THE MAGNUS METHOD FOR SOLVING OSCILLATORY LIE-TYPE ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

This work presents a new method for solving *Lie*-type equations $X'_\omega = A_\omega(t)X_\omega$, $X_\omega(0) = I$. The solution to the equations of this type have the following representation $X_\omega = \exp(\Omega)$. Here Ω stands for an infinite *Magnus* series of integral commutators. We assume that the matrix A_ω has large imaginary spectrum and ω is a real parameter describing the frequency of oscillation. The novel method, called the *FM* method, combines the *Magnus* method, which is based on iterative techniques to approximate Ω, and the *Filon*-type method, an efficient quadrature rule for solving oscillatory integrals. We show that the *FM* method has superior performance compared to the classical *Magnus* method when applied to oscillatory differential equations.

1 Introduction

We proceed with the matrix ordinary differential equation

$$X'_\omega = A_\omega X_\omega, \qquad X_\omega(0) = I, \qquad X_\omega = \mathrm{e}^{\Omega}, \tag{1}$$

where Ω represents the *Magnus* expansion, an infinite recursive series,

$$\begin{aligned}
\Omega(t) \;=\; & \int_0^t A_\omega(t)dt \\
& +\frac{1}{2}\int_0^t [A_\omega(\tau), \int_0^\tau A_\omega(\xi)d\xi]d\tau \\
& +\frac{1}{4}\int_0^t [A_\omega(\tau), \int_0^\tau [A_\omega(\xi), \int_0^\xi A_\omega(\zeta)d\zeta]d\xi]d\tau \\
& +\frac{1}{12}\int_0^t [[A_\omega(\tau), \int_0^\tau A_\omega(\xi)d\xi], \int_0^\tau A_\omega(\zeta)d\zeta]d\tau \\
& +\cdots .
\end{aligned}$$

We make the following assumptions: $A_\omega(t)$ is a smooth matrix-valued function, the spectrum $\sigma(A_\omega)$ of the matrix A_ω has large imaginary eigenvalues and that $\omega \gg 1$ is a real parameter describing frequency of oscillation.

It has been shown by *Hausdorff*, [Hau06], that the solution of the linear matrix equation in (1) is a matrix exponential $X_\omega(t) = \mathrm{e}^{\Omega(t)}$, where $\Omega(t)$ satisfies the following nonlinear differential equation,

$$\Omega' = \sum_{j=0}^{\infty} \frac{B_j}{j!}\mathrm{ad}_\Omega^j(A_\omega) = \mathrm{dexp}_\Omega^{-1}(A_\omega(t)), \quad \Omega_0 = 0. \tag{2}$$

Here B_j are Bernoulli numbers and the adjoint operator is defined as

$$\begin{aligned}
\mathrm{ad}_\Omega^0(A_\omega) &= A_\omega \\
\mathrm{ad}_\Omega(A_\omega) &= [\Omega, A_\omega] = \Omega A_\omega - A_\omega \Omega \\
\mathrm{ad}_\Omega^{j+1}(A_\omega) &= [\Omega, \mathrm{ad}_\Omega^j A_\omega].
\end{aligned}$$

Later it was observed by *W. Magnus*, [Mag54], that solving equation (2) with Picard iteration it is possible to obtain an infinite recursive series for Ω and then the truncated expansion can be used in approximation of Ω.

Magnus methods preserve peculiar geometric properties of the solution, ensuring that if X_ω is in matricial Lie group G and A_ω is in associated Lie algebra g of G, then the numerical solution after discretization stays on the manifold. Moreover, *Magnus* methods preserve time-symmetric properties of the solution after discretization, for any $a > b$, $X_\omega(a,b)^{-1} = X_\omega(b,a)$ yields $\Omega(a,b) = -\Omega(b,a)$. These properties appear to be valuable in applications.

Employing the *Magnus* method to solve highly oscillatory differential equations we develop it in combination with appropriate quadrature rules, such as

Filon-type methods, proved to be more efficient for oscillatory equations than for example classical *Gaussian* quadrature. Below we provide a brief background on the *Filon*-type method and the *asymptotic* method. The *asymptotic* method provides theoretical background for the *Filon*-type method. At this point of discussion we find it appropriate to state the two important theorems from [IN05], describing the *asymptotic* and the *Filon*-type methods used to approximate highly oscillatory integrals of the form

$$I[f] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x. \tag{3}$$

Suppose $f,g \in C^\infty$ are smooth, g is strictly monotone in $[a,b], a \le x \le b$ and the frequency is $\omega \gg 1$.

