$$

$$\lesssim m^{-1}\|\phi\|_\infty \sum_{r=0}^{l} \left(1 + |\sin \tfrac{\pi}{2}(x - x_r)|^{-1}\right) \sum_{s=0}^{n_r - 1} \left(\frac{\bar{n}^2}{m}\right)^s.$$

Hence it remains to show (3.8). Without loss of generality, we consider $q_{0,s}(x)$. Using (3.4) and noticing that $\mathcal{P}_m q_{0,s}(x)$ converges to $q_{0,s}(x)$ whenever $-1 < x < 1$, we obtain

$$|q_{0,s}(x) - \mathcal{P}_m q_{0,s}(x)| = \frac{1}{2\pi^{s+1}} \left| \sum_{|j|>m} \frac{(-1)^j}{j^{s+1}} \exp(\mathrm{i}j\pi x) \right| \lesssim \left| \sum_{j>m} \frac{z^j}{j^{s+1}} \right|,$$

where $z = -\exp(\mathrm{i}\pi x)$ (replacing $\lfloor \frac{m}{2} \rfloor$ by m in the summation does not effect results and slightly simplifies the notation). Thus, it suffices to consider the sum

$$S_{s,m}(z) = \sum_{j>m} \frac{z^j}{j^{s+1}}.$$

We now use Abel summation:

$$S_{s,m}(z) = \sum_{j>m} \frac{z^j}{j^s} \left(\frac{1}{j} - \frac{1}{j+1} \right) + \sum_{j>m} \frac{z^j}{j^s(j+1)} = \sum_{j>m} \frac{z^j}{j^{s+1}(j+1)} + \sum_{j>m} \frac{z^j}{j^s(j+1)}.$$

The first term is $\mathcal{O}\left(m^{-s-1}\right)$ uniformly in $|z| = 1$. Now consider the second sum. We have

$$z^{-1} \sum_{j>m+1} \frac{z^j}{j(j-1)^s} = z^{-1} \sum_{j>m+1} \frac{z^j}{j^{s+1}} \left(1 + \mathcal{O}\left(j^{-1}\right)\right) = z^{-1} \sum_{j>m+1} \frac{z^j}{j^{s+1}} + \mathcal{O}\left(m^{-s-1}\right),$$

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which holds uniformly in $|z| = 1$. Hence

$$S_{s,m}(z) = z^{-1} \left(S_{s,m}(z) - \frac{z^m}{m^{s+1}} \right) + \mathcal{O}\left(m^{-s-1} \right).$$

Rearranging, this gives

$$S_{s,m}(z) = \frac{z^m}{1-z} m^{-s-1} + \mathcal{O}\left(m^{-s-1} \right).$$

Note that $x \neq \pm 1$, therefore $z \neq 1$. Moreover, since $z = -\exp(i\pi x)$, we have that

$$|z - 1| = 2|\cos \tfrac{\pi}{2} x| = 2|\sin \tfrac{\pi}{2}(x - x_0)|,$$

(recall that $x_0 = -1$) and consequently

$$|S_{s,m}(z)| \lesssim \left(1 + |\sin \tfrac{\pi}{2}(x - x_0)|^{-1} \right) m^{-s-1},$$

which gives (3.8). $\qquad\square$

We are now able to prove the main result of this section:

Proof of Theorem 3.7. It suffices to show that, given $\theta \in (0,1)$, there exists c such that $E_{n,m} < 1 - \theta$ for all $m \geq c\bar{n}^r$, where $E_{n,m}$ is given by (2.6). Let $\phi \in \mathrm{T}_n$ with $\|\phi\|_R = 1$. Note that

$$\|\phi - \mathcal{P}_m\phi\|_{\alpha,\beta}^2 = \sum_{r=0}^{l} \int_{I_r} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x.$$

