

Gibbs phenomenon and its removal for a class of orthogonal expansions

Ben Adcock

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Abstract We detail the Gibbs phenomenon and its resolution for the family of orthogonal expansions consisting of eigenfunctions of univariate polyharmonic operators equipped with homogeneous Neumann boundary conditions. As we establish, this phenomenon closely resembles the classical Fourier Gibbs phenomenon at interior discontinuities. Conversely, a weak Gibbs phenomenon, possessing a number of important distinctions, occurs near the domain endpoints. Nonetheless, in both cases we are able to completely describe this phenomenon, including determining exact values for the size of the overshoot.

Next, we demonstrate how the Gibbs phenomenon can be both mitigated and completely removed from such expansions using a number of different techniques. As a by-product, we introduce a generalisation of the classical Lidstone polynomials.

Keywords Polyharmonic expansions · Gibbs phenomenon · Accelerating convergence

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1 Introduction

Fourier series lie at the heart of countless methods in computational mathematics. Unfortunately, whenever a piecewise smooth function is represented by its Fourier series, the approximation suffers from the well-known Gibbs phenomenon [24, 38]. Several characteristics of this phenomenon include the slow convergence of the expansion away from the discontinuity locations, the lack of uniform convergence and

B. Adcock
Department of Mathematics
Simon Fraser University
8888 University Drive
Burnaby, BC V5A 1S6
Canada
E-mail: ben_adcock@sfu.ca

the presence of $\mathcal{O}(1)$ oscillations near discontinuities [40]. In particular, the maximal overshoot of the Fourier series of a function f near any discontinuity x_0 is of size $c[f(x_0^+) - f(x_0^-)]$, where $c \approx 0.0895$.

It is a testament to the importance of the Gibbs phenomenon that the development of techniques for its amelioration, and indeed, complete removal, remains an active area of inquiry. The list of existing methods includes filtering [38], Gegenbauer reconstruction [20, 21], techniques based on extrapolation [14–16], Padé methods [13] and Fourier extension/continuation methods [10, 22], to name but a few (for a more comprehensive survey see [11, 38] and references therein). All such methods rely on one common principle: the Gibbs phenomenon is so regular, and so well understood mathematically, that it is possible to devise techniques to circumvent it. As also discussed in [20], the Gibbs phenomenon is certainly not restricted to Fourier series. Other notable examples include spherical harmonics, Fourier–Bessel series and radial basis functions [17]. In the same spirit, the intent of this paper is to study the Gibbs phenomenon in a particular family of orthogonal expansions, and techniques therein for its removal.

Recently, the idea of expanding functions in eigenfunctions of univariate polyharmonic operators equipped with homogeneous Neumann boundary conditions was proposed as an alternative to classical Fourier series [6, 23]. Denoting the n^{th} such eigenfunction by ϕ_n , where

$$\begin{aligned} (-1)^q \phi_n^{(2q)}(x) &= \mu_n \phi_n(x), \quad x \in [-1, 1], \\ \phi_n^{(q+r)}(\pm 1) &= 0, \quad r = 0, \dots, q-1, \quad q \in \mathbb{N}_+, \end{aligned} \quad (1.1)$$

the collection of sets $\{\phi_n\}_{n=1}^\infty$ forms a one-parameter family of orthogonal bases, indexed by q . When $q = 1$, (1.1) corresponds to eigenfunctions of the Laplace–Neumann operator, resulting in the basis $\{\cos n\pi x : n \in \mathbb{N}\} \cup \{\sin(n - \frac{1}{2})\pi x : n \in \mathbb{N}_+\}$. The close proximity to the standard Fourier basis $\{\cos n\pi x : n \in \mathbb{N}\} \cup \{\sin n\pi x : n \in \mathbb{N}_+\}$, suggests that a similar Gibbs phenomenon ought to occur in so-called *Laplace–Neumann expansions* (or *modified Fourier expansions* [23]). One may speculate that such a phenomenon ought also to be present for arbitrary $q \geq 2$ (even though the eigenfunctions (1.1) are no longer simple trigonometric functions). However, Fourier series and polyharmonic–Neumann expansions have a number of important disparities. In particular, whenever the function considered is smooth but nonperiodic, the polyharmonic–Neumann expansion, when truncated after N terms, converges uniformly at a rate of $\mathcal{O}(N^{-q})$. Moreover, away from $x = \pm 1$, pointwise convergence occurs at the increased rate of $\mathcal{O}(N^{-q-1})$ (see [32] for $q = 1$ and [5] for $q \geq 2$). Conversely, the Fourier series of a nonperiodic function lacks uniform convergence and suffers from the Gibbs phenomenon near $x = \pm 1$.

Faster convergence of polyharmonic–Neumann expansions over Fourier expansions has led to their application in a number of problems. Indeed, such expansions allow for considerably cheaper and faster computation of certain Fredholm integral equations [12], and lead to better conditioned algorithms for the numerical solution of partial differential equations (PDEs) than more standard polynomial-based methods [2, 4]. In [36] such series (predominantly the $q = 1$ case) were also employed as part of a novel approach for the solution of Helmholtz problems in polygonal domains.

Despite these advantages, there are two questions which motivate this particular paper. First, if q is fixed, what if one wishes to increase the convergence rate? Whilst in some applications it may be possible to increase q to obtain faster convergence, in others, the numerical solution of PDEs, for example, this parameter may be fixed (e.g. if q were chosen to match the boundary conditions of the particular problem). In addition, whilst theoretically possible to increase q arbitrarily, in practice this may become too costly beyond $q = 4$ [5,6]. Second, what if the approximated function is only piecewise smooth? As we shall show in this paper, the advantageous convergence of polyharmonic–Neumann expansions is no longer present in this scenario, and a Gibbs phenomenon occurs near interior discontinuity locations. Hence, a technique for increasing the convergence rate (equivalently, ameliorating or removing the Gibbs phenomenon) is also required in this case.

For these reasons, we devote the first part of this paper to studying the convergence, or lack thereof, of polyharmonic–Neumann expansions. As we establish, the interior Gibbs phenomenon occurring for a piecewise smooth function is strikingly similar to that of classical Fourier series. Moreover, for smooth functions, the barrier to faster convergence (i.e. faster than $\mathcal{O}(N^{-q})$) can be explained in terms of weak Gibbs phenomenon occurring in the q^{th} derivative of the expansion. Equivalently, this phenomenon can be understood using certain duality arguments, and by introducing related expansions corresponding to eigenfunctions subject to homogeneous Dirichlet boundary conditions. Upon comparison with the classical Fourier case, this weak Gibbs phenomenon is found to possess a number of important distinctions. Nonetheless, as we subsequently establish, we are still able to exactly determine the overshoot constant c in this case.

Having described this phenomenon, in the second part of this paper we consider both its mitigation and complete removal. As we discuss, factors such as smoothness and periodicity that determine the convergence rate of Fourier series have natural analogues for these expansions. Once such factors are understood, it is possible to generalise a number of known techniques to the polyharmonic–Neumann setting. In doing so, we highlight the broad applicability of such methods, beyond their original purpose. The culmination of this work is a spectrally accurate approximation scheme based on such eigenfunctions, similar in character to the Fourier extension method [10,22]. Moreover, as a by-product, families of polynomials that generalise the classical Lidstone polynomials are introduced and discussed.

The outline of the remainder of this paper is as follows. In Section 2 we introduce expansions in polyharmonic–Neumann eigenfunctions and recap salient aspects of [5,6,23]. Pointwise convergence of such expansions away from discontinuities is established in Section 3, and in Section 4 we detail the Gibbs phenomenon. Techniques to remove the Gibbs phenomenon are studied in Section 5.

2 Expansions in polyharmonic eigenfunctions

Polyharmonic eigenfunctions have been studied systematically in [23] (the case $q = 1$) and [6] (arbitrary $q \geq 1$). Since the polyharmonic operator, when equipped with homogeneous Neumann boundary conditions, is semi-positive definite, it has a count-

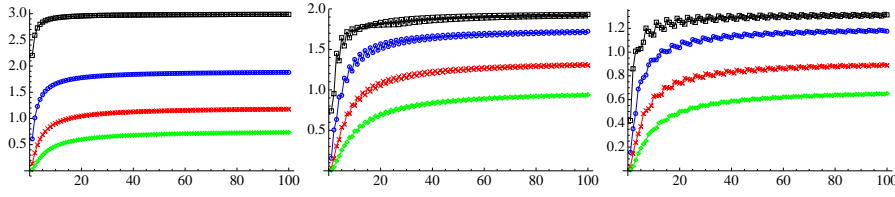


Fig. 2.1 Scaled errors $N^q \|f - f_N\|_{L^\infty[-1,1]}$ (left), $N^{q+1} \|f - f_N\|_{L^\infty[-\frac{1}{2}, \frac{1}{2}]}$ (middle) and $N^{q+1} \|f - f_N\|_{L^\infty[-\frac{1}{4}, \frac{1}{4}]}$ (right) against $N = 1, \dots, 100$ for $q = 1, 2, 3, 4$ (squares, circles, crosses and diamonds respectively).

able number of real, nonnegative eigenvalues μ_n , $n \in \mathbb{N}$, having no finite limit point in \mathbb{R} . A simple argument finds that the zero eigenvalue $\mu_0 = 0$ has multiplicity q . Denoting the eigenfunctions corresponding to this eigenvalue by $\phi_{0,n}$, $n = 0, \dots, q-1$, and the eigenfunctions corresponding to μ_n by ϕ_n , $n \in \mathbb{N}_+$, standard spectral theory establishes that the set $\{\phi_{0,n} : n = 0, \dots, q-1\} \cup \{\phi_n : n \in \mathbb{N}_+\}$ forms an orthogonal basis of $L^2(-1, 1)$. For this reason, any function $f \in L^2(-1, 1)$ may be expanded in polyharmonic–Neumann eigenfunctions

$$f(\cdot) \sim \sum_{n=0}^{q-1} \frac{\hat{f}_{0,n}}{\|\phi_{0,n}\|^2} \phi_{0,n}(\cdot) + \sum_{n=1}^{\infty} \frac{\hat{f}_n}{\|\phi_n\|^2} \phi_n(\cdot),$$

with identification in the usual L^2 sense. Here $\hat{f}_n = \int_{-1}^1 f(x) \overline{\phi_n(x)} dx$ (respectively $\hat{f}_{0,n}$) is the coefficient of f with respect to ϕ_n ($\phi_{0,n}$), $\|g\|^2 = \int_{-1}^1 |g(x)|^2 dx$ is the standard L^2 norm of $g \in L^2(-1, 1)$ and \bar{z} denotes the complex conjugate of $z \in \mathbb{C}$.

In practice, this infinite sum must be truncated, leading to the approximation

$$f(x) \approx f_N(x) = \sum_{n=0}^{q-1} \frac{\hat{f}_{0,n}}{\|\phi_{0,n}\|^2} \phi_{0,n}(x) + \sum_{n=1}^N \frac{\hat{f}_n}{\|\phi_n\|^2} \phi_n(x). \quad (2.1)$$

Our interest in the first part of this paper lies with the convergence of the approximation f_N to f (or lack thereof) and, in particular, the nature of the Gibbs phenomenon whenever f is only piecewise smooth. To illustrate, in Figure 2.1 we consider the approximation of the smooth function $f(x) = e^{2x}$ by $f_N(x)$ for $q = 1, 2, 3, 4$. Confirming the result proved in [5, 32], the uniform error $\|f - f_N\|_{L^\infty(I)}$ is $\mathcal{O}(N^{-q})$ when $I = [-1, 1]$ is the whole interval, and $\mathcal{O}(N^{-q-1})$ whenever I is compactly contained in $(-1, 1)$.

Conversely, in Figure 2.2 we illustrate the approximation of the Heaviside function H . Herein we witness a Gibbs phenomenon near $x = 0$: $\mathcal{O}(1)$ oscillations occur, and, away from $x = 0$ the approximation f_N converges only linearly in N , regardless of q . However, whilst the Gibbs phenomenon at $x = 0$ strongly resembles that occurring in Fourier series (a fact we fully explain and quantify later), there is no endpoint phenomenon, even though the function H , when considered as a function of the unit torus $\mathbb{T} = [-1, 1)$, has a discontinuity at $x = -1$. For this reason, we can expect, and it turns out to be the case, that the endpoint Gibbs phenomenon in polyharmonic–Neumann series (whenever it occurs), differs from that of Fourier series.

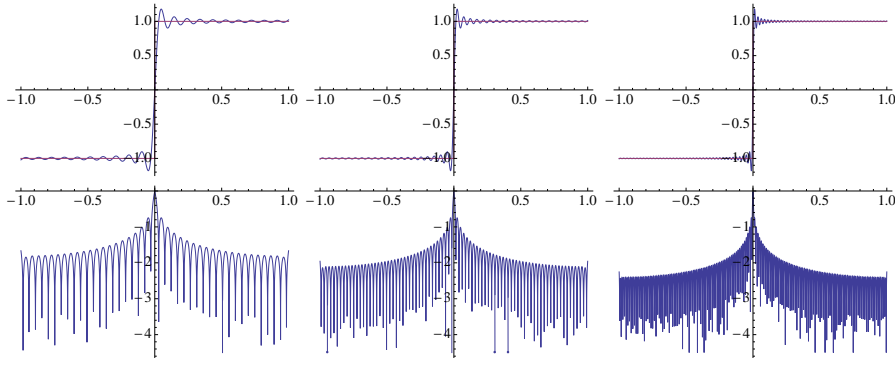


Fig. 2.2 The functions $f_N(x)$ (top row) and $\log_{10}|f(x) - f_N(x)|$ (bottom row) against $-1 \leq x \leq 1$ for $N = 20, 40, 80$ (left to right) and $q = 2$.

