Accuracy of the Fourier extension method for oscillatory phenomena

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Outline

Introduction

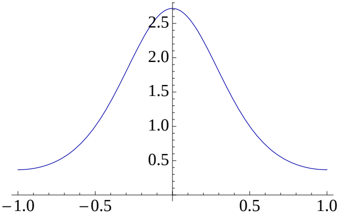
Fourier extensions

Resolution power

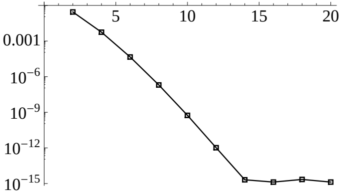
Fourier series

Fourier series are an extremely powerful tool for approximating periodic functions in intervals and (hyper)rectangles.

- Easy to compute/manipulate via the FFT.
- Converge spectrally fast for smooth and periodic functions, and exponentially fast for analytic and periodic functions.



$f(x) = \mathrm{e}^{\cos \pi x}$

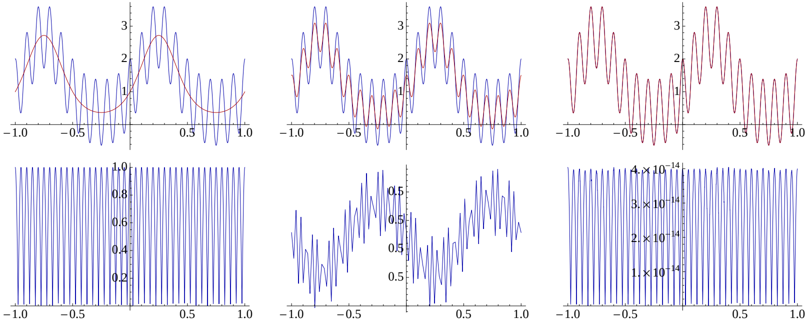


$\|f - f_N\|$

Resolution power

Fourier series are good at resolving periodic oscillations.

- Obtain the optimal resolution constant of 2 d.o.f. per wavelength.



$N = 10$ $N = 20$ $N = 40$

Top: Graphs of $f(x) = \cos 20\pi x + \exp(\sin 2\pi x)$ (blue) and its $2N$-term Fourier series $f_N(x)$ (red). Bottom: the error $f(x) - f_N(x)$.

Conversely, expansions in orthogonal polynomials (e.g. Chebyshev polynomials) have a higher resolution constant equal to π.

Problems

1. Most functions are not periodic.
    - Very poor convergence for nonperiodic functions.
    - Also the Gibbs phenomenon – $\mathcal{O}(1)$ oscillations near the domain boundary.
    - Thus, Fourier series cannot resolve nonperiodic oscillations.

2. Fourier series are limited to simple domains.
    - Typically, intervals, (hyper)rectangles and parallelopipeds.
    - Some extensions to certain triangles and tetrahedra – require somewhat unphysical notions of periodicity.

Question: *is there a way to retain the good properties of Fourier series (in particular, rapid convergence and resolution power) for nonperiodic functions, and functions in general bounded domains?*

- *Can one improve on the resolution power of polynomial expansions?*

Outcomes of the talk

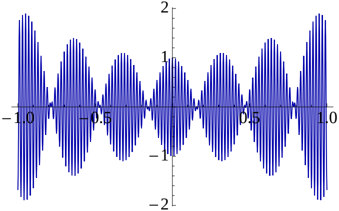
Yes! One can compute approximations of analytic, nonperiodic functions which

1. converge exponentially fast,
2. have a resolution constant that can be made arbitrarily close to 2 by an appropriate (function-independent) choice of a certain parameter,
3. are expressed in terms of Fourier series.

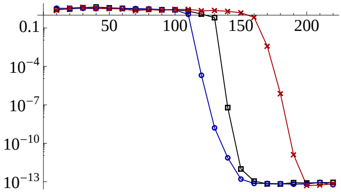
The method is based on computing Fourier extensions.

Numerical example

Consider $f(x) = (1 + x^2)\cos 10x \cos 100\pi x$.



$f(x)$



$\|f - f_N\|$

Here f_N represent the Chebyshev series of f (red), and the Fourier extension (black and blue).

Introduction

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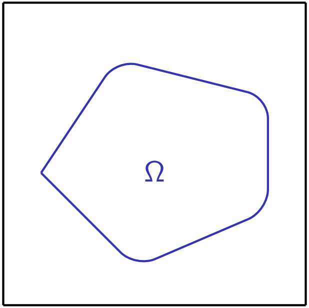
An (old) idea

Seek to approximate a function $f : \Omega \to \mathbb{R}$ by a Fourier series on a larger, (hyper)rectangular domain.