We consider each term of this sum separately. Write $I_r = I_r^+ \cup I_r^-$, where $I_r^+ = (d_r, x_{r+1})$, $I_r^- = (x_r, d_r)$ and $d_r = \frac{1}{2}(x_r + x_{r+1})$ is the midpoint of I_r. In a similar manner we also define, for arbitrary $0 < \epsilon < c_r = \frac{1}{2}(x_{r+1} - x_r)$,

$$I_{r,\epsilon}^+ = (d_r, x_{r+1} - \epsilon), \quad I_{r,\epsilon}^- = (x_r + \epsilon, d_r),$$
$$J_{r,\epsilon}^+ = (x_{r+1} - \epsilon, x_{r+1}), \quad J_{r,\epsilon}^- = (x_r, x_r + \epsilon).$$

We now write

$$\int_{I_r} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x$$

$$= \int_{I_{r,\epsilon}^+} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x + \int_{I_{r,\epsilon}^-} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x$$

$$+ \int_{J_{r,\epsilon}^+} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x + \int_{J_{r,\epsilon}^-} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x.$$

Consider the integral over $I_{r,\epsilon}^+$. By construction $w_{\alpha,\beta}(x) \lesssim (x_{r+1} - x)^{\alpha_r}$ uniformly for $x \in I_r^+$. Using this and Lemma 3.10 we find that

$$\int_{I_{r,\epsilon}^+} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x \lesssim m^{-2} \|\phi\|_\infty^2 \sum_{s=0}^{l} \int_{I_{r,\epsilon}^+} \left(1 + |\sin \tfrac{\pi}{2}(x - x_s)|^{-1} \right) (x_{r+1} - x)^{\alpha_r} \, \mathrm{d}x.$$

Since $|\sin \tfrac{\pi}{2}(x - x_{r+1})| \gtrsim |x - x_{r+1}|$ and $|\sin \tfrac{\pi}{2}(x - x_s)| \gtrsim 1$, $s \neq r+1$, whenever $-1 < |x - x_{r+1}| < 1$ we deduce that

$$\int_{I_{r,\epsilon}^+} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x \lesssim m^{-2} \|\phi\|_\infty^2 \left(1 + \int_{I_{r,\epsilon}^+} (x_{r+1} - x)^{\alpha_r - 2} \, \mathrm{d}x \right)$$

$$\lesssim \max\{1, \epsilon^{\alpha_r - 1}\} m^{-2} \|\phi\|_\infty^2.$$

Similarly, for $I_{r,\epsilon}^-$, we have

$$\int_{I_{r,\epsilon}^-} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x) \, \mathrm{d}x \lesssim \max\{1, \epsilon^{\beta_r - 1}\} m^{-2} \|\phi\|_\infty^2.$$

We now require an estimate for $\|\phi\|_\infty$. First let $p \in \mathbb{P}_n$ be a polynomial and suppose that $\alpha, \beta > -1$. Then, it can be shown that

$$\|p\|_\infty \lesssim n^{\max\{\alpha,\beta\}+1} \|p\|_{\alpha,\beta}.$$

(this follows directly from the decomposition $\|p\|_{\alpha,\beta}^2 = \sum_{j=0}^n |\langle p, p_j \rangle_{\alpha,\beta}|^2$, where $\{p_j\}$ are the orthonormal polynomials with respect to the weight function $w_{\alpha,\beta}$, and the fact that $\|p_j\|_\infty \sim j^{\max\{\alpha,\beta\}+\frac{1}{2}}$ [48]).

Since $\phi \in \mathrm{T}_n$, and therefore $\phi|_{I_r} \in \mathbb{P}_{n_r-1}$, we have

$$\|\phi\|_\infty = \max_{r=0,\ldots,l} \|\phi\|_{\infty,I_r} \lesssim \max_{r=0,\ldots,l} \left\{ n_r^{\max\{\alpha_r,\beta_r\}+1} \|\phi\|_{\alpha_r,\beta_r,I_r} \right\} \leq \bar{n}^{p+1} \|\phi\|_{\alpha,\beta},$$

where $p = \max\{\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_l\}$. Hence,

$$\sum_{r=0}^l \left\{ \int_{I_{r,\epsilon}^+} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x)\,\mathrm{d}x + \int_{I_{r,\epsilon}^-} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x)\,\mathrm{d}x \right\}$$
$$\lesssim \max\{1, \epsilon^{p-1}\} \left(\frac{\bar{n}^{p+1}}{m} \right)^2 \|\phi\|_{\alpha,\beta}^2.$$