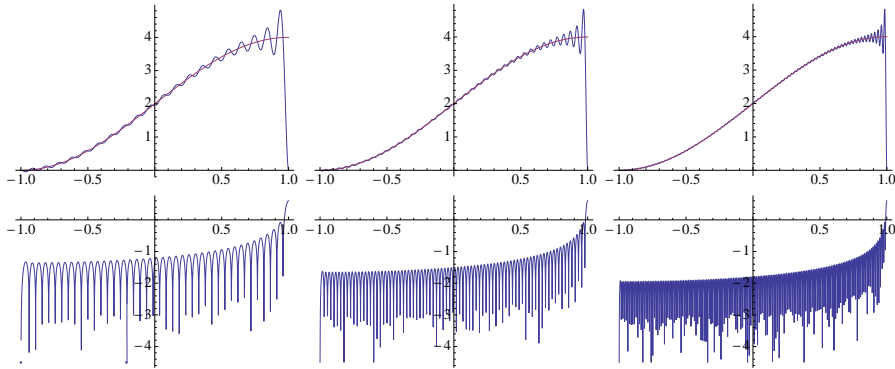


Fig. 2.3 The functions $f_N^{(q)}(x)$ (top row) and $\log_{10}|f^{(q)}(x) - f_N^{(q)}(x)|$ (bottom row) against $-1 \leq x \leq 1$ for $N = 20, 40, 80$ (left to right).

To further illustrate this point, consider a smooth function f . As noted, there is no Gibbs phenomenon in f_N itself. However, as we will establish, a weak Gibbs phenomenon occurs in the q^{th} derivative $f_N^{(q)}$ at the endpoints $x = \pm 1$. In Figure 2.3 we consider such phenomenon for $q = 2$ and the example $f(x) = x^2 - \frac{8}{\pi^2} \sin \frac{\pi}{2}x$. Immediately, we notice one important distinction between this and the Fourier case: namely, the phenomenon is local. In particular, at $x = -1$, where f vanishes, there is no Gibbs phenomenon, whilst the phenomenon indeed occurs at $x = 1$. Notice also pointwise convergence (at rate N^{-1}) in $(-1, 1)$. The aim of Sections 3 and 4 is to confirm these observations.

Before doing so, however, we first require convenient, explicit expressions for the polyharmonic–Neumann eigenfunctions, valid for arbitrary q . Our interest primarily lies with those eigenfunctions ϕ_n corresponding to nonzero eigenvalues: trivially, the eigenfunction $\phi_{0,n}$ is precisely P_n , where P_n is the n^{th} Legendre polynomial.

For $n \in \mathbb{N}_+$ we may write the n^{th} eigenfunction as

$$\phi_n(x) = \sum_{r=0}^{2q-1} c_{r,n} e^{\lambda_r \alpha_n x}, \quad (2.2)$$

where $\lambda_r = -ie^{\frac{i\pi}{q}}$, and the constants $\alpha_n^{2q} = \mu_n$ and the $c_{r,n} \in \mathbb{C}$ are specified by enforcing boundary conditions. This results in an algebraic eigenproblem (for each n), from which the coefficients $c_{r,n}$ and the value μ_n can be computed. As discussed in [6], this expression is usually reduced to a real form for computations. However, for the purposes of analysis, it is more convenient to retain the complex exponential version (2.2).

In [5] the asymptotic nature (as $n \rightarrow \infty$) of the eigenfunctions ϕ_n and the values α_n were considered. It was found that such quantities have known asymptotic expressions up to exponentially small terms in n . In particular, if $\gamma_q = \sin \frac{\pi}{q}$, then

$$\alpha_n = \frac{1}{4}(2n + q - 1)\pi + \mathcal{O}(e^{-n\pi\gamma_q}), \quad n \rightarrow \infty, \quad (2.3)$$

and

$$\phi_n(x) = \sum_{r=0}^{q-1} c_r \left[e^{\lambda_r \alpha_n (x-1)} + (-1)^{n+q+1} e^{-\lambda_r \alpha_n (x+1)} \right] + \mathcal{O}(e^{-n\pi\gamma_q}), \quad (2.4)$$

uniformly for $x \in [-1, 1]$. Here the values c_r are given explicitly as particular minors of the matrix $V \in \mathbb{C}^{q \times q}$ with $(r, s)^{\text{th}}$ entry λ_r^s . Specifically, $c_r = (-1)^q (\det V) (V^{-1})_{r,q}$. Several other results concerning such eigenfunctions were also obtained in [5]. In particular, away from the endpoints $x = \pm 1$, the n^{th} eigenfunction ϕ_n is exponentially close to a regular oscillator:

$$\phi_n(x) = c_0 \left[e^{-i\alpha_n (x-1)} + (-1)^{n+q+1} e^{i\alpha_n (x+1)} \right] + \mathcal{O}\left(e^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right).$$

Furthermore, for $r \in \mathbb{N}$, the derivative $\phi_n^{(r)}(x)$ is given by

$$c_0 (-i)^r \alpha_n^r \left[e^{-i\alpha_n (x-1)} + (-1)^{n+r+q+1} e^{i\alpha_n (x+1)} \right] + \mathcal{O}\left(n^r e^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right), \quad (2.5)$$

and, at the endpoints $x = \pm 1$, the function ϕ_n and its derivatives satisfy

$$\phi_n^{(r)}(\pm 1) = (\pm 1)^{n+r+q+1} d_r \alpha_n^r, \quad d_r = (-i)^{q-r-1} c_0 + \sum_{s=0}^{q-1} c_s \lambda_s^r. \quad (2.6)$$

Finally, concerning the eigenfunction norm, we have $\|\phi_n\| = c + \mathcal{O}(e^{-\frac{1}{2}n\pi\gamma_q})$, where $c = 2|c_0|$ [5].

We remark at this point that such exponential asymptotics are vital to this paper. We are able to detail the Gibbs phenomenon for the family of polyharmonic–Neumann expansions precisely because such remainder terms decay so rapidly with n (for convenience, from this point onwards we drop any exponentially small terms). Having said this, whilst the classical Gibbs phenomenon for Fourier series is usually studied by analysing finite sums with indices $n = 1, \dots, N$, we need to consider infinite sums with $n > N$ so as to exploit (2.3)–(2.6). Although additional care is necessary to ensure convergence, few technical issues arise from this approach.

3 Pointwise convergence for piecewise smooth functions

The first facet of the classical Gibbs phenomenon is the lack of uniform convergence on $[-1, 1]$ of the truncated Fourier sum of a piecewise smooth function f . Moreover, whilst such expansions converge pointwise away from the discontinuities, the rate of convergence is only linear in the truncation parameter N .

The intent of this section is to demonstrate identical convergence for the expansion of a piecewise smooth function in polyharmonic–Neumann eigenfunctions (we could impose lower regularity in each subdomain, yet, for simplicity, we shall assume smoothness throughout). To this end, suppose that $f : [-1, 1] \rightarrow \mathbb{R}$ has jump discontinuities at $-1 < x_1 < \dots < x_k < 1$. To study the expansion f_N of f , it is first necessary to obtain explicit expressions for the coefficients \hat{f}_n . These are provided by first replacing ϕ_n by $(-1)^q \alpha_n^{-2q} \phi_n^{(2q)}$ in $\int_{-1}^1 f(x) \phi_n(x) dx$ and integrating by parts. Taking care of the discontinuities and noticing that $\phi_n^{(2q-1)}(\pm 1) = 0$, this gives

$$\begin{aligned} \int_{-1}^1 f(x) \phi_n(x) dx &= \frac{(-1)^{q+1}}{\alpha_n^{2q}} \sum_{j=1}^k [f](x_j) \phi_n^{(2q-1)}(x_j) \\ &\quad + \frac{(-1)^{q+1}}{\alpha_n^{2q}} \sum_{j=0}^k \int_{x_j}^{x_{j+1}} f'(x) \phi_n^{(2q-1)}(x) dx, \end{aligned} \quad (3.1)$$

where $[g](x) = g(x^+) - g(x^-)$. Integrating by parts a further $2q - 1$ times and applying the boundary conditions $\phi_n^{(q+s)}(\pm 1) = 0$, $s = 0, \dots, q - 1$, we then find that

$$\begin{aligned} &\int_{-1}^1 f(x) \phi_n(x) dx \\ &= \frac{1}{\alpha_n^{2q}} \left\{ \sum_{s=0}^{q-1} (-1)^s \left[f^{(q+s)}(1) \phi_n^{(q-s-1)}(1) - f^{(q+s)}(-1) \phi_n^{(q-s-1)}(-1) \right] \right. \\ &\quad \left. - \sum_{s=0}^{2q-1} (-1)^{q+s} \left[f^{(s)} \right] (x_j) \phi_n^{(2q-s-1)}(x_j) \right\} + \frac{(-1)^q}{\alpha_n^{2q}} \sum_{j=0}^k \int_{x_j}^{x_{j+1}} f^{(2q)}(x) \phi_n(x) dx. \end{aligned}$$

Furthermore, after iterating this process, we obtain

$$\begin{aligned} &\int_{-1}^1 f(x) \phi_n(x) dx \\ &\sim \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{q-1} \frac{(-1)^{rq+s}}{\alpha_n^{2(r+1)q}} \left[f^{((2r+1)q+s)}(1) \phi_n^{(q-s-1)}(1) - f^{((2r+1)q+s)}(-1) \phi_n^{(q-s-1)}(-1) \right] \right. \\ &\quad \left. - \sum_{s=0}^{2q-1} \frac{(-1)^{(r+1)q+s}}{\alpha_n^{2(r+1)q}} \sum_{j=1}^k \left[f^{(2rq+s)} \right] (x_j) \phi_n^{(2q-s-1)}(x_j) \right\}, \quad n \rightarrow \infty. \end{aligned} \quad (3.2)$$

In [5], it was shown that $\|\phi_n^{(r)}\|_{\infty} = \mathcal{O}(n^r)$. Since $\alpha_n = \mathcal{O}(n)$, (3.2) presents an asymptotic expansion for the coefficient \hat{f}_n in inverse powers of n (in the usual Poincaré sense). Hence, the use of the symbol \sim . Naturally, the right hand side of (3.2) will

not typically converge for fixed n . Note that (3.2) is very similar, and is derived using the same technique of repeated integration by parts, to the corresponding well-known asymptotic expansion for Fourier coefficients (known as the *Fourier Coefficient Asymptotic Expansion (FCAE)*), as found in Lyness [27, 29], and as considered at length in the monograph of Gottlieb and Orszag [19].

On closer inspection, (3.2) provides several indications as to the nature of the Gibbs phenomenon for polyharmonic–Neumann expansions. First, whilst all derivatives of f at $x = x_j$ appear in (3.2), at the endpoints only those derivatives $f^{(l)}$ with $l = (2r + 1)q + s$ and $s = 0, \dots, q - 1$ are present. In particular, the lowest derivative is $f^{(q)}(\pm 1)$, indicating that the Gibbs phenomenon at the endpoints is weak (it only appears in the q^{th} derivative of f). Second, the endpoints $x = \pm 1$ are represented in a fundamentally different manner to the internal discontinuities x_j , $j = 1, \dots, k$. In particular, whilst the contribution from interior discontinuities involves only the jump values $[f^{(2rq+s)}](x_j)$, the values $f^{(2rq+s)}(\pm 1)$ occur separately. We may therefore expect, and it turns out to be the case, that the maximum overshoot size in the q^{th} derivative at each endpoint depends only on the value of f at that endpoint. Conversely, in a manner akin to Fourier series, the overshoot at any internal discontinuity x_j involves the jump value $[f](x_j)$. This last point comes as no great surprise. As highlighted in (2.5), the eigenfunctions ϕ_n are (up to exponentially small terms) regular oscillators in $(-1, 1)$. Hence, intuition suggests that the error $f(x) - f_N(x)$ will behave similarly to a tail of a standard Fourier series.

The expansion (3.2) also provides at least one further insight. Notice that $\hat{f}_n = \mathcal{O}(n^{-1})$, thus we cannot expect uniform convergence of f_N to f . However, whenever f is smooth, $\hat{f}_n = \mathcal{O}(n^{-q-1})$, and we can expect uniform convergence at a rate of $\mathcal{O}(N^{-q})$ (as shown in [6]). Moreover, this coefficient decay is determined by the values $f^{(q)}(\pm 1)$, thus indicating a weak Gibbs phenomenon at $x = \pm 1$ in the q^{th} derivative $f_N^{(q)}$. In Section 4.1 we describe this phenomenon in more detail.

Returning to pointwise convergence for a function with only piecewise smoothness, suppose that $x_0 = -1$, $x_{k+1} = 1$ and let $U_j \subseteq [-1, 1]$, $j = 0, \dots, k + 1$, be compact with $U_j \cap \{x_j\} = \emptyset$. Given $N \in \mathbb{N}_+$, we now define

$$\Theta_N(j, r, s; x) = \frac{1}{c^2} \sum_{n \geq N} \frac{\overline{\phi_n^{(2q-s-1)}(x_j)}}{\alpha_n^{2(r+1)q}} \phi_n(x), \quad r \in \mathbb{N}, \quad s = 0, \dots, 2q - 1. \quad (3.3)$$

It is not immediately apparent that such functions are well-defined. However,

Lemma 3.1 *For each $j = 0, \dots, k + 1$, $r \in \mathbb{N}$ and $s = 0, \dots, 2q - 1$, the function $\Theta_N(j, r, s; \cdot)$ is well-defined and continuous on U_j . Moreover,*

$$\Theta_N(j, r, s; x) = \mathcal{O}(N^{-2rq-s-1}), \quad (3.4)$$

uniformly for $x \in U_j$.