Known as the Fourier extension problem.

The Fourier extension problem

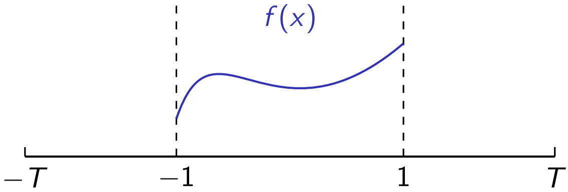
Existence/construction of extensions:

- Whitney (1934), Hestenes (1941), Fefferman (2005),...
- However, typically cannot obtain exponential convergence this way – no analytic and periodic extension of an arbitrary analytic function.
- Throughout, we shall never explicitly calculate extensions.

Computation of extensions:

- Boyd (2002), Bruno (2003), Bruno et al (2007), Bruno & Lyon (2010, 2011), Huybrechs (2010), BA & Huybrechs (2011).
- Obtain exponential convergence, but only in the original domain Ω.

One-dimensional Fourier extensions



Let

$$\mathcal{S}_N = \mathrm{span}\left\{ \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\frac{n\pi}{T}x} : n = -N, \ldots, N \right\}.$$

We seek an approximation $f_N \in \mathcal{S}_N$.

Questions:

1. How do we compute f_N?
2. How fast does $f_N \to f$?
3. What is the resolution power?
4. How do 2. and 3. depend on T?

Warning!

The set

$$\left\{ \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\frac{n\pi}{T}x} : n \in \mathbb{Z} \right\},$$

is not an orthonormal basis for $\mathrm{L}^2(-1,1)$. It is a tight frame.

- Redundancy of the frame leads to severe ill-conditioning.
- Nonetheless, the ill-conditioning is natural – hence it typically does not lead to numerical instability.

However, we warn that the result of any numerical implementation may differ from the proposed approximation f_N.

- Any standard solver will seek to regularise the computation.
- Numerically one always obtains extensions with bounded coefficients – theoretical extensions may well have unbounded coefficients.

Computing f_N

Most straightforward approach (Boyd, Bruno):

$$f_N = \arg\min_{\phi \in \mathcal{S}_N} \|f - \phi\|_{\mathrm{L}^2(-1,1)}.$$

Theorem (Huybrechs (2010), BA & Huybrechs (2011))

Suppose that f is analytic in a sufficiently large region around $[-1,1]$. Then

$$\|f - f_N\|_{\mathrm{L}^\infty(-1,1)} \leq c_f E(T)^{-N},$$

where

$$E(T) = \frac{3 + c(T) + 2\sqrt{2 + 2c(T)}}{1 - c(T)}, \qquad c(T) = \cos\frac{\pi}{T},$$

and c_f depends only on f.

- One can also show spectral/algebraic convergence for $\mathrm{C}^\infty/\mathrm{C}^k$ functions.

Idea behind the proof

Note that f_N consists of the functions

$$\cos \tfrac{k\pi}{T} x, \quad \sin \tfrac{k\pi}{T} x, \qquad k = 0, \ldots, N.$$

If $y = \cos \frac{\pi}{T} x$, then

$$\cos \tfrac{k\pi}{T} x = T_k(y) \in \mathbb{P}_k, \qquad \sin \tfrac{(k+1)\pi}{T} x / \sin \tfrac{\pi}{T} x = U_k(y) \in \mathbb{P}_k.$$

Thus, f_N is a sum of two polynomials expansions of degree N in y, corresponding to the even and odd parts of f respectively.

Standard results on polynomial expansions can now be used.

- The map $y = \cos \frac{\pi}{T} x$ introduces a square-root type singularity at $y = -1$. This gives $E(T)$ for the rate of exponential convergence.

Fourier extensions and polynomial expansions

domain	x	y
interval	$[0, 1]$	$[c(T), 1]$
usage	numerics	analysis
basis	Fourier	polynomial
functions	$f_e(x)$ $f_o(x)$	$f_1(y) = f_e(\cos^{-1} \frac{T}{\pi} y)$ $f_2(y) = f_o(\cos^{-1} \frac{T}{\pi} y)/\sqrt{1-y^2}$
weight functions	1	$1/\sqrt{1-y^2}$ $\sqrt{1-y^2}$

- Note the similarity to Chebyshev polynomials.