Now consider the integral over $J_{r,\epsilon}^\pm$. After an application of Lemma 3.9, we obtain

$$\int_{J_{r,\epsilon}^+} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x)\,\mathrm{d}x \lesssim \|\phi\|_\infty^2 \int_{1-\epsilon}^1 (1-x)^{\alpha_r}\,\mathrm{d}x \lesssim \|\phi\|_\infty^2 \epsilon^{\alpha_r+1},$$

and likewise for $J_{r,\epsilon}^-$ with the exponent α_r replaced by β_r. Since $\epsilon < 1$, this gives

$$\sum_{r=0}^l \left\{ \int_{J_{r,\epsilon}^+} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x)\,\mathrm{d}x + \int_{J_{r,\epsilon}^-} |\phi(x) - \mathcal{P}_m\phi(x)|^2 w_{\alpha_r,\beta_r}(x)\,\mathrm{d}x \right\} \lesssim \epsilon^{q+1} \bar{n}^{2(p+1)} \|\phi\|_{\alpha,\beta}^2.$$

Setting $\epsilon = cm^{-1}$ for $c < \min\{c_r : r = 0, \ldots, l\}$ and combining terms, we obtain

$$\|\phi - \mathcal{P}_m\phi\|_{\alpha,\beta}^2 \lesssim \left[\max\{1, m^{1-p}\} \left(\frac{\bar{n}^{p+1}}{m} \right)^2 + \frac{\bar{n}^{2(p+1)}}{m^{q+1}} \right] \|\phi\|_{\alpha,\beta}^2.$$

Hence

$$E_{n,m}^2 = \sup_{\substack{\phi \in \mathrm{T}_n \\ \|\phi\|_{\alpha,\beta}=1}} \|\phi - \mathcal{P}_m\phi\|_{\alpha,\beta}^2 \lesssim \left[\max\{1, m^{1-p}\} \left(\frac{\bar{n}^{p+1}}{m} \right)^2 + \frac{\bar{n}^{2(p+1)}}{m^{q+1}} \right]. \tag{3.9}$$

The second term can be made arbitrarily small by selecting

$$m \gtrsim \bar{n}^{2\frac{p+1}{q+1}} = \bar{n}^{2+2\frac{p-q}{q+1}}.$$

Now consider the first term. If $p < 1$ then this is

$$m^{1-p} \left(\frac{\bar{n}^{p+1}}{m} \right)^2 = \left(\frac{\bar{n}^2}{m} \right)^{p+1},$$

and therefore we require $m \gtrsim \bar{n}^2$. Conversely, when $p \geq 1$, we have

$$\max\{1, m^{1-p}\} \left(\frac{\bar{n}^{p+1}}{m} \right)^2 = \left(\frac{\bar{n}^{p+1}}{m} \right)^2,$$

and we therefore stipulate that $m \gtrsim \bar{n}^{p+1}$. Hence, the first term of the right-hand side of (3.9) can be made arbitrarily small by the choice $m \gtrsim \bar{n}^{\max\{2,p+1\}}$. This completes the proof. $\qquad\square$

One downside of Theorem 3.7 is it does not give an explicit upper bound for the scaling constant c. In simpler case of Legendre polynomials explicit, and fairly sharp, bounds were derived in [7]. It is, in theory, possible to repeat the proof of Theorem 3.7 to give an explicit bound. However, because the proof is rather involved, this is likely to be grossly pessimistic. On the other hand, since $\Theta(n;\theta)$ can always be computed, we may always numerically estimate c. In fact, as shown in Figure 1 one can take a fairly small value $c = \frac{1}{4}$ in implementations. We use this value in the next section.

4 Numerical implementation and examples

4.1 Implementation

As mentioned in §2.3, the main issue in implementation is the computation of the entries of the matrices U and C. Computation of the entries of U can be carried out using the iterative procedure of [7, Lemma 3.4]. The total cost incurred is $\mathcal{O}(mn)$ operations. Note that, as commented in [7], one may in practice need to use a two-phase algorithm (such as that described in [21]) to ensure stability of the computation of these entries. However, this does not increase the overall cost.