To prove this lemma, it is first useful to note the following:

Lemma 3.2 *Suppose that V is a compact subset of $\{z \in \mathbb{C} : |z| \leq 1, z \neq 1\}$. Then the function Φ , the Lerch transcendental function [37], given by*

$$\Phi(z, r, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^r}, \quad a > 0, \quad r \geq 1,$$

is well-defined and continuous on V . Moreover, if $\sigma_{z,r}(t) = \frac{t^{r-1}}{1-ze^{-t}}$, then

$$\Phi(z, r, a) \sim \frac{1}{\Gamma(r)} \sum_{s=r-1}^{\infty} \frac{\sigma_{z,r}^{(r-1)}(0)}{a^{s+1}}, \quad a \rightarrow \infty.$$

In particular, $\Phi(z, r, a) = a^{-r}(1-z)^{-1} + \mathcal{O}(a^{-r-1})$.

Proof When $r > 1$ the infinite sum converges uniformly for all $z \in V$. Hence, for the first part of the proof it suffices to consider $r = 1$. We use Abel summation. Let

$$I_N(z) = \sum_{n=0}^N \frac{z^n}{n+a},$$

be the partial sum, and notice that

$$\begin{aligned} I_N(z) &= \sum_{n=0}^N z^n \left[\frac{1}{n+a} - \frac{1}{n+a+1} \right] + \sum_{n=0}^N \frac{z^n}{n+a+1} \\ &= \sum_{n=0}^N \frac{z^n}{(n+a)(n+a+1)} + \frac{1}{z} I_N - \frac{1}{za} + \frac{z^{N+1}}{N+a+1}. \end{aligned}$$

Since $z \neq 1$, it follows that

$$I_N(z) = \frac{z}{z-1} \sum_{n=0}^N \frac{z^n}{(n+a)(n+a+1)} + \frac{1}{a(1-z)} + \frac{z^{N+1}}{(N+a+1)(z-1)}.$$

Hence I_N converges (as $N \rightarrow \infty$) uniformly in V to a continuous function.

The second part of this lemma is very similar to a result proved in [32]. The only generalisations are allowing $|z| \leq 1$, as opposed to $|z| = 1$, and $r \geq 1$ instead of $r > 1$. The proof is virtually identical, hence is omitted. \square

Proof (Lemma 3.1) Consider first the case $j = 1, \dots, k$. By (2.5), it follows that

$$\overline{\phi_n^{(2q-s-1)}}(x_j) = \overline{c_0} i^{2q-s-1} \alpha_n^{2q-s-1} \left[e^{i\alpha_n(x_j-1)} + (-1)^{n+q+s} e^{-i\alpha_n(x_j+1)} \right].$$

Now examine the partial sums

$$\sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \left[e^{i\alpha_n(x_j-1)} + (-1)^{n+q+s} e^{-i\alpha_n(x_j+1)} \right] \phi_n(x).$$

Replacing ϕ_n by its asymptotic expression (2.4), we obtain

$$\begin{aligned} \sum_{l=0}^{q-1} \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \left[e^{i\alpha_n(x_j-1)} + (-1)^{n+q+s} e^{-i\alpha_n(x_j+1)} \right] \\ \times \left[e^{\lambda_l \alpha_n(x-1)} + (-1)^{n+q+1} e^{-\lambda_l \alpha_n(x+1)} \right], \end{aligned}$$

up to exponentially small terms in N . Thus, it suffices to consider separately the following four sums

$$\begin{aligned} & \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} e^{[i(x_j-1)+\lambda_l(x-1)]\alpha_n}, & \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} (-1)^n e^{[-i(x_j+1)+\lambda_l(x-1)]\alpha_n} \\ & \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} (-1)^n e^{[i(x_j-1)-\lambda_l(x+1)]\alpha_n}, & \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} e^{[-i(x_j+1)-\lambda_l(x+1)]\alpha_n}. \end{aligned}$$

Since all cases are similar, we study the first sum only. As $\alpha_n \sim \frac{1}{4}(2n+q-1)\pi$, we see that this reduces to a constant multiple of

$$z^{(N+\frac{q-1}{2})\pi} \sum_{m=0}^M \frac{z^m}{(m+N+\frac{q-1}{2})^{2rq+s+1}},$$

where $z = e^{\frac{1}{2}[i(x_j-1)+\lambda_l(x-1)]\pi}$. Now, since $\operatorname{Re} \lambda_l \geq 0$ and $x \leq 1$, we conclude that $|z| \leq 1$. Moreover, if $l = 1, \dots, q-1$, then $\operatorname{Re} \lambda_l > 0$. Hence, $|z| < 1$ in this case since $x \neq 1$. Now suppose that $l = 0$, so that $z = e^{\frac{1}{2}i(x_j-x)\pi}$. Since $x \neq x_j$, it follows that $z \neq 1$. Thus, an application of Lemma 3.2 now confirms that this sum converges uniformly on U_j to a continuous function. In addition, we also obtain the estimate (3.4).

It remains to demonstrate the result when $j = 0, k+1$. Both cases are similar, so we assume that $j = k+1$, whence $x_j = 1$. Consider the partial sum

$$\begin{aligned} & \sum_{n=N}^{N+M} \frac{\overline{\phi_n^{(2q-s-1)}}(1)}{\alpha_n^{2(r+1)q}} \phi_n(x) \\ & = d_{2q-s-1} \sum_{l=0}^{q-1} \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \left[e^{\lambda_l \alpha_n (x-1)} + (-1)^n e^{-\lambda_l \alpha_n (x+1)} \right]. \end{aligned}$$

We now proceed in an identical manner. □

With this lemma to hand, we are now able to provide the key result of this section:

Theorem 3.1 *Suppose that $U \subseteq [-1, 1]$ is compact and $\{x_1, \dots, x_k\} \cap U = \emptyset$. Then f_N converges uniformly to f in U . In particular, $\|f - f_N\|_{L^\infty(U)} = \mathcal{O}(N^{-1})$.*

Proof Recall (3.1). Integrating the remainder term by parts once more, we have

$$\hat{f}_n = \frac{(-1)^{q+1}}{\alpha_n^{2q}} \sum_{j=1}^k [f](x_j) \overline{\phi_n^{(2q-1)}}(x_j) + \mathcal{O}(n^{-2}).$$

Substituting this into (2.1), we find that

$$\begin{aligned} & f_{N+M}(x) - f_N(x) \\ & = (-1)^{q+1} \sum_{j=1}^k [f](x_j) [\Theta_{N+M}(j, 0, 0; x) - \Theta_N(j, 0, 0; x)] + \mathcal{O}(N^{-1}). \end{aligned}$$

Using Lemma 3.1, we deduce that $\{f_N(x)\}_{N=1}^\infty$ forms a Cauchy sequence uniformly for $x \in U$. Hence f_N converges uniformly on U to some continuous function \tilde{f} . Now suppose that $\tilde{f}(y) \neq f(y)$ for some $y \in U$. Then, by continuity, these functions must differ on some neighbourhood $U' \subseteq U$. Hence

$$0 < \int_{U'} |f(x) - \tilde{f}(x)|^2 dx \leq \lim_{N \rightarrow \infty} \int_{U'} |f(x) - f_N(x)|^2 dx = 0,$$

a contradiction (the rightmost equality follows from the fact that $\{\phi_n\}$ is an orthogonal basis of $L^2(-1, 1)$ and f_N is the orthogonal projection). Hence $\tilde{f} = f$ and we conclude uniform convergence of f_N to f in U .

Since convergence is assured, we may now write the error $f(x) - f_N(x)$ as an infinite sum. In an identical manner, we find that

$$f(x) - f_N(x) = (-1)^{q+1} \sum_{j=1}^k [f](x_j) \Theta_N(j, 0, 0; x) + \mathcal{O}(N^{-1}). \quad (3.5)$$

In view of Lemma 3.1, we conclude that the rate of convergence is $\mathcal{O}(N^{-1})$. \square

This theorem confirms the numerical results in Figure 2.2. Moreover, although pointwise convergence has now been confirmed, we can actually provide a far more detailed assessment. Trivially, using (3.2), we find that

$$\begin{aligned} f(x) - f_N(x) \sim \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{2q-1} (-1)^{(r+1)q+s+1} \sum_{j=1}^k [f^{(2rq+s)}](x_j) \Theta_N(j, r, s; x) \right. \\ \left. + \sum_{s=0}^{q-1} (-1)^{rq+s} \left\{ f^{((2r+1)q+s)}(1) \Theta_N(k+1, r, s+q; x) \right. \right. \\ \left. \left. - f^{((2r+1)q+s)}(-1) \Theta_N(0, r, s+q; x) \right\} \right\}, \quad N \rightarrow \infty. \quad (3.6) \end{aligned}$$

Due to Lemma 3.1, this is an asymptotic expansion for $f(x) - f_N(x)$ (in inverse powers of N), valid uniformly in U . In particular, for smooth f , the error $f(x) - f_N(x) = \mathcal{O}(N^{-q-1})$ away from $x = \pm 1$, thus confirming the result of Figure 2.1.

With sufficient effort, we could derive exact expressions for each Θ_N in terms of the Lerch transcendental function $\Phi(\cdot, \cdot, \cdot)$. However, we shall not do this (this is described in further detail in [5, 32]). Nonetheless, it is of interest to determine the precise leading order asymptotic behaviour of such functions. In turn, this provides an exact expression for the leading order behaviour of the error $f(x) - f_N(x)$. Recalling (3.5), we note that this behaviour is determined solely by the functions $\Theta_N(j, 0, 0; x)$, $j = 1, \dots, k$. The only contribution of f occurs in the values $[f](x_j)$, $j = 1, \dots, k$. Hence, we conclude that, aside function dependent constants, the local behaviour of the error is independent of the approximated function (naturally, it is also dependent on the singularity locations x_1, \dots, x_k). For this reason, it suffices to consider only the functions $\Theta_N(j, 0, 0; x)$, $j = 1, \dots, k$. We have

Lemma 3.3 For $j = 1, \dots, k$ the function $\Theta_N(j, 0, 0; x)$, $x \in U_j$, satisfies

$$\begin{aligned} & \Theta_N(j, 0, 0; x) \\ &= \frac{(-1)^{q+1}}{2} \left\{ \frac{\cos(\alpha_N - \frac{\pi}{4})(x - x_j)}{\sin \frac{\pi}{4}(x - x_j)} + (-1)^N \frac{\sin(\alpha_N - \frac{\pi}{4})(x + x_j)}{\cos \frac{\pi}{4}(x + x_j)} \right\} \alpha_N^{-1} + \mathcal{O}(N^{-2}). \end{aligned}$$

Proof Suppose that $w \in \mathbb{C}$ with $|w| \leq 1$ and $w \neq 1$. Consider $\sum_{n \geq N} (\pm 1)^n w^{\alpha_n} \alpha_n^{-1}$. Since $\alpha_{n+N} = \alpha_N + \frac{1}{2}n\pi$, we find that

$$\begin{aligned} \sum_{n \geq N} \frac{(\pm 1)^n w^{\alpha_n}}{\alpha_n} &= \frac{2(\pm 1)^N w^{\alpha_N}}{\pi} \sum_{n=0}^{\infty} \frac{(\pm w^{\frac{1}{2}\pi})^n}{n + N + \frac{q-1}{2}} \\ &= \frac{2(\pm 1)^N w^{\alpha_N}}{\pi} \Phi\left(\pm w^{\frac{1}{2}\pi}, 1, N + \frac{q-1}{2}\right). \end{aligned}$$

It now follows from Lemma 3.2 that

$$\sum_{n \geq N} \frac{(\pm 1)^n w^{\alpha_n}}{\alpha_n} = \frac{(\pm 1)^N w^{\alpha_N}}{\alpha_N(1 \mp w^{\frac{1}{2}\pi})} + \mathcal{O}(N^{-2}). \quad (3.7)$$

The full result is obtained upon substituting the asymptotic formulae (2.4)–(2.6) for ϕ_n into (3.3) and applying (3.7) to each term. To simplify the various expressions, we use the equalities

$$\frac{e^{ia}}{1 + e^{ib}} + \frac{e^{-ia}}{1 + e^{-ib}} = \frac{\cos(a - \frac{1}{2}b)}{\cos \frac{1}{2}b}, \quad \frac{e^{ia}}{1 + e^{ib}} - \frac{e^{-ia}}{1 + e^{-ib}} = i \frac{\sin(a - \frac{1}{2}b)}{\cos \frac{1}{2}b},$$

which are valid for all $a, b \in \mathbb{C}$, $b \neq (2n+1)\pi$. □

In Figure 3.1 we display the pointwise error in approximating the function

$$f(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ -e^x & \text{otherwise} \end{cases} \quad (3.8)$$

with polyharmonic–Neumann eigenfunctions corresponding to $q = 3$. A number of key features of polyharmonic–Neumann expansions are now apparent. First, as established in Theorem 3.1, the expansion converges away from the singularities at $x = \pm \frac{1}{2}$ at a rate of N^{-1} . Second, the error oscillates with $\mathcal{O}(N)$ frequency, and, third, the particular bounding curve for the error increases (like $(x - x_j)^{-1}$) as x approaches x_j , $j = 1, \dots, k$. These final two features are predicted by the previous lemma, with the former being a consequence of the terms $e^{\pm i\alpha_n x}$, and latter being due to the denominator $\sin \frac{\pi}{4}(x - x_j)$, which is unbounded as $x \rightarrow x_j$.