Condition number

To implement, we solve a linear system of the form $Aa = b$.

Theorem (BA, Huybrechs (2011))

The condition number $\kappa(A)$ is $E(T)^{2N}$ for large N.

In practice, we usually solve the discrete least squares problem

$$f_N = \arg \min_{\phi \in \mathcal{S}_N} \left\{ \sum_{n=0}^{N} |f(x_n) - \phi(x_n)^2 + |f(-x_n) - \phi(-x_n)|^2 \right\},$$

where

$$x_n = \frac{T}{\pi} \cos^{-1} \left\{ \frac{1}{2}(1 - c(T)) \cos \left[\frac{(2n+1)\pi}{2N+2} \right] + \frac{1}{2}(1 + c(T)) \right\}.$$

- These are the image of the Chebyshev nodes in $y \in [c(T), 1]$ under $y = \cos \frac{\pi}{T} x$.

Leads to a significantly smaller condition number $\kappa(A) \sim E(T)^N$.

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The Main Result

By analysing the behaviour of the Fourier extension of

$$f(x) = \mathrm{e}^{\mathrm{i}\pi\omega x}, \quad x \in [-1, 1],$$

for large $\omega \gg 1$, we obtain:

Theorem (BA, Huybrechs (2011))

The resolution constant $r = r(T)$ satisfies

$$r(T) \leq T\sqrt{2 - 2c(T)}, \quad T > 1.$$

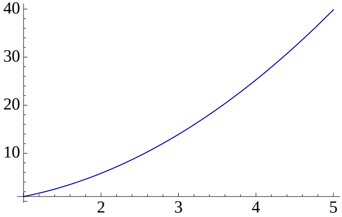
In particular,

$$r(T) \sim 2 + \mathcal{O}\left(T - 1\right), \quad T \to 1,$$
$$r(T) \sim \pi + \mathcal{O}\left(T^{-2}\right), \quad T \to \infty.$$

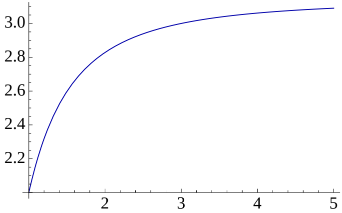
- Thus, the Fourier and polynomial resolution constants are the limiting values for $r(T)$.

Resolution vs convergence

The parameter T can be chosen by the user.



$E(T)$



$r(T)$

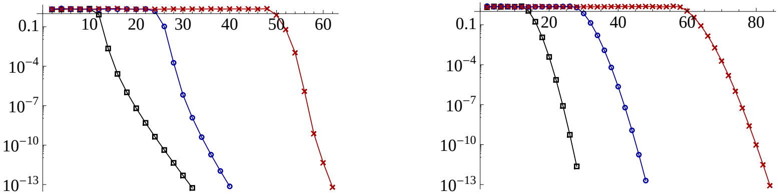
Large T: faster exponential convergence, but worse resolution

Small T: better resolution, but slower exponential convergence.

Varying T: (i.e. $T = 1 + N^{-\alpha}$, $0 < \alpha < 1$) formally optimal resolution, but much slower convergence $\sim \mathrm{e}^{-\pi N^{1-\alpha}}$.

Numerical results

Using (very) high precision one sees:



$T = \sqrt{2}$

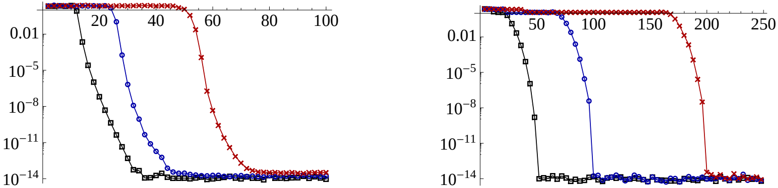
$T = 5$

The error $\|f - f_N\|$ for $f(x) = \mathrm{e}^{\mathrm{i}\pi\omega x}$ and $\omega = 10$ (black), $w = 20$ (blue), $w = 40$ (red).

	$\omega = 10$	$\omega = 20$	$\omega = 40$
$\frac{1}{2}\omega r(T)$ $(T = \sqrt{2})$	12.7	25.3	50.7
$\frac{1}{2}\omega r(T)$ $(T = 5)$	15.5	30.9	61.8

Numerical results

However, in standard precision:



$T = \sqrt{2}$ $T = 5$

The error $\|f - f_N\|$ for $f(x) = \mathrm{e}^{\mathrm{i}\pi\omega x}$ and $\omega = 10$ (black), $w = 20$ (blue), $w = 40$ (red).