For the computation of C, recall that this matrix has $(j, k)^{\text{th}}$ entry $\langle \psi_j, \psi_k \rangle_R$. In other words, if $w_{\alpha,\beta}$ is the corresponding weight function, then

$$\langle \psi_j, \psi_k \rangle_R = \frac{1}{2} \int_{-1}^{1} e^{i\pi(j-k)x} w_{\alpha,\beta}(x) \, dx = c_{j-k}.$$

Hence C is always a Toeplitz matrix, and to compute C we only need to determine c_j for $j = 0, \ldots, m$ (note that $c_{-j} = \overline{c_j}$). Fortunately, since

$$\int_{-1}^{1} e^{izx}(1-x^2)^\alpha \, dx = \sqrt{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} (\alpha+\tfrac{1}{2}) \left(\frac{2}{z}\right)^{\alpha+\frac{1}{2}} J_{\alpha+\frac{1}{2}}(z), \quad z \in \mathbb{R},$$

where Γ is the Gamma function and J_ν is the Bessel function of the first kind [35], we can always determine c_j explicitly for the case where $w_{\alpha,\beta}$ is a piecewise ultraspherical weight function. Unfortunately, there is no such closed form expression when $w_{\alpha,\beta}$ is a piecewise Jacobi weight. However, in this case one could always compute c_j using a Gaussian quadrature based on the given weight function, with the total cost of this approach being $\mathcal{O}(m^2)$. We shall not discuss this further.

For evaluation of the quantity $\Theta(n; \theta)$ it is also necessary to compute the matrix $V \in \mathbb{C}^{m \times n}$ with $(j, k)^{\text{th}}$ entry $\langle \phi_k, \psi_j \rangle_R$. Whenever w corresponds to a piecewise ultraspherical weight function, the entries of V are actually known explicitly. This follows from the fact that the integrals

$$\int_{-1}^{1} e^{izx} \phi_j(x)(1-x^2)^\alpha \, dx, \quad z \in \mathbb{R},$$

have an explicit expression whenever ϕ_j is the jth normalized ultraspherical polynomial corresponding to the parameter $\alpha > -1$ (see, for example, [35, eqn. (3.7)]).

Note that the above arguments establish the final part of Theorem 1.4 (the first parts were addressed in the previous sections) concerning the cost of the preprocessing step in the computation of $f_{n,m}$ (i.e. forming the matrices U and C).

4.2 Examples

In Figure 2 we consider the reconstruction of the function $f(x) = e^x \cos 8x$ using Legendre polynomials and Chebyshev polynomials of the first and second kinds. As is evident, all three approaches give near identical approximation errors. In fact, regardless of the polynomial basis used, one attains roughly machine accuracy using only $m = 225$ Fourier samples (by means of comparison, direct expansion of this function in a Fourier series gives an error of order 10^{-1} for this value of m). Also, the condition number of the matrix $A = U^* C U$ remains bounded in n, as predicted by the analysis in §2.

In Figure 3 we consider the piecewise analytic function

$$f(x) = \begin{cases} (2e^{2\pi(x+1)} - 1 - e^\pi)(e^\pi - 1)^{-1} & x \in [-1, -\frac{1}{2}) \\ -\sin(\frac{2\pi x}{3} + \frac{\pi}{3}) & x \in [-\frac{1}{2}, 1] \end{cases} \tag{4.1}$$