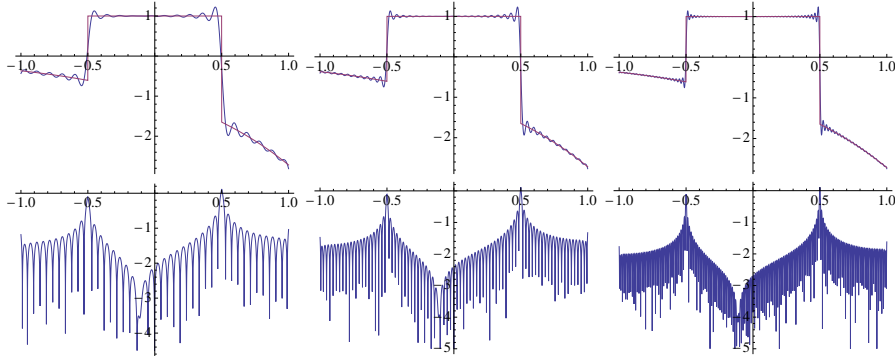


Fig. 3.1 The functions $f_N(x)$ (top row) and $\log_{10}|f(x) - f_N(x)|$ (bottom row) against $-1 \leq x \leq 1$ for $N = 20, 40, 80$ (left to right).

4 The Gibbs phenomenon

We now study the Gibbs phenomenon in polyharmonic–Neumann expansions. There are two cases: first, in Section 4.1 we consider the interior phenomenon occurring at the discontinuity locations of a piecewise smooth function. Second, we detail the weak endpoint Gibbs phenomenon in the q^{th} derivative of the polyharmonic–Neumann expansion of a smooth function. This is the content of Section 4.2. In both instances, our main task is to determine the maximal overshoot of the truncated expansion $f_N(x)$ in a neighbourhood of the discontinuity.

4.1 Interior Gibbs phenomenon

As before, let f be piecewise smooth with jump discontinuities at $-1 < x_1 < \dots < x_l < 1$. Suppose that $x \in U \setminus \{x_j\}$, where U is a compact neighbourhood of x_j (for some $j = 1, \dots, k$) with $x_l \notin U$ for $l \neq j, l = 1, \dots, k$. Using (3.5) and Lemma 3.1, we find that

$$f(x) - f_N(x) = \frac{(-1)^{q+1}[f](x_j)}{c^2} \sum_{n>N} \frac{\overline{\phi_n^{(2q-1)}(x_j)}}{\alpha_n^{2q}} \phi_n(x) + \mathcal{O}(N^{-1}). \quad (4.1)$$

By (2.5) it follows that

$$\begin{aligned} \overline{\phi_n^{(2q-1)}(x_j)} \phi_n(x) &= |c_0|^{2q-1} \alpha_n^{2q-1} \left[e^{i\alpha_n(x_j-1)} + (-1)^{n+q} e^{-i\alpha_n(x_j+1)} \right] \\ &\quad \times \left[e^{-i\alpha_n(x-1)} + (-1)^{n+q+1} e^{i\alpha_n(x+1)} \right]. \end{aligned}$$

After some simplification, this reduces to

$$\overline{\phi_n^{(2q-1)}(x_j)} \phi_n(x) = 2|c_0|^{2q-2} \alpha_n^{2q-1} \left[\sin \alpha_n(x - x_j) + (-1)^{n+q} \sin \alpha_n(x + x_j) \right].$$

Recall that $c^2 = 4|c_0|^2$. Upon substituting this into (4.1), and noticing that the term involving $(-1)^{n+q} \sin \alpha_n(x+x_j)$ is $\mathcal{O}(N^{-1})$ (by Lemma 3.2), we obtain

$$f(x) - f_N(x) = \frac{1}{2}[f](x_j) \sum_{n>N} \frac{1}{\alpha_n} \sin \alpha_n(x-x_j) + \mathcal{O}(N^{-1}). \quad (4.2)$$

We are now able to prove the main result of this section:

Theorem 4.1 *For $j = 1, \dots, k$ let U be a compact neighbourhood of x_j not containing x_l for $l \neq j$, $l = 1, \dots, k$. Suppose that*

$$\tilde{f}_N(x) = \frac{1}{2}\hat{f}_0^C + \sum_{n=1}^N [\hat{f}_n^C \cos n\pi x + \hat{f}_n^S \sin n\pi x],$$

is the truncated Fourier sum of f , where \hat{f}_n^C and \hat{f}_n^S are given by $\int_{-1}^1 f(x) \cos n\pi x dx$ and $\int_{-1}^1 f(x) \sin n\pi x dx$ respectively. Then $f_N(x) = \tilde{f}_N(x) + R(x)$, where

$$R(x) = [1 - g_1(x)] \left[f(x) - \tilde{f}_{\frac{N}{2}}(x) \right] - [f](x_j) g_2(x) \left[h(x) - \tilde{h}_{\frac{N}{2}}(x) \right] + \mathcal{O}(N^{-1}), \quad (4.3)$$

$h(x) = -\frac{1}{\pi} \log [2 |\sin \frac{1}{2} \pi(x-x_j)|]$, and

$$g_1(x) = \cos \frac{1}{4} q \pi(x-x_j) \cos \frac{1}{4} \pi(x-x_j), \quad g_2(x) = \sin \frac{1}{4} q \pi(x-x_j) \cos \frac{1}{4} \pi(x-x_j).$$

Proof First consider the Fourier sum of f . Since

$$\hat{f}_n^C = \int_{-1}^1 f(x) \cos n\pi x dx = -\frac{1}{n\pi} \sum_{l=1}^k \sin n\pi x_l [f](x_l) + \mathcal{O}(n^{-2}),$$

and

$$\begin{aligned} \hat{f}_n^S &= \int_{-1}^1 f(x) \sin n\pi x dx = \frac{1}{n\pi} \sum_{l=1}^k \cos n\pi x_l [f](x_l) \\ &\quad + \frac{(-1)^{n+1}}{n\pi} [f(1) - f(-1)] + \mathcal{O}(n^{-2}), \end{aligned}$$

we find that

$$\begin{aligned} f(x) - \tilde{f}_N(x) &= \sum_{l=1}^k [f](x_l) \sum_{n>N} \frac{1}{n\pi} [-\cos n\pi x \sin n\pi x_l + \sin n\pi x \cos n\pi x_l] \\ &\quad + \sum_{n>N} \frac{(-1)^{n+1}}{n\pi} [f(1) - f(-1)] + \mathcal{O}(N^{-1}), \end{aligned}$$

for $x \in U \setminus \{x_j\}$. The second sum is $\mathcal{O}(N^{-1})$. Hence, after simplifying, we obtain

$$f(x) - \tilde{f}_N(x) = \sum_{l=1}^k [f](x_l) \sum_{n>N} \frac{1}{n\pi} \sin n\pi(x-x_l) + \mathcal{O}(N^{-1}).$$

Since U is compact and $x_l \notin U$ for $l \neq j$, this reduces to

$$f(x) - \tilde{f}_N(x) = [f](x_j) \sum_{n>N} \frac{1}{n\pi} \sin n\pi(x - x_j) + \mathcal{O}(N^{-1}), \quad x \in U \setminus \{x_j\}. \quad (4.4)$$

Now consider the polyharmonic–Neumann expansion. Using (4.2) and the fact that $\alpha_n \sim \frac{1}{2}n\pi + \frac{1}{4}(q-1)\pi$, we have

$$\begin{aligned} f(x) - f_N(x) &= \frac{1}{2}[f](x_j) \sum_{n>\frac{N}{2}} \frac{1}{n\pi} \left\{ \sin\left[n\pi + \frac{1}{4}(q-1)\pi\right](x - x_j) + \sin\left[n\pi + \frac{1}{4}(q+1)\pi\right](x - x_j) \right\}, \end{aligned}$$

up to $\mathcal{O}(N^{-1})$. In view of (4.4), this now gives

$$f(x) - f_N(x) = g_1(x) \left[f(x) - \tilde{f}_{\frac{N}{2}}(x) \right] + g_2(x)[f](x_j) \sum_{n>\frac{N}{2}} \frac{1}{n\pi} \cos n\pi(x - x_j).$$

Therefore, to prove (4.3), it suffices to show that

$$\int_{-1}^1 h(x) \cos n\pi x \, dx = \frac{1}{n\pi}. \quad (4.5)$$

The full result then follows immediately from periodicity and standard estimates. To establish (4.5), it is useful to introduce the Clausen function $\mathcal{C}_2(\theta)$ [1], given by

$$\mathcal{C}_2(\theta) = - \int_0^\theta \log |2 \sin \frac{1}{2}t| \, dt = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n\theta.$$

Note that the infinite sum on the right-hand side converges uniformly for θ in any compact subset of \mathbb{R} . Returning to (4.5), it is readily seen that $h(x) = \frac{1}{\pi^2} \frac{d}{dx} \mathcal{C}_2(\pi x)$. Hence, substituting this into (4.5) and integrating by parts, we obtain

$$\int_{-1}^1 h(x) \cos n\pi x \, dx = \frac{1}{\pi^2} \mathcal{C}_2(\pi x) \cos n\pi x \Big|_{-1}^1 + \frac{n}{\pi} \int_{-1}^1 \mathcal{C}_2(\pi x) \sin n\pi x \, dx.$$

Since \mathcal{C}_2 has a uniformly convergent series expression, the result now follows immediately from orthogonality of the functions $\sin n\pi x$ on $[-1, 1]$ and the fact that $\mathcal{C}_2(\pm\pi) = 0$. \square

It is at first somewhat surprising that the expression (4.3) involves a term possessing a logarithmic singularity when the original function f has a jump discontinuity. Yet, this singularity is removable, since the function $g_2(x) = \mathcal{O}(x - x_j)$ for $x - x_j \ll 1$. Moreover, the appearance of Clausen functions is no great surprise given that, in general, the s^{th} such function, denoted \mathcal{C}_s , is equivalently defined as the unique odd function with n^{th} Fourier sine coefficient equal to n^{-s} [1]. Note that, although we have used such functions as a theoretical tool, their practical application to the removal of the Gibbs phenomenon in certain Fourier series has been considered in [11].

Returning to the problem at hand, consider the main conclusion of Theorem 4.1: polyharmonic–Neumann series are well approximated by Fourier series in neighbourhoods of interior singularities. Closer inspection of the remainder term $R(x)$ confirms

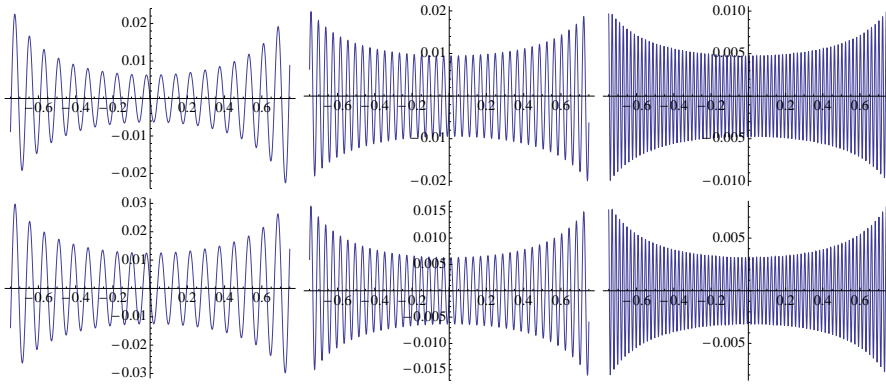


Fig. 4.1 The error $f_N(x) - \tilde{f}_{N/2}(x)$ against $x \in [-\frac{3}{4}, \frac{3}{4}]$, where $N = 25, 50, 100$ (left to right), $q = 2$ (top) and $q = 3$ (bottom), and $f(x)$ is the Heavyside function.

this fact. Indeed, if $|x - x_j| = \mathcal{O}(N^{-1})$, then $|R(x)| = \mathcal{O}(N^{-1} \log N)$. On the other hand, for $|x - x_j| \gg N^{-1}$, $|R(x)| \leq c|f(x) - \tilde{f}_{N/2}(x)| + \mathcal{O}(N^{-1})$. In Figure 4.1 we confirm these estimates by plotting the error between the two truncated sums f_N and $\tilde{f}_{N/2}$ when $f(x)$ is the Heavyside function. Notice both the linear decay in N and the growth of the bounding curve as $|x|$ increases, the latter being due to the term $|f(x) - \tilde{f}_{N/2}(x)|$.