- For large T, one sees much worse resolution in finite precision.

Resolution power of the numerical method

Numerical results suggest that the numerical approximation, computed in finite precision, has resolution constant

$$r(T) = 2T, \quad \forall T > 1.$$

- The theorem is not wrong per se: one just never sees the true solution in finite precision.

Indeed, recall that we solve $Aa = b$, where A is the Gram matrix for

$$\left\{ \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\frac{n\pi}{T}x} : n = -N, \dots, N \right\}.$$

- For large N, the columns of A become nearly linearly dependent.
- Thus $Aa = b$ is approximately undetermined.
- Any standard solver will therefore seek a solution a with small norm.

Coefficients of a

Lemma (BA, Huybrechs (2011))

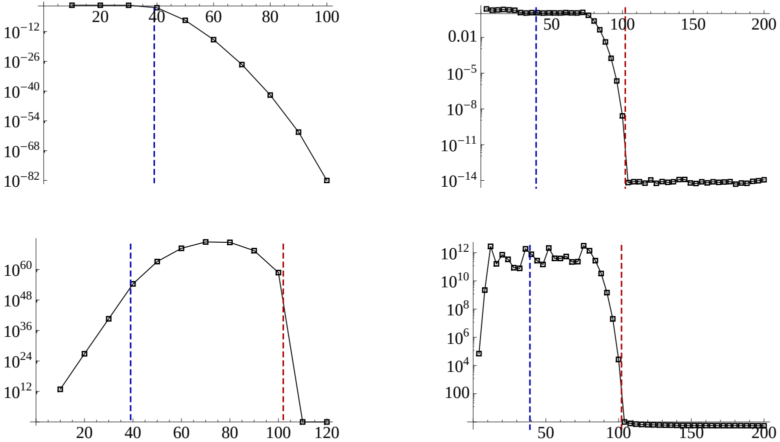
Let a be the exact solution to $Aa = b$, where $f(x) = \exp(\mathrm{i}\pi\omega x)$. Then, for $N < \frac{1}{2}r(T)\omega$, we have

$$\|a\|_{l^2} \sim E(T)^N.$$

- The exact solution has large coefficients in the unresolved regime – in particular, when T is large.
- Hence one cannot expect to obtain the exact solution in finite precision.

Numerical example

Consider $f(x) = \cos \frac{51}{2}\pi x$ and $T = 4$ (i.e. $E(T) = 25.3$).



theoretical method

numerical method

Top: the error $\|f - f_N\|$. Bottom: the coefficient norm $\|a\|$.

Existence of bounded approximate solutions

Nonetheless, there are infinitely many candidate a's with bounded coefficients:

Theorem (BA, Huybrechs (2011))

Let $k \in \mathbb{N}$ be given. Then there exist $a^{[N]}$, $N = 1, 2, \ldots$ such that

$$\|a^{[N]}\|_{l^2} \leq c_k \omega^k, \quad N = 1, 2, \ldots .$$

and

$$\|Aa^{[N]} - b\|_{l^2} \leq c'_k \left(\frac{\omega T}{N} \right)^k, \quad N = 1, 2, \ldots, \qquad (\star)$$

for constants c_k, c'_k independent of N and ω.

- Note that $(\star)$ implies a resolution constant $r(T) = 2T$.

Conclusions

1. Fourier extensions successfully recover both the rapid convergence of Fourier series and their good resolution properties.

2. There is a full theory based on interpreting the Fourier extension as a polynomial expansion in the transformed variable y.

3. However, care is necessary: due to the ill-conditioning, theoretical results need not be witnessed in finite precision.

Future work

1. Is the method stable?
    - The method is theoretically unstable (condition number is $E(T)^N$).
    - Empirical results suggest that the numerical method is stable.
    - Thus ill-conditioning regularises a theoretically unstable method to give a stable numerical method.

2. Breaking the ‘impossibility’ barrier.
    - Platte, Trefethen & Kuijlaars (2011) proved that no method based on equispaced grid values could be exponentially convergent and numerically stable.
    - Fourier extensions appear to contradict this result.

3. Extensions to higher dimensions.
    - Arbitrary domains – extend to rectangular regions. Can be very expensive.
    - Triangles, tetrahedra – close connection to Lie groups and representation theory.