This function was put forth in [53] to test algorithms for overcoming the Gibbs phenomenon. Aside from the discontinuity, its sharp peak makes it a challenging function to reconstruct accurately. However, as shown in this figure, using a polynomial of degree 16 in each subinterval of smoothness, we recover the function to around 14 digits of accuracy. As in the previous example, all three polynomial bases used (Legendre and Chebyshev) give roughly the same approximation error, and as before, lead to bounded condition numbers with $\kappa(A)$ being at worst ≈ 10 for all cases.
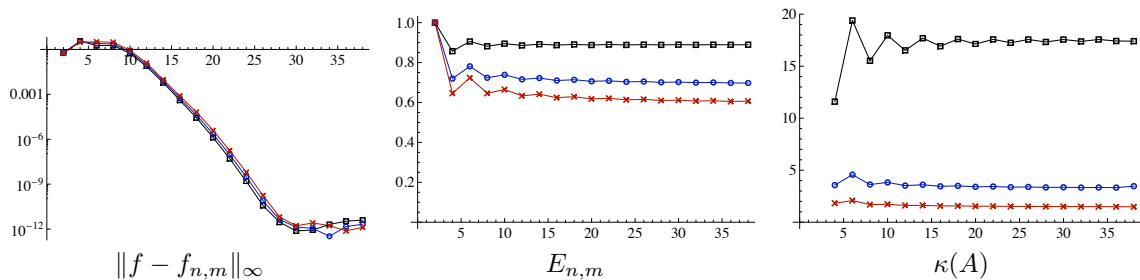
Figure 2: The quantities $\|f - f_{n,m}\|_\infty$, $E_{n,m}$ and $\kappa(A)$ (left to right) against $n = 1, 2, \ldots, 40$, for $f(x) = e^x \cos 8x$, $m = \lceil \frac{1}{4} n^2 \rceil$ and $\gamma = -\frac{1}{2}, 0, \frac{1}{2}$ (squares, circles and crosses respectively).
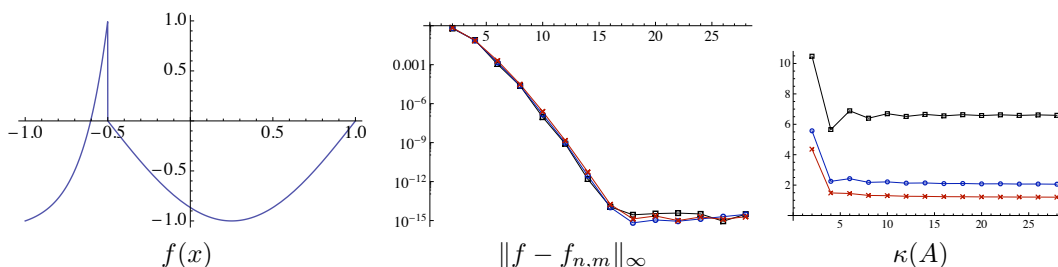

Figure 3: The function (4.1) (left), and the quantities $\|f - f_{n,m}\|_\infty$ and $\kappa(A)$ against $n = 1, 2, \ldots, 30$, with $n_0 = n_1$, $m = n_0^2$ and $\gamma = -\frac{1}{2}, 0, \frac{1}{2}$ (squares, circles and crosses respectively).

This function was used in [30] to test spectral reprojection. Therein it was found that more than 1024 Fourier coefficients were required to obtain close to machine precision. Conversely, with the GS approach we achieve the same value using only around 256 such coefficients – a factor of four increase (a similar experiment was reported in [7]). Moreover, it is worth recalling that (see §1), with this approach one has near-complete freedom to choose the polynomial basis for reconstruction, whereas with spectral reprojection one must use a specific, Gibbs complementary, basis.

5 Other applications

As mentioned, the extended GS framework of §2 is far more general than the application considered in the previous section of recovering a function from its Fourier coefficients. In this section we briefly discuss a family of other related problem to which this framework can also be successfully applied.

Let $f : [-1, 1] \to \mathbb{R}$ br piecewise analytic, and suppose now that we have access to the first m coefficients of f with respect to a basis of orthogonal polynomials (e.g. Jacobi or ultraspherical). Since f is only piecewise analytic, direct expansion in the given basis of polynomials yields an extremely poor approximation (in fact, it suffers from a Gibbs-type phenomenon). Therefore, much as in the Fourier case, the problem is to recover f to high accuracy (other variants of this problem have also been considered, such as the reconstruction of functions from Fourier–Bessel [47] and spherical harmonic coefficients [27]). We remark that this problem occurs notably in spectral methods for hyperbolic PDEs [31, 34, 41]. Although spectral reprojection can also be used in this setting [32, 33, 34], it suffers from the same drawbacks as those in the Fourier case (see §1). Thus, we now propose to use GS instead.