Theorem 4.1 can be viewed as an equiconvergence theorem. In general, we say that two sequences a_n and b_n are *equiconvergent* if their difference $a_n - b_n \rightarrow 0$ as $n \rightarrow \infty$. Equiconvergence occurs most notably in nonharmonic Fourier series [39], as well as certain Birkhoff series [30]. For example, in [39, p.197] it is shown that, under certain conditions, a nonharmonic Fourier series and a classical Fourier series are uniformly equiconvergent inside any compact subset of $(-1, 1)$. In particular, the nonharmonic Fourier series inherits the same convergence and summability properties as the standard Fourier series, for example. Arguing as in the previous paragraph, we conclude that f_N and $\tilde{f}_{N/2}$ are also uniformly equiconvergent. For this reason, we can precisely determine of the nature of the interior Gibbs phenomenon occurring in polyharmonic–Neumann expansions. In particular, we have:

Corollary 4.1 *Let f have an interior jump discontinuity at x_j . Then, for sufficiently large N , the truncated polyharmonic–Neumann expansion $|f_N|$ has maximal overshoot in a neighbourhood of x_j occurring at $x_j \pm \frac{2}{N}$. Moreover, $f_N(x_j) \rightarrow [f](x_j)$ as $N \rightarrow \infty$ and*

$$f_N(x_j \pm \frac{2}{N}) = f(x_j^\pm) \pm c^*[f](x_j) + \mathcal{O}(N^{-1}),$$

where $c^* = \frac{1}{\pi} \int_0^\pi \frac{\sin x}{x} dx - \frac{1}{2} \approx 0.08949$.

We conclude that interior Gibbs phenomena for the polyharmonic–Neumann and Fourier expansions of a piecewise smooth function are identical. In Figure 4.2 we highlight this result for the case $q = 2$. Note that the maximal overshoot value, as predicted by Corollary 4.1 and corroborated by this figure, is $1 + 2c^* \approx 1.17898$.

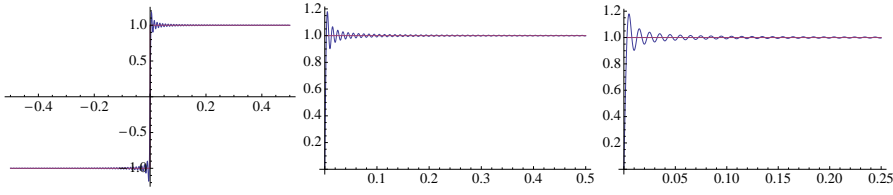


Fig. 4.2 The error $f(x) - f_{200}(x)$ against $x \in [-\frac{1}{2}, \frac{1}{2}]$ (left), $x \in [0, \frac{1}{2}]$ (middle) and $x \in [0, \frac{1}{4}]$ (right).

4.2 A weak endpoint Gibbs phenomenon

We now consider the case of a smooth function f . As observed in Figure 2.3, the approximation f_N suffers from a weak Gibbs phenomenon. To analyse this phenomenon, let us introduce eigenfunctions of the polyharmonic operator subject to homogeneous Dirichlet boundary conditions

$$(-1)^q \psi_n^{(2q)}(x) = \mu_n \psi_n(x), \quad x \in [-1, 1], \quad \psi_n^{(r)}(\pm 1) = 0, \quad r = 0, \dots, q-1. \quad (4.6)$$

Like their Neumann counterparts, the eigenfunctions ψ_n form an orthogonal basis of $L^2(-1, 1)$. The corresponding positive eigenvalues μ_n are identical to those of the Neumann case, but there is no zero eigenvalue. In the same manner, we may expand any function f in terms of polyharmonic–Dirichlet eigenfunctions

$$f(\cdot) \sim \sum_{n=1}^{\infty} \frac{\check{f}_n}{\|\psi_n\|^2} \psi_n(\cdot), \quad \check{f}_n = \int_{-1}^1 f(x) \overline{\psi_n(x)} dx.$$

Truncation of this series leads to the polyharmonic–Dirichlet expansion \tilde{f}_N (from this point on we no longer deal with Fourier series, so we shall allow this overlap in notation from the previous section). The following lemma, found in [5], describes the duality between polyharmonic–Dirichlet and polyharmonic–Neumann expansions, and explains why a study of the Gibbs phenomenon in the latter can be reduced to a consideration of the former:

Theorem 4.2 *The q^{th} derivative of the polyharmonic–Neumann expansion f_N of a function $f \in C^q[-1, 1]$ is precisely the polyharmonic–Dirichlet expansion of $f^{(q)}$.*

As a result of this theorem, to study the weak Gibbs phenomenon in polyharmonic–Neumann expansions it suffices to consider the polyharmonic–Dirichlet expansion \tilde{f}_N of an arbitrary smooth function f . For this, we derive two results. First, we confirm pointwise convergence away from $x = \pm 1$, and second, we provide an exact expression for the maximal overshoot near $x = \pm 1$. We commence with the former:

Theorem 4.3 *Suppose that $U \subseteq (-1, 1)$ is compact and f is smooth. Then \tilde{f}_N converges uniformly to f in U . Moreover,*

$$f(x) - \tilde{f}_N(x) \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} (-1)^{(r+1)q+s} \left\{ f^{(2rq+s)}(1) \Omega_+(r, s; x) - f^{(2rq+s)}(-1) \Omega_-(r, s; x) \right\},$$

for large N , where $\Omega_{\pm}(r, s; x)$ is given by

$$\Omega_{\pm}(r, s; x) = \frac{1}{c^2} \sum_{n>N} \frac{\overline{\psi_n^{(2q-s-1)}(\pm 1)}}{\alpha_n^{2(r+1)q}} \psi_n(x), \quad r \in \mathbb{N}, \quad s = 0, \dots, q-1.$$

In particular, $f(x) - \tilde{f}_N(x) = \mathcal{O}(N^{-1})$ for $x \in U$.

Note that the functions $\Omega_{\pm}(r, s; x)$ are very similar to $\Theta(k+1, r, s; x)$ and $\Theta(0, r, s; x)$ given in (3.3). In fact, although we shall not prove it, Theorem 4.2 can be used to confirm that

$$\Omega_{+}(r, s; x) = \frac{d^q}{dx^q} \Theta(k+1, r, s; x), \quad \Omega_{-}(r, s; x) = \frac{d^q}{dx^q} \Theta(0, r, s; x).$$

To prove Theorem 4.3, it is necessary to show that these functions are well defined. For this, we first note that virtually identical exponential asymptotics holds for polyharmonic–Dirichlet eigenfunctions as for their Neumann counterparts (see Section 2). In particular, we have $\|\psi_n\| = c$,

$$\psi_n^{(r)}(\pm 1) = (\pm 1)^{n+r} d_r \alpha_n^r, \quad d_r = i^{q+r+1} \tilde{c}_0 + \sum_{s=0}^{q-1} \tilde{c}_s \lambda_s^r, \quad r \in \mathbb{N},$$

and

$$\psi_n(x) = \sum_{r=0}^{q-1} \tilde{c}_r \left[e^{\lambda_r \alpha_n(x-1)} + (-1)^{n+1} e^{-\lambda_r \alpha_n(x+1)} \right] + \mathcal{O}(e^{-n\pi\gamma_q}), \quad (4.7)$$

where $\tilde{c}_r = \lambda_r^q c_r$. We are now able to prove the following lemma regarding the functions $\Omega_{\pm}(r, s; \cdot)$:

Lemma 4.1 *The functions $\Omega_{\pm}(r, s; x)$ are well-defined and continuous on any compact set $U \subseteq (-1, 1)$. Moreover, $\Omega_{\pm}(r, s; x) = \mathcal{O}(N^{-2rq-s-1})$ for $r \in \mathbb{N}$ and $s = 0, \dots, q-1$. In particular,*

$$\begin{aligned} \Omega_{+}(0, 0; x) &= c_{\Omega} \frac{\cos \left[\alpha_N x - \frac{\pi}{4}(x+1) \right]}{\cos \frac{\pi}{4}(x+1)} e^{i\alpha_N} \alpha_N^{-1} + \mathcal{O}(N^{-2}), \\ \Omega_{+}(0, 0; x) &= c_{\Omega} \frac{\sin \left[\alpha_N x - \frac{\pi}{4}(x+1) \right]}{\sin \frac{\pi}{4}(x+1)} (-1)^N e^{i\alpha_N} \alpha_N^{-1} + \mathcal{O}(N^{-2}), \quad N \rightarrow \infty, \end{aligned}$$

where $c_{\Omega} = -i \overline{\tilde{d}_{2q-1}} \tilde{c}_0 c^{-2}$.

Proof The first part of this proof is virtually identical to that of Lemma 3.1. For example, writing

$$\begin{aligned} & \sum_N^{N+M} \frac{\overline{\psi_n^{(2q-s-1)}(1)}}{\alpha_n^{2(r+1)q}} \psi_n(x) \\ &= \overline{\tilde{d}_{2q-s-1}} \sum_{l=0}^{q-1} \tilde{c}_l \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \left[e^{\lambda_l \alpha_n(x-1)} + (-1)^n e^{-\lambda_l \alpha_n(x+1)} \right], \end{aligned}$$

we find that the partial sums form a Cauchy sequence. Hence uniform convergence. To derive the exact asymptotic expression we proceed exactly as in Lemma 3.3. \square

Proof (Theorem 4.3) As in the Neumann case, consider the coefficient \check{f}_n . Replacing ψ_n by $(-1)^q \psi_n^{(2q)}$ and integrating by parts gives

$$\check{f}_n = \frac{(-1)^q}{\alpha_n^{2q}} \left\{ f(1) \psi_n^{(2q-1)}(1) - f(-1) \psi_n^{(2q-1)}(-1) \right\} + \mathcal{O}(n^{-2}).$$

The proof of the first part of the theorem is now identical to the proof of Theorem 3.1. For the second part, we once again consider the coefficient \check{f}_n . Integrating by parts repeatedly and applying the boundary conditions gives

$$\check{f}_n \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} \frac{(-1)^{(r+1)q+s}}{\alpha_n^{2(r+1)q}} \left[f^{(2rq+s)}(1) \psi_n^{(2q-s-1)}(1) - f^{(2rq+s)}(-1) \psi_n^{(2q-s-1)}(-1) \right],$$

in a similar fashion to (3.2). The result now follows immediately. \square

Having confirmed pointwise convergence of polyharmonic–Dirichlet expansions in $(-1, 1)$ we now turn our attention to describing the Gibbs phenomenon near the endpoints. To this end, assume that U is a compact neighbourhood of $x = 1$, with $-1 \notin U$. We therefore write

$$f(x) - \check{f}_N(x) = \frac{(-1)^q \overline{\check{d}_{2q-1}} f(1)}{c^2} \sum_{n>N} \frac{1}{\alpha_n} \psi_n(x) + \mathcal{O}(N^{-1}).$$

Let $x = 1 - \frac{2a}{N}$, where $a > 0$ is fixed. Substituting the asymptotic estimate (4.7), we obtain

$$\begin{aligned} & f\left(1 - \frac{2a}{N}\right) - \check{f}_N\left(1 - \frac{2a}{N}\right) \\ &= \frac{(-1)^q \overline{\check{d}_{2q-1}} f(1)}{c^2} \sum_{n>N} \frac{1}{\alpha_n} \left\{ \tilde{c}_0 \left[e^{i \frac{2a\alpha_n}{N}} - i^{q-1} e^{-i \frac{2a\alpha_n}{N}} \right] + \sum_{r=1}^{q-1} \tilde{c}_r e^{-\lambda_r \frac{2a\alpha_n}{N}} \right\}. \end{aligned}$$

up to $\mathcal{O}(N^{-1})$. Hence,

$$\begin{aligned} & f\left(1 - \frac{2a}{N}\right) - \check{f}_N\left(1 - \frac{2a}{N}\right) \\ &= \frac{2(-1)^q \overline{\check{d}_{2q-1}} f(1)}{\pi c^2} \int_{\pi}^{\infty} \frac{1}{x} \left\{ \tilde{c}_0 \left[e^{iax} - i^{q-1} e^{-iax} \right] + \sum_{r=1}^{q-1} \tilde{c}_r e^{-\lambda_r ax} \right\} dx. \end{aligned}$$

This now gives $f_N\left(1 - \frac{2a}{N}\right) = [1 + G(a)] f(1) + \mathcal{O}(N^{-1})$, where

$$\begin{aligned} G(a) &= \frac{2(-1)^{q+1} \overline{\check{d}_{2q-1}}}{\pi c^2} \int_{\pi}^{\infty} \frac{1}{x} \left\{ \tilde{c}_0 \left[e^{iax} - i^{q-1} e^{-iax} \right] + \sum_{r=1}^{q-1} \tilde{c}_r e^{-\lambda_r ax} \right\} dx \\ &= \frac{2(-1)^{q+1} \overline{\check{d}_{2q-1}}}{\pi c^2} \left\{ \tilde{c}_0 \left[\Gamma(0, -ia\pi) - i^{q-1} \Gamma(0, ia\pi) \right] + \sum_{r=1}^{q-1} \tilde{c}_r \Gamma(0, \lambda_r \pi a) \right\}, \end{aligned}$$

and $\Gamma(\cdot, \cdot)$ is the incomplete gamma function [1]. From this we conclude

	$q = 1$	$q = 2$	$q = 3$	$q = 4$
a^*	1	1.25437	1.52315	1.74643
$1 + G(a^*)$	1.1798	1.20705	1.21958	1.22792

Table 4.1 The values a^* and $1 + G(a^*)$ for $q = 1, 2, 3, 4$.