Let $\{\psi_j\}$ be orthonormal polynomials with respect to the (modified) Jacobi weight function w_{α^S, β^S}, where $\alpha^S, \beta^S > -1$, and let $\{\phi_j\}$ be piecewise orthonormal polynomials with respect to the piecewise (modified) Jacobi weight function w_{α^R, β^R}, where $\alpha^R = \{\alpha_0, \ldots, \alpha_l\}$ and $\beta^R = \{\beta_0, \ldots, \beta_l\}$. We shall consider the following two important cases:

$$(\text{i}) \ \alpha^S = \beta^S = \gamma^S = 0, \qquad (\text{ii}) \ \alpha^S = \beta^S = \gamma^S = -\frac{1}{2} \ \text{or} \ \alpha^S = \beta^S = \gamma^S = \frac{1}{2},$$

corresponding to sampling f with Legendre and Chebyshev polynomials (of the first and second kind) respectively.

5.1 Legendre polynomial sampling

As in the Fourier sampling case, to apply the GS framework of §2 we need to verify (A1) and (A2) for this example. Note that $H^S = L^2(-1, 1)$ in this setting, and $H^R = L^2_{\gamma^R}(-1, 1)$. We have:

Theorem 5.1. *Suppose that $H^S = L^2(-1, 1)$ and $\mathcal{P}_m$ is the Legendre polynomial projection operator. Let $\alpha^R = \{\alpha_0, \ldots, \alpha_l\}$ and $\beta^R = \{\beta_0, \ldots, \beta_l\}$ with $\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_l > -1$ arbitrary, and define $H^R = L^2_{\alpha^R, \beta^R}(-1, 1)$. Let*

$$U = \left\{ f : f|_{I_r} \in H^1(I_r), \ r = 0, \ldots, l \right\},$$

be the space of all piecewise H^1 functions, and define $\|\cdot\|$ to be the standard norm on U. Then (A1) and (A2) hold.

Proof. For $\alpha > -1$ it can be shown that

$$\int_{-1}^{1} |g(x)|^2 (1 \pm x)^\alpha \, dx \lesssim \begin{cases} \|g\|^2 & \alpha \geq 0 \\ \|g\|^{2(1+\alpha)} \|g\|_\infty^{-2\alpha} & -1 < \alpha < 0 \end{cases}$$

(see [7, Lemma 3.6]), provided $g \in L^\infty(-1, 1)$. Now, for $I_r = (x_r, x_{r+1})$ define $I_r^- = (x_r, d_r)$, $I_r^+ = (d_r, x_{r+1})$, where $d_r = \frac{1}{2}(x_r + x_{r+1})$, so that $I_r = I_r^+ \cup I_r^-$. Also, let

$$J_1^+ = \{r : -1 < \alpha_r < 0\}, \quad J_1^- = \{r : -1 < \beta_r < 0\}, \quad J_2^+ = \{r : \alpha_r \geq 0\}, \quad J_2^- = \{r : \beta_r \geq 0\}.$$

Then, for $g \in U$,

$$\|g\|_{\alpha^R, \beta^R} \lesssim \sum_{r \in J_1^+} \|g\|_{0, I_r^+}^{1+\alpha_r} \|g\|_{\infty, I_r^+}^{-\alpha_r} + \sum_{r \in J_1^-} \|g\|_{0, I_r^-}^{1+\beta_r} \|g\|_{\infty, I_r^-}^{-\beta_r} + \sum_{r \in J_2^+} \|g\|_{0, I_r^+} + \sum_{r \in J_2^-} \|g\|_{0, I_r^-},$$

where $\|\cdot\|_{0, I}$ and $\|\cdot\|_{\infty, I}$ are the L^2 and uniform norms on an interval I respectively. It follows that

$$\|g\|_{\alpha^R, \beta^R} \lesssim \sum_{r \in J_1^+} \|g\|^{1+\alpha_r} \|g\|_\infty^{-\alpha_r} + \sum_{r \in J_1^-} \|g\|^{1+\beta_r} \|g\|_\infty^{-\beta_r} + \|g\| \tag{5.1}$$