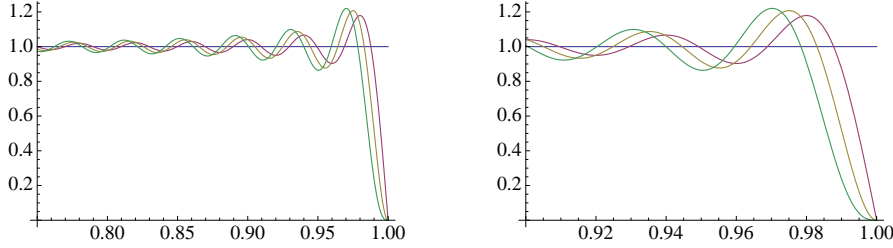


Fig. 4.3 The function $f_{50}(x)$ for $x \in [\frac{3}{4}, 1]$ (left) and $x \in [\frac{9}{10}, 1]$ and $q = 1, 2, 3$, where $f(x) = 1$.

Theorem 4.4 For sufficiently large N , $|\tilde{f}_N|$ has maximal overshoot in an $\mathcal{O}(N^{-1})$ neighbourhood of the endpoint $x = \pm 1$ occurring at $x = \pm(1 - \frac{2a^*}{N})$, where $a^* = \operatorname{argmax}_{a \geq 0} G(a)$. In addition,

$$f_N\left(\pm\left(1 - \frac{2a^*}{N}\right)\right) = [1 + G(a^*)]f(\pm 1) + \mathcal{O}(N^{-1}), \quad N \rightarrow \infty.$$

Aside from the $q = 1$ case, where $a^* = 1$, the value a^* must be found numerically. Note that a^* is a zero of the function

$$H(a) = \frac{1}{a} \left\{ \tilde{c}_0 \left[e^{i\pi a} - i^{q-1} e^{-i\pi a} \right] + \sum_{r=1}^{q-1} c_r e^{-\lambda_r \pi a} \right\}.$$

In Table 4.1 we report the values of a^* and $G(a^*)$ for various q . Figure 4.3 gives a plot of $f_N(x)$ near $x = 1$, confirming these results. Notice that, as q increases, so does the value a^* . Thus the overshoot moves further away from the endpoint $x = 1$.

If required, we could also compute the successive local maxima and minima of $G(a)$. As shown in Figure 4.3, these correspond to overshoots and undershoots of the approximation of f by f_N . In the Fourier case, these occur precisely at the values $x = 1 - \frac{2k-1}{N}$ and $x = 1 - \frac{2k}{N}$, $k \in \mathbb{N}_+$, respectively. Though this is not true in the polyharmonic–Dirichlet setting, the exponential decay (as a increases) of the terms $e^{-\lambda_r a \pi}$ appearing in $G'(a)$, indicates that successive maxima and minima will become increasingly equispaced away from $x = 1$, as in the Fourier scenario.

Another consequence of Theorem 4.4 is that, unlike the interior case, the endpoint Gibbs phenomenon is local: it involves only the value of f at the particular endpoint. This explains the lack of a Gibbs phenomenon at $x = -1$ in the example of Figure 2.3. In contrast, periodicity ensures that the Fourier series Gibbs phenomenon is identical regardless of where the discontinuity is located in $[-1, 1]$.

5 Removal of the Gibbs phenomenon

Having detailed the Gibbs phenomenon for polyharmonic–Neumann expansions, we now develop a number of techniques to first ameliorate and then completely remove this effect. By the former we mean that, given the first N polyharmonic–Neumann coefficients of a function f , we seek a new approximation f_N that suffers from the Gibbs phenomenon only in some higher derivative r , say, and correspondingly delivers uniform convergence at the increased rate of N^{-r} . Similarly, for the complete removal of the Gibbs phenomenon, we desire an approximation in which no derivative suffers from this phenomenon, and which possesses *spectral* convergence (i.e. convergence faster than any algebraic power of N^{-1}) or *exponential* convergence (convergence at a rate ρ^{-N} for some $\rho > 1$).

For a piecewise smooth function there are two components to this task. First, we remove the Gibbs phenomenon occurring at any jump discontinuities, and second we remove the weak Gibbs phenomenon occurring at both endpoints $x = \pm 1$. Naturally, the former task requires the locations of the singularities to be known. Hence, a complete method would also require a procedure for singularity detection. In the context of Fourier series, with potential for extension to this case, a number of such methods exist [38]. However, we shall not consider this issue further. From now on, we assume that the discontinuity locations are known exactly.

The techniques we consider in this section are all generalisations of known methods in the context of Fourier series. However, since the polyharmonic–Neumann expansion is well approximated by a Fourier series near any internal discontinuity, we expect that few modifications need to be made to remove the Gibbs phenomenon from such singularities. This turns out to be the case, as we document in Section 5.4. Conversely, near the endpoints the (weak) Gibbs phenomenon is completely different in character, thereby requiring genuine extensions of existing methods. For this reason, in Sections 5.1–5.3 we discuss only the case of smooth functions f . Section 5.4 is devoted to the general case.

5.1 Polynomial subtraction

Let f be smooth. To mitigate the Gibbs phenomenon near the endpoints, we first consider the asymptotic expansion (3.6) for the error $f(x) - f_N(x)$. Since f has no interior discontinuities, this reads

$$f(x) - f_N(x) \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} (-1)^{(r+1)q+s} \left\{ f^{((2r+1)q+s)}(1) \Theta_N(1, r, s+q; x) - f^{((2r+1)q+s)}(-1) \Theta_N(0, r, s+q; x) \right\}. \quad (5.1)$$

Recall from Lemma 3.1 that the functions $\Theta_N(j, r, s; x)$ are $\mathcal{O}(N^{-2rq-s-1})$ for $x \neq x_j$. In fact, with a little effort it can be shown that $\|\Theta_N(j, r, s; \cdot)\|_{\infty} = \mathcal{O}(N^{-2rq-s})$. With this in mind, (5.1) provides an important observation: the derivatives $f^{((2r+1)q+s)}(\pm 1)$ completely determine the rate of convergence of f_N . Had such derivatives vanished

(up to a certain order), faster convergence would have been witnessed. Specifically, suppose that $f^{(l)}(\pm 1) = 0$ whenever $l \in D_m$, where

$$D_m = \{l \in \mathbb{N} : l = (2r+1)q + s < m, r \in \mathbb{N}, s = 0, \dots, q-1\}, \quad m \in \mathbb{N},$$

then the expansion f_N satisfies the error estimates

$$\begin{aligned} \|f - f_N\|_\infty &= \mathcal{O}\left(N^{-(2k+1)q-p}\right), \\ f(x) - f_N(x) &= \mathcal{O}\left(N^{-(2k+1)q-p-1}\right), \quad x \in (-1, 1), \end{aligned} \quad (5.2)$$

where $m = (2k+1)q + p$, $p = 1, \dots, q-1$, $k \in \mathbb{N}$, and $m = 2kq$ when $p = 0$. Note that these derivative conditions can be viewed as the natural analogue of periodicity for polyharmonic–Neumann expansions (in particular, had we been concerned with regularity in this paper, we could have introduced an analogue of the periodic Sobolev spaces $H^k(\mathbb{T})$ for polyharmonic–Neumann expansions using such conditions).

Unfortunately, the assumption of vanishing derivatives is unrealistic. However, with the understanding that it is those derivatives with indices in D_m which determine the convergence rate, we are able to develop a simple technique to obtain faster convergence. This approach is a generalisation of a well-known method in the context of Fourier series: namely, the *polynomial subtraction* device [19, 28] (also known as Krylov’s method [25] or the Bernoulli method [18]).

Suppose that a function g satisfies

$$g^{(l)}(\pm 1) = f^{(l)}(\pm 1), \quad \forall l \in D_m, \quad (5.3)$$

where $m = (2k+1)q + p$ ($p \neq 0$) or $m = 2kq$ ($p = 0$). Then the m^{th} polynomial subtraction approximation $f_{N,m}$ is defined by $f_{N,m} = (f_N - g_N) + g$. Since the error $f - f_{N,m} = (f - g) - (f - g)_N$, and $f - g$ has vanishing derivatives with indices $l \in D_m$, we immediately see that the approximation $f_{N,m}$ obtains the faster convergence rates given by (5.2). Thus, by choosing m suitably, we can obtain algebraic convergence in N of arbitrarily high, fixed order. As a result, the Gibbs phenomenon can be ameliorated. In fact, though we shall not show this, it is only appears in the derivative $(f_{N,m})^{(l)}$, where $l = (2k+1)q + p$. Additionally, those derivatives $(f_{N,m})^{(l)}$ with $l < (2k+1)q + p$ converge uniformly to the corresponding derivatives of f [5].

The main question remaining is how to construct the function g . Typically, this is achieved with a polynomial (hence the name *polynomial subtraction*). For $q = 1$ it is well-known (see [8, 34]) that such a function g has the explicit representation

$$g(x) = \sum_{r=0}^{k-1} 2^{2r+1} \left[\Xi_r\left(\frac{1+x}{2}\right) f^{(2r+1)}(1) - \Xi_r\left(\frac{1-x}{2}\right) f^{(2r+1)}(-1) \right], \quad (5.4)$$

where $\Xi_r \in \mathbb{P}_{2r+2}$ is given by $\Xi_r(x) = \int_0^x \Lambda_r(x) dx$ and Λ_r is r^{th} Lidstone polynomial [7], defined by $\Lambda_0 = x$ and

$$\Lambda_r'' = \Lambda_{r-1}, \quad \Lambda_r(0) = \Lambda_r(1) = 0, \quad r = 1, 2, \dots \quad (5.5)$$

Note that g , as given by (5.4), is a polynomial of degree $2k$ and is the unique Birkhoff–Hermite interpolating polynomial satisfying $g(0) = 0$ and $g^{(2r+1)}(\pm 1) = f^{(2r)}(\pm 1)$,

$r = 0, \dots, k-1$. We mention in passing that Birkhoff–Hermite problems (interpolation problems based on lacunary derivatives) need not have solutions in general (unlike pure Hermite problems) [26]. However, in this case, as evidenced by (5.4), the problem is uniquely solvable.

Let us now consider the general setting $q \geq 1$. Given f , we seek a function g that satisfies the interpolation conditions (5.3). Notice that the Lidstone polynomials (5.5) are defined as solutions of Poisson’s equation. This suggests the following generalisation. For $r = 0, \dots, q-1$ define $\Lambda_r \in \mathbb{P}_{2r+1}$ by

$$\Lambda_r^{(s)}(0) = \Lambda_r^{(s)}(1) = 0, \quad s = 0, \dots, r-1, \quad \Lambda_r^{(r)}(0) = 0, \quad \Lambda_r^{(r)}(1) = 1, \quad (5.6)$$

and, for arbitrary $r \geq q$, let $\Lambda_r \in \mathbb{P}_{2r+1}$ be given by

$$\Lambda_r^{(2q)} = \Lambda_{r-q}, \quad \Lambda_r^{(s)}(0) = \Lambda_r^{(s)}(1) = 0, \quad s = 0, \dots, q-1. \quad (5.7)$$

We refer to $\{\Lambda_r\}_{r=1}^\infty$ as q -Lidstone polynomials. Note that the existence and uniqueness of such polynomials is an immediate consequence of the positive definiteness of the polyharmonic–Dirichlet operator and standard results regarding Hermite interpolation. With these polynomials in hand, we define

$$\Xi_r(x) = \int_0^x \int_0^x \cdots \int_0^x \Lambda_r(x) dx \dots dx dx, \quad r \in \mathbb{N},$$

as the q -fold integral of Λ_r . As an aside, note that the polynomial $G(x) = \Xi_{rq+s}(\frac{1+x}{2})$ has polyharmonic–Neumann coefficients

$$\hat{G}_{0,n} = 0, \quad n = 0, \dots, q-1, \quad \hat{G}_n = \frac{(-1)^{rq+s} \phi_n^{(q-s-1)}}{\alpha_n^{2(r+1)q}} (\pm 1), \quad n = 1, 2, \dots$$

Returning to the construction of g , we have

Lemma 5.1 *Let $c_{r,s} = 2^{(2r+1)q+s}$. Then the polynomial*

$$\begin{aligned} g(x) = & \sum_{r=0}^{k-1} \sum_{s=0}^{q-1} c_{r,s} \left[\Xi_{rq+s} \left(\frac{1+x}{2} \right) f^{((2r+1)q+s)}(1) + (-1)^{q+s} \Xi_{rq+s} \left(\frac{1-x}{2} \right) f^{((2r+1)q+s)}(-1) \right] \\ & + \sum_{s=0}^{p-1} c_{k,s} \left[\Xi_{kq+s} \left(\frac{1+x}{2} \right) f^{((2k+1)q+s)}(1) + (-1)^{q+s} \Xi_{kq+s} \left(\frac{1-x}{2} \right) f^{((2k+1)q+s)}(-1) \right], \end{aligned}$$

is the unique polynomial of degree $(2k+1)q+2p-1$ satisfying (5.3) and $g^{(l)}(0) = 0$, $l = 0, \dots, q-1$.

Proof This follows immediately from the definition of the polynomials Ξ_r . \square

In Figure 5.1 we demonstrate polynomial subtraction for $q = 1, 2$. Note the higher accuracy gained from increasing the degree of the subtraction polynomial g . In particular, using only $N = 30$, $q = 2$ and $m = 8$ we obtain an error of order 10^{-14} .

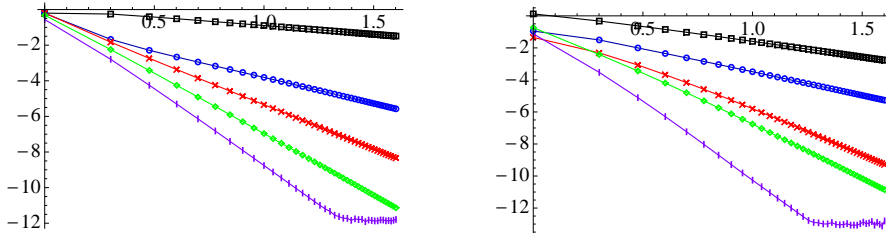


Fig. 5.1 Error in polynomial subtraction applied to $f(x) = e^x \cos 4x$. Left: log error $\log_{10} \|f - f_{N,m}\|_\infty$ against $\log_{10} N$ for $q = 1$ with $m = 0, 2, 4, 6, 8$ (in descending order). Right: the error $\log_{10} \|f - f_{N,m}\|_\infty$ for $q = 2$ with $m = 0, 1, 4, 7, 8$.

5.2 Extrapolation-based techniques

The polynomial subtraction device is widely used in the context of Fourier series. As considered, once the particular factors that determine the convergence rate of polyharmonic–Neumann expansions are understood, it can be readily generalised to this setting. Unfortunately, this technique suffers from the restriction of requiring exact derivative values. In general these are not readily available, and approximation via finite differences is not recommended for this task [28]. Fortunately, for Fourier series at least, a technique to circumvent this problem is also known. This approach, referred to as Eckhoff’s method [15, 16], is based on the idea that the coefficients \hat{f}_n themselves contain sufficient information to approximate such derivative values.

Eckhoff’s method can be extended to polyharmonic–Neumann expansions in a straightforward manner. The starting point is the asymptotic expansion (3.2) for the coefficient \hat{f}_n :

$$\hat{f}_n \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} \frac{(-1)^{r+q+s}}{\alpha_n^{2(r+1)q}} \left[f^{((2r+1)q+s)}(1) \phi_n^{(q-s-1)}(1) - f^{((2r+1)q+s)}(-1) \phi_n^{(q-s-1)}(-1) \right].$$

Suppose now that the function g interpolates exactly those derivatives $f^{(l)}(\pm 1)$ with $l \in D_m$. Then, it is readily seen that $\hat{f}_n = \hat{g}_n + \mathcal{O}(n^{-2kq-p-1})$. To avoid the use of derivatives in the construction of the function g , we seek to enforce this relation in the asymptotic limit $n \rightarrow \infty$. To do so, we define the new function g by

$$\hat{f}_n = \hat{g}_n, \quad n = N+1, N+2, \dots, N+2(kq+p), \quad (5.8)$$

a $(2kq+2p) \times (2kq+2p)$ linear system for the coefficients of g . As before, we introduce the new approximation via $f_{N,m} = (f_N - g_N) + g$. Since this procedure is reminiscent of (but not identical to) the Richardson extrapolation method [35], we refer to it as an *extrapolation-based* technique.

When $q = 1$, this method has been thoroughly studied in [3]. In fact, it has been shown that this process does not lead to a deterioration in the convergence rate over polynomial subtraction. In particular, the uniform error $\|f - f_{N,m}\|_\infty$ remains

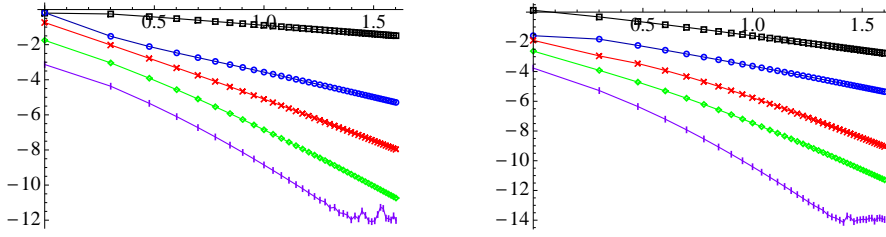


Fig. 5.2 Error in Eckhoff's method applied to $f(x) = e^x \cos 4x$. Left: $\log_{10} \|f - f_{N,m}\|_\infty$ against $\log_{10} N$ for $q = 1$ with $m = 0, 2, 4, 6, 8$ (in descending order). Right: the error $\log_{10} \|f - f_{N,m}\|_\infty$ for $q = 2$ with $m = 0, 1, 4, 7, 8$.

$\mathcal{O}(N^{-(2k+1)q-p})$. Thus, exact derivatives are not necessary to obtain faster convergence of polyharmonic–Neumann expansions.

The main drawback of this device is that the linear system to be solved is extremely ill-conditioned. Nonetheless, as discussed in [3], there are a number of ways to mitigate this effect. First, we replace the linear system (5.8) with an overdetermined least squares problem. Second, instead of forming g as a linear combination of the polynomials \mathcal{E}_r , we employ a set consisting of, for example, Chebyshev or Legendre polynomials (nonpolynomial choices, such as trigonometric functions, may also be used [3]). In Figure 5.2 we give numerical results for Eckhoff's method applied to the function $f(x) = e^x \cos 4x$. Upon comparison with Figure 5.1, we notice that the ill-conditioning has little effect on the resultant approximation. Furthermore, as previously commented, there is no deterioration in the convergence rate.

Whilst Figure 5.2 confirms that the approximation $f_{N,m}$ performs as expected, we will not provide any analysis of Eckhoff's method in this setting. Instead, we detail an approach to completely remove the Gibbs phenomenon (as opposed to ameliorating it to a certain order). As we prove, the resulting method delivers spectral accuracy.

5.3 Least squares methods

An alternative to extrapolation techniques is to augment the approximation space suitably and use a least squares criterion to compute the approximation. Suppose that we consider the system $\mathcal{H} = \{\phi_n : n \in \mathbb{N}_+\} \cup \{\psi_n : n \in \mathbb{N}_+\}$ consisting of both polyharmonic–Dirichlet and polyharmonic–Neumann eigenfunctions, which we denote by ϕ_n and ψ_n respectively (for simplicity, from this point onwards we will not denote the polyharmonic–Neumann eigenfunctions $\phi_{0,n}$ corresponding to the zero eigenvalue separately, we just write $\{\phi_n\}_{n=1}^\infty$, with the understanding that this is the set of all eigenfunctions, with some suitable enumeration). Let \mathcal{H}_N be the finite subset $\{\phi_n : n = 1, \dots, N\} \cup \{\psi_n : n = 1, \dots, N\}$. We seek an approximation

$$\tilde{f}_N(x) = \sum_{n=1}^N a_n \theta_n(x) \in \text{span} \mathcal{H}_N,$$

defined by the least squares criterion

$$\tilde{f}_N = \arg \min_{g \in \mathcal{H}_N} \|f - g\|, \quad (5.9)$$

where $\theta_{2n} = \phi_n$ and $\theta_{2n-1} = \psi_n$. Note that this is equivalent to the condition

$$\int_{-1}^1 \tilde{f}_N(x) \overline{\theta_n(x)} dx = f_n = \int_{-1}^1 f(x) \overline{\theta_n(x)} dx, \quad n = 1, \dots, 2N, \quad (5.10)$$

which results in a linear system $Ax = y$, where $x = (a_1, \dots, a_{2N})^\top$, $y = (f_1, \dots, f_{2N})^\top$ and $A \in \mathbb{R}^{2N \times 2N}$ has (n, m) th entry $\int_{-1}^1 \theta_n(x) \overline{\theta_m(x)} dx$. Observe that $f_{2n} = \hat{f}_n$, the polyharmonic–Neumann coefficient of f , whereas $f_{2n-1} = \check{f}_n$ is the polyharmonic–Dirichlet coefficient of f .

As with Eckhoff’s method, ill-conditioning also occurs with this approach. Hence we typically overdetermine the problem in practice. This corresponds to replacing the square matrix A with an augmented $2M \times 2N$ matrix and the vector y with a vector of length $2M$ (here $M \geq N$). This issue aside, however, we can now prove spectral convergence of the approximation \tilde{f}_N , and thus confirm the complete removal of the Gibbs phenomenon by this approach. We have

Theorem 5.1 *The approximation \tilde{f}_N converges spectrally fast to f . In particular, $\|f - \tilde{f}_N\| \leq c_k(f) N^{-(2k+1)q}$, $\forall k \in \mathbb{N}$, for some positive constant $c_k(f)$ depending only on f and k .*

Proof Since \tilde{f}_N is defined by (5.9), we have

$$\|f - \tilde{f}_N\| \leq \|f - h_N\|, \quad \forall h_N \in \text{span} \mathcal{H}_N. \quad (5.11)$$

Therefore, to prove convergence of \tilde{f}_N to f at a rate of $N^{-(2k+1)q}$ it suffices to find a function $h_N \in \text{span} \mathcal{H}_N$ for which $\|f - h_N\| \leq cN^{-(2k+1)q}$. To do so, let $N > 2kq$. Suppose that we can find a function $\psi \in \text{span}\{\psi_1, \dots, \psi_N\}$ such that

$$\psi^{((2r+1)q+s)}(\pm 1) = f^{((2r+1)q+s)}(\pm 1), \quad r = 0, \dots, k-1, \quad s = 0, \dots, q-1.$$

Note that $\psi^{(2rq+s)}(\pm 1) = 0$ for all r, s , since ψ is a sum of polyharmonic–Dirichlet eigenfunctions. However, it is not necessarily the case that $\psi^{((2r+1)q+s)}(\pm 1) = 0$, thus this interpolation problem at least makes sense. Assuming that such a ψ exists (a nontrivial assumption, which requires proof—see below), we now define $h_N = f_N - \psi_N + \psi$, where f_N and ψ_N are the expansions of f and ψ in polyharmonic–Neumann eigenfunctions respectively (note that $h_N \in \text{span} \mathcal{H}_N$). To show that h_N converges to f at a rate of $N^{-(2k+1)q}$ we note that $f - h_N = (f - \psi) - (f - \psi)_N$ and $(f - \psi)^{(l)}(\pm 1) = 0$ for $l = (2r+1)q + s$, $r = 0, \dots, k-1$ and $s = 0, \dots, q-1$. Thus, this rate of convergence is guaranteed by the arguments of Section 5.1 (indeed, h_N can be viewed as a polynomial subtraction approximation to f , but using a nonpolynomial interpolating function ψ consisting of polyharmonic–Dirichlet eigenfunctions).

Hence, to complete the proof we wish to show that it is always possible to find such a function ψ . Suppose that $M > 0$ and that $2kq + 2M \leq N$. Set

$$\psi(x) = \sum_{n=2M}^{2kq+2M-1} a_n \psi_n(x). \quad (5.12)$$

We claim that, for sufficiently large M , it is always possible to find a function ψ of this form satisfying $\psi^{((2r+1)q+s)}(\pm 1) = c_{rq+s}^\pm$, $r = 0, \dots, k-1$, $s = 0, \dots, q-1$, for arbitrary constants c_{rq+s}^\pm .

To establish this claim, recall the asymptotics for polyharmonic–Dirichlet eigenfunctions (see Section 4.2). In particular, we have

$$\psi_n^{(r)}(\pm 1) = \alpha_n^r \tilde{d}_r(\pm 1)^{n+r} + \mathcal{O}(n^r e^{-n\pi\gamma_q}).$$

Hence we seek a_n such that, for $r = 0, \dots, k-1$ and $s = 0, \dots, q-1$,

$$\sum_{n=2M}^{2kq+2M-1} a_n \left[\alpha_n^{(2r+1)q+s} (\pm 1)^n + E_{rq+s,n}^\pm \right] = \frac{(\pm 1)^r}{\tilde{d}_{2rq+s}} c_{2rq+s}^\pm,$$

where $E_{rq+s,n}^\pm$ is of magnitude $n^{(2r+1)q+s} e^{-n\pi\gamma_q}$. After separating terms corresponding to $(\pm 1)^n$, we find that

$$\begin{aligned} \sum_{n=M}^{kq+M-1} a_{2n} \left[\alpha_{2n}^{(2r+1)q+s} + E_{rq+s,2n} \right] &= C_{rq+s}, \\ \sum_{n=M}^{kq+M-1} a_{2n+1} \left[\alpha_{2n+1}^{(2r+1)q+s} + E_{rq+s,2n+1} \right] &= D_{rq+s}, \end{aligned}$$

for arbitrary values C_{2rq+s} and D_{2rq+s} , where $E_{rq+s,n} = \mathcal{O}(n^{(2r+1)q+s} e^{-n\pi\gamma_q})$. Consider the first system of equations. The claim is now seen to hold, provided the matrix with entries $\alpha_{2n}^{(2r+1)q+s}$ is nonsingular and has condition number growing only algebraically with M . Since $\alpha_n = \mathcal{O}(n)$, it is trivial to see that the condition number must be only at worst algebraically large in M . Hence, we need only show that this matrix is nonsingular.

Consider the transpose of this matrix. Seeking a contradiction, we assume that

$$\sum_{r=0}^{k-1} \sum_{s=0}^{q-1} b_{rq+s} \alpha_{2(n+M)}^{2rq+s} = 0, \quad n = 0, \dots, kq-1.$$

Let $P(x)$ be the polynomial $\sum_{r=0}^{k-1} \sum_{s=0}^{q-1} b_{rq+s} x^{2rq+s}$, so that $P(\alpha_{2(n+M)}) = 0$ for $n = 0, \dots, kq-1$. We claim that P must be identically zero.