Now, for a function $g \in U$, it can be shown that its Legendre polynomial expansion, whilst exhibiting a Gibbs-type phenomenon, is uniformly bounded and satisfies $\|\mathcal{P}_m g\|_\infty \lesssim \|g\|$ [46]. Now, suppose we replace g by $\mathcal{P}_m g$ in (5.1). Then, using this bound and Bessel's inequality $\|\mathcal{P}_m g\| \leq \|g\| \leq \|g\|$, we obtain $\|g\|_{\alpha^R, \beta} \lesssim \|g\|$, which gives (A2). Similarly, replacing g by $g - \mathcal{P}_m g$ and using the observations that $\|g - \mathcal{P}_m g\| \to 0$ as $m \to \infty$ and that $\|g - \mathcal{P}_m g\|_\infty \leq \|g\|_\infty + \|\mathcal{P}_m g\|_\infty \leq c\|g\|$, we obtain (A1). $\qquad\square$

This theorem shows that, provided we restrict to a space of sufficiently regular functions (which, in practice, will always be the case), we can always recover a function f in any polynomial basis from its Legendre polynomial samples. Note that the choice of U here is not unique, and the smoothness assumption could most likely be lowered with a deeper understanding of the convergence of Legendre polynomial expansions for piecewise functions. However, since we mainly consider piecewise analytic functions (which always belong to this choice of U), this consideration is of little importance.

We shall omit two further considerations from this discussion. First, it is unknown at this moment under what conditions the stronger assumption (A2′) hold for this problem. Second, we also have no estimate for the quantity $\Theta(n; \theta)$. In [7] the case $\gamma^R \equiv 0$ (that is, we both sample and reconstruct using (piecewise) Legendre polynomials) was considered. It was shown numerically that $\Theta(n; \theta) = \mathcal{O}(n^2)$ when $\gamma^R \equiv 0$. Establishing this result remains an open problem, as does determining $\Theta(n; \theta)$ for other values of γ_R.

5.2 Chebyshev polynomial sampling

We now consider the case where $\gamma^S = \pm\frac{1}{2}$. In this case, unlike that of Legendre polynomial sampling, we are actually able to provide a complete answer, at least regards assumptions (A1) and (A2) (or (A2′)). This follows directly from the fact that the Chebyshev polynomial expansion of a function can be expressed as a Fourier expansion under the transformation $x = \cos\theta$:

Theorem 5.2. *Suppose that* $\mathrm{H}^S = \mathrm{L}^2_\gamma(-1,1)$ *and* $\mathcal{P}_m$ *is the projection operator corresponding to Chebyshev polynomials of the first ($\gamma = -\frac{1}{2}$) or second ($\gamma = \frac{1}{2}$) kind. Let w be an integrable weight function and set* $\mathrm{H}^R = \mathrm{L}^2_w(-1,1)$. *Then (A1) and (A2$'$) hold if and only if the function $w(\cos\theta)\sin\theta$ (first kind) or $w(\cos\theta)/\sin\theta$ (second kind) is a Helson–Szegö weight function on $[0,\pi]$.*

Proof. Consider the case of first kind Chebyshev polynomials. The projection $\mathcal{P}_m f$ of a function $f \in \mathrm{L}^2_{-\frac{1}{2}}(-1,1)$ is precisely the Fourier expansion the function $F(\theta) = f(\cos\theta)$ on $[0,\pi]$. Note that $f \in \mathrm{L}^2_{-\frac{1}{2}}(-1,1)$ if and only if $F \in \mathrm{L}^2(0,\pi)$. Hence, the result follows immediately from Corollary 3.2. The case where $\gamma = \frac{1}{2}$ is similar. $\qquad\square$

Corollary 5.3. *Suppose that* $\mathrm{H}^S = \mathrm{L}^2_\gamma(-1,1)$ *and* $\mathcal{P}_m$ *is the projection operator corresponding to Chebyshev polynomials of the first ($\gamma = -\frac{1}{2}$) or second ($\gamma = \frac{1}{2}$) kind. Then (A1) and (A2$'$) hold for the piecewise (modified) Jacobi weight function $w_{\alpha,\beta}$ if and only if*

$$-1 < \alpha_r, \beta_{r+1} < 1, \quad r = 0, \dots, l-1,$$

and $-1 < \alpha_l, \beta_0 < 0$ ($\gamma = -\frac{1}{2}$) or $0 < \alpha_l, \beta_0 < 2$ ($\gamma = \frac{1}{2}$).

Proof. Let $\gamma = -\frac{1}{2}$. By Theorem 5.2 it suffices to consider the weight function $W(\theta) = w_{\alpha,\beta}(\cos\theta)\sin\theta$ on $[0,\pi]$. Let $\theta_r = \arccos(x_r)$, $r = 1, \dots, l$. Then

$$W(\theta) = \mathcal{O}\left((\theta - \theta_r)^{\alpha_r}\right), \quad \theta \to \theta_r^+, \qquad W(\theta) = \mathcal{O}\left((\theta_r - \theta)^{\beta_{r+1}}\right), \quad \theta \to \theta_r^-.$$

Also

$$W(\theta) = \mathcal{O}\left(\theta^{1+\alpha_l}\right), \quad \theta \to 0^+, \qquad W(\theta) = \mathcal{O}\left((\pi - \theta)^{1+\beta_0}\right), \quad \theta \to \pi^-.$$

Hence, as in the proof of Lemma 3.4, W is a Helson–Szegö weight function if and only if $-1 < \alpha_r, \beta_{r+1} < 1$, $r = 0, \dots, l-1$ and $-1 < \alpha_l, \beta_0 < 0$. The case of $\gamma = \frac{1}{2}$ is similar. $\qquad\square$

Due to this theorem, it is always possible to reconstruct a function in a basis of piecewise Jacobi polynomials from its Chebyshev polynomial samples. In particular, we can recover in terms of piecewise Chebyshev polynomials $\alpha_0 = \dots = \alpha_l = \beta_0 = \dots = \beta_l = \gamma = \pm\frac{1}{2}$ from Chebyshev polynomial samples. Note that, unlike the Legendre polynomial case (see §5.1), we have (A2$'$) for this problem, as opposed to the weaker condition (A2).

6 Conclusions and challenges

In this paper we have introduced a general framework for sampling and reconstruction in Hilbert spaces, and analyzed in detail the resulting numerical method for reconstructing a piecewise analytic function in a orthogonal basis of (piecewise) polynomials from its Fourier coefficients. The method can be implemented with a wide variety of different polynomial bases, including Chebyshev and Legendre polynomial bases. In all cases the numerical method is stable and exponentially convergent (in the polynomial degree n). Whenever Chebyshev or Legendre polynomials are employed, for example, the method is also root-exponentially convergent in the number of Fourier coefficients m. Finally, we have given a first insight into how the same framework can be applied to the related problem of reconstructing a piecewise analytic function from its orthogonal polynomial coefficients.

Purposefully we have not described the implementation of this framework for the polynomial coefficient reconstruction problem described in §5. Nor have we presented numerical examples. The main stumbling block herein is the computation of the entries of the matrices U and C. This involves constructing particular quadratures to approximate the inner products $\langle \psi_j, \phi_k \rangle_S$ and $\langle \psi_j, \psi_k \rangle_R$, and is beyond the scope of this paper. This aside, we have also not addressed the behaviour of the quantity $\Theta(n;\theta)$ for this problem. In [7] numerical results were presented suggesting quadratic growth in the case $\mathrm{H}^R = \mathrm{H}^S = \mathrm{L}^2(-1,1)$ (i.e. Legendre polynomial sampling and reconstruction). This, and the corresponding extension to other orthogonal polynomial systems, remains a conjecture.

This aside, it is worth repeating that the GS framework developed in this paper is far more general than the problems considered in §3–§5. Namely, it allows one to reconstruct any element $f \in \mathrm{U}$ from its samples in an arbitrary orthonormal basis, even in the case that the sampling and reconstruction vectors are

orthogonal with respect to different inner products. In fact, this paper forms part of a much larger project on numerical methods for sampling and reconstruction (see [4, 5, 6, 7, 8]), with a long list of potential applications. Currently we are investigating a number of such applications, including spline and wavelet-based reconstructions of images, the solution of certain inverse and ill-posed problems (see [8]), and reconstructions from other types of integral transforms. It is also worth mentioning that the key ideas for GS originated from new tools for the computational spectral problem [38, 39]. Furthermore, similar ideas, when combined with convex optimization techniques, can be used to extended the current theory of compressed sensing to formally infinite-dimensional problems. This topic is described in [5].

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