To establish this claim, we use induction on k . For $k = 1$, $P(x) = \sum_{s=0}^{q-1} b_s x^s$, and the result follows immediately. Now assume that the result holds up to and including k . Define P as above, with k replaced by $k+1$, and assume that $P(x)$ vanishes at $x = \alpha_{2(n+M)}$, $n = 0, \dots, (k+1)q-1$. A simple argument concludes that the q^{th} derivative $P^{(q)}$ has at least kq simple zeros in the region $[\alpha_{2M}, \infty)$. However,

$$P^{(q)}(x) = x^q \sum_{r=0}^{k-1} \sum_{s=0}^{q-1} \tilde{b}_{rq+s} x^{2rq+s} = x^q Q(x),$$

for some constants \tilde{b}_{rq+s} . It follows that the Q must have at least kq simple zeros in $[\alpha_{2M}, \infty)$. However $Q \equiv 0$ by induction, and thus $P \equiv 0$, therefore completing the proof. \square

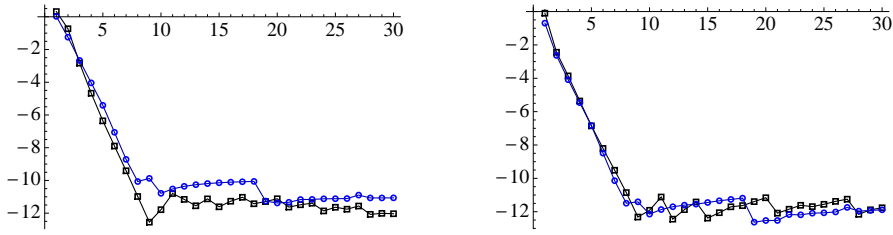


Fig. 5.3 Log error $\log_{10} \|f - \tilde{f}_N\|_\infty$ against $N = 1, \dots, 30$ for $q = 1$ (squares) and $q = 2$ (circles), where $f(x) = e^x \cos 4x$ (left) and $f(x) = x \cos(x+1)$ (right).

In Figure 5.3 we present numerical results for this method in the cases $q = 1$ and $q = 2$. As predicted, spectral convergence occurs. Indeed, these examples indicate that the approximation actually converges exponentially fast; an observation which, as we next discuss, has been confirmed in the $q = 1$ case.

This method can be viewed as a generalisation of the so-called *Fourier extension* method [10, 22] to arbitrary $q \geq 1$. Indeed, the $q = 1$ case corresponds precisely to this method. As the name suggests, the Fourier extension method is intimately related to Fourier series. In fact, the approximation \tilde{f}_N , being of the form

$$\tilde{f}_N(x) = a_0 + \sum_{n=1}^{2N} \left[a_n \cos \frac{1}{2} n \pi x + b_n \sin \frac{1}{2} n \pi x \right], \quad (5.13)$$

is readily identified as a truncated Fourier series on the extended domain $[-2, 2]$. Thus, this procedure numerically computes a smooth periodic extension of the original function f on $[-2, 2]$ via the least squares criterion (5.9) (alternative, potentially more effective, ways in which to do this are also discussed in [22], yet the least squares approach remains the most standard). In light of standard approximation properties of Fourier series of periodic functions, spectral convergence is expected.

The Fourier extension method has been thoroughly analysed in [22] by relating the solution \tilde{f}_N computed via the least squares procedure (5.9) to a certain orthogonal polynomial expansion in the variables $\cos \frac{1}{2} \pi x$ and $\sin \frac{1}{2} \pi x$. The principal result confirms exponential convergence in N (for analytic functions f) at a rate of E^{-N} , where $E \approx 5.828$. Unfortunately, when $q \geq 2$ the analogy with Fourier series is lost. However, we are still able to verify spectral convergence in this case (Theorem 5.1), and therefore the removal of the Gibbs phenomenon.

Note that this approach requires both the coefficients \hat{f}_n and \check{f}_n to be known (or computed) explicitly. However, a relatively minor adjustment can be made to tackle the case where only the polyharmonic–Neumann coefficients \hat{f}_n are given. In this setting, we enforce the conditions

$$\int_{-1}^1 \tilde{f}_N(x) \overline{\phi_n(x)} dx = \int_{-1}^1 f(x) \overline{\phi_n(x)} dx = \hat{f}_n, \quad n = 1, \dots, 2N,$$

instead of of (5.10). Note that the resultant approximation is no longer the solution of the least squares problem (5.9). Nonetheless, although we shall not prove it, this scheme also converges spectrally fast.

5.4 Piecewise smooth functions

The previous three techniques were all designed to overcome the weak Gibbs phenomenon occurring in polyharmonic–Neumann expansions at $x = \pm 1$. Now suppose that the function $f(x)$ has jump discontinuities at $-1 < x_1 < \dots < x_k < 1$. Aside from the weak Gibbs phenomenon at $x = \pm 1$, we now also seek to remove the interior phenomenon occurring at x_1, \dots, x_k .

Let us first consider the most simple approach: namely, polynomial subtraction. Once more we appeal to the expression (3.6) for the pointwise error $f(x) - f_N(x)$. Arguing as before, we notice that arbitrarily fast algebraic convergence can be achieved if we replace f_N with $f_{N,\kappa,m} = f_N - g_N + g$, where $\kappa = (\kappa_1, \dots, \kappa_k)$ and the function g satisfies both

$$g^{(l)}(\pm 1) = f^{(l)}(\pm 1), \quad \forall l \in D_m, \quad (5.14)$$

and

$$g^{(l)}(x_j) = f^{(l)}(x_j), \quad \forall l = 0, \dots, \kappa_j, \quad j = 1, \dots, k. \quad (5.15)$$

In this case, the error $\|f - f_{N,\kappa,m}\|_\infty = N^{-K}$, where $K = \min\{\kappa_1, \dots, \kappa_k, (2k' + 1)q + p\}$, where k' and p are such that $m = (2k' + 1)q + p$ ($p \neq 0$) or $m = 2k'q$ ($p = 0$).

To obtain such a function g , we first find a function g_1 satisfying (5.15). The following construction is standard (see [16] for example). Let $\tilde{C}_r(x)$ be the periodic extension of the function $-\frac{2^{r+1}}{(r+1)!}B_{r+1}(x)$ to the real line, where B_{r+1} is $(r+1)$ th Bernoulli polynomial. We now define $C_r(x) = \tilde{C}_r(\frac{1}{2}x + 1)$. It follows that C_r is piecewise smooth with

$$\left[C_r^{(s)} \right] (0) = \delta_{r,s}, \quad r, s \in \mathbb{N}.$$

With this to hand, an appropriate function g_1 is given by

$$g_1(x) = \sum_{r=1}^m \sum_{l=0}^{\kappa_r-1} \left[f^{(l)} \right] (x_r) C_l(x - x_r).$$

Given g_1 , we now construct g_2 . We require that (5.14) holds, i.e. $g_1^{(l)}(\pm 1) = f^{(l)}(\pm 1) - g_2^{(l)}(\pm 1)$, $l \in D_m$. For this, we merely use the polynomials Ξ_r once more.

In Figure 5.4(a) we consider the approximation of the function

$$f(x) = \begin{cases} \cos 4x & -\frac{1}{4} \leq x < \frac{1}{2} \\ -e^x & \text{otherwise} \end{cases} \quad (5.16)$$

via this approach. Note that with $N \approx 30$ and $\kappa_0 = \kappa_1 = 9$, $m = 8$, we obtain approximately 12 digits of accuracy.

Since it is straightforward to extend the extrapolation ideas to this case, thereby avoiding the use of derivatives, we now describe the least-squares procedure in this setting. For this, suppose that $\theta_1, \dots, \theta_{2N}$ are as in Section 5.3. Assume that we restrict each such function θ_j so that $\theta_j \equiv 0$ for $|x| > 1$. We now define the mapping $\gamma_r(x) = 2\frac{x-x_r}{x_{r+1}-x_r} - 1$, and let $\theta_{n,r}(x) = \theta_n(\gamma_r(x))$ for $n = 1, \dots, 2N$ and $r = 0, \dots, m$, where $x_0 = -1$ and $x_{m+1} = 1$. In other words, the functions $\theta_{n,r}$ are local polyharmonic eigenfunctions with support in each interval of smoothness $[x_r, x_{r+1}]$.

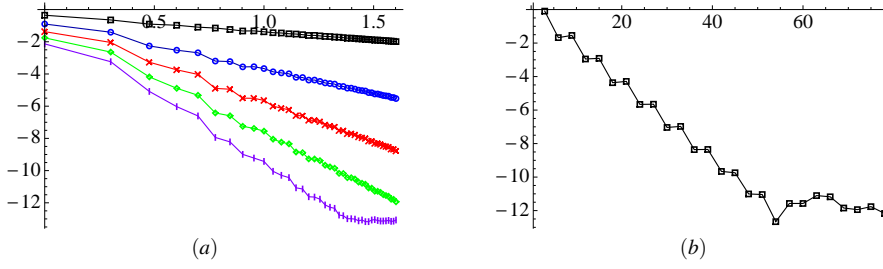


Fig. 5.4 (a) the error $\log_{10} \|f - f_{N, \kappa, m}\|_\infty$ against $\log_{10} N$ for $N = 1, \dots, 40$, where $\kappa_0 = \kappa_1 = m + 1$, $m = 0, 2, 4, 6, 8$ (top to bottom) and $q = 1$. (b) the error $\log_{10} \|f - \tilde{f}_N\|_\infty$ against $N = 1, \dots, 80$, where $N_0 = N_1 = N_2 = \frac{1}{3}N$ and $q = 1$.

With this to hand, we let

$$\tilde{f}_N(x) = \sum_{r=0}^m \sum_{n=1}^{2N_r} a_{n,r} \theta_{n,r}(x), \quad N = \sum_{r=0}^m N_r,$$

where unknowns $a_{n,r}$ are specified by the conditions

$$\int_{-1}^1 \tilde{f}_N(x) \theta_n(x) dx = \int_{-1}^1 f(x) \theta_n(x) dx, \quad n = 1, \dots, 2N.$$

This results in a $2N \times 2N$ linear system. In Figure 5.4(b) we consider the approximation of (5.16) via this method. Once more, exponential convergence is witnessed.

Conclusions

The intent of this paper was to describe the Gibbs phenomenon in polyharmonic–Neumann expansions and consider techniques for its removal. In particular, we have shown that the Gibbs phenomenon is identical at internal singularities to that occurring in standard Fourier series, whereas near the endpoints the corresponding weak phenomenon has a different character. Next, we developed technique for removal of this phenomenon, culminating in a method which delivered spectral accuracy using combinations of polyharmonic–Neumann and polyharmonic–Dirichlet eigenfunctions.

Potential applications of this work are the subject of current investigations. One obvious area is in the numerical solution of fourth and higher-order boundary value problems (the $q = 1$ eigenfunctions have been applied to the solution of second-order problems in [2, 4]). For example, if u is the solution of the biharmonic problem $u^{(4)}(x) + bu(x) = f(x)$, $u(\pm 1) = u'(\pm 1) = 0$, where $b > 0$, then u can be immediately expanded in its biharmonic–Dirichlet series. Indeed, the n^{th} biharmonic–Dirichlet coefficient of u is precisely $\check{u}_n = (b + \alpha_n^4)^{-1} \check{f}_n$. With this observation to hand, we can immediately apply a version of the technique of Section 5.3, for example, to compute an approximation \tilde{u}_N to u . In Table 5.1 we provide numerical results for the example with $f(x) = 4 + x$ and $b = 1$. Using only $N = 8$ we obtain an error of order 10^{-14} .

N	1	2	4	8	16
$\ u - \tilde{u}_N\ _\infty$	1.56×10^{-3}	3.70×10^{-5}	2.08×10^{-8}	1.68×10^{-14}	9.85×10^{-15}

Table 5.1 Error in approximating u , where $u^{(4)}(x) + u(x) = 4 + x$, $u(\pm 1) = u'(\pm 1) = 0$, by \tilde{u}_N .

Encouraged by this particular example, future work will address the application of the approach to a broader variety of problems.

Nonetheless, it seems preferable to use a small value of q , unless the particular problem at hand lends itself naturally to a specific value (e.g. solving the aforementioned boundary value problem). As discussed in [5, 6], complications arise for larger q (computational cost and round-off error). In addition, the examples in Figures 5.1–5.3 indicate that there is little or no advantage gained in general from larger values of q . However, even when $q = 1$ there remain a number open problems. In particular, all known techniques to remove the Gibbs phenomenon from Fourier (or Fourier-like) series suffer from ill-conditioning. A theoretical justification of this observation has been established in [33]: any exponentially convergent scheme based on Fourier data must possess exponentially poor conditioning. However, there may be ways to circumvent this issue if the condition of exponential convergence was sufficiently relaxed.

Outside the issue its removal, it is of independent theoretical interest that the Gibbs phenomenon can be so accurately described in both this and many other instances (see Section 1). Even when there is no obvious connection to Fourier series, we still observe a similar phenomenon. A natural question to ask is whether the work of this paper can be generalised to even larger families of eigenfunction expansions. For example, polyharmonic–Neumann expansions can be viewed as a particular type of so-called *Birkhoff series* [6, 9, 31]. Despite typically lacking the exponential asymptotics of the polyharmonic case [5], it may be possible to extend the results of this paper to this setting.

A final topic for future investigation involves the q -Lidstone polynomials introduced in Section 5.1. Standard Lidstone polynomials (and, more generally, Lidstone series) have been extensively studied [7]. Much of this work exploits the close relation between such polynomials and expansions in Laplace–Dirichlet eigenfunctions. It may be possible to generalise this theory to q -Lidstone polynomials with the aid of polyharmonic–Dirichlet expansions.

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