LEMMA 1.1 *[A. Iserles & S. Nørsett][IN05] Let* $f,g \in C^\infty$, $g'(x) \neq 0$ *on* $[a,b]$ *and*

$$\begin{aligned} \sigma_0[f](x) &= f(x), \\ \sigma_{k+1}[f](x) &= \frac{d}{dx}\frac{\sigma_k[f](x)}{g'(x)}, \qquad k = 0,1,\ldots. \end{aligned}$$

Then, for $\omega \gg 1$,

$$I[f] \sim -\sum_{m=1}^{\infty} \frac{1}{(-iw)^m}\left[\frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{g'(1)}\sigma_{m-1}[f](1) - \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)}\sigma_{m-1}[f](0)\right].$$

The *asymptotic* method is defined as follows,

$$Q_s^A[f] = -\sum_{m=1}^{s} \frac{1}{(-i\omega)^m}\left[\frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{g'(1)}\sigma_{m-1}[f](1) - \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)}\sigma_{m-1}[f](0)\right].$$

THEOREM 1.2 *[A. Iserles & S. Nørsett][IN05] For every smooth* f *and* g*, such that* $g'(x) \neq 0$ *on* $[a,b]$*, it is true that*

$$Q_s^A[f] - I[f] \sim \mathscr{O}(\omega^{-s-1}), \qquad \omega \to \infty.$$

The *Filon*-type method was first pioneered in the work of L. N. G. Filon in 1928 and later developed by A. Iserles and S. Nørsett in 2004, [IN05]. The method works as follows. First interpolate function f in (3) at chosen node points. For instance, take Hermite interpolation

$$\tilde{v}(x) = \sum_{l=1}^{\nu} \sum_{j=0}^{\theta_l - 1} \alpha_{l,j}(x) f^{(j)}(c_l), \tag{4}$$

which satisfies $\tilde{v}^{(j)}(c_l) = f^{(j)}(c_l)$, at node points $a = c_1 < c_2 < \ldots < c_\nu = b$, with $\theta_1, \theta_2, \ldots, \theta_\nu \geq 1$ associated multiplicities, $j = 0, 1, \ldots, \theta_l - 1, l = 1, 2, \ldots, \nu$ and $r = \sum_{l=1}^{\nu} \theta_l - 1$ being the order of approximation polynomial. Then, for the *Filon*-type method, by definition,

$$Q_s^F[f] = I[\tilde{v}] = \int_a^b \tilde{v}(x) \mathrm{e}^{i\omega g(x)} \mathrm{d}x = \sum_{l=1}^{\nu} \sum_{j=0}^{\theta_l - 1} b_{l,j}(w) f^{(j)}(c_l),$$

where

$$b_{l,j} = \int_0^1 \alpha_{l,j}(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x, \qquad j = 0, 1, \ldots, \theta_l - 1, \qquad l = 1, 2, \ldots, \nu.$$

THEOREM 1.3 *[A. Iserles & S. Nørsett][IN05] Suppose that $\theta_1, \theta_\nu \geq s$. For every smooth f and g, $g'(x) \neq 0$ on $[a, b]$, it is true that*

$$I[f] - Q_s^F[f] \sim \mathscr{O}(\omega^{-s-1}), \quad \omega \to \infty.$$

The *Filon*-type method has the same asymptotic order as the *asymptotic* method. For formal proof of the asymptotic order of the *Filon*-type method for univariate integrals, we refer the reader to [IN05] and for matrix and vector-valued functions we refer to [Kha08]. The following theorem from [Kha08] is on the numerical order of the *Filon*-type method. It was also shown in [Kha08] that the numerical solution obtained after discretisation of the integral with *Filon*-type method is convergent.

THEOREM 1.4 *[MK][Kha08] Let $\theta_1, \theta_\nu \geq s$. Then the numerical order of the Filon-type method is equal to $r = \sum_{l=1}^{\nu} \theta_l - 1$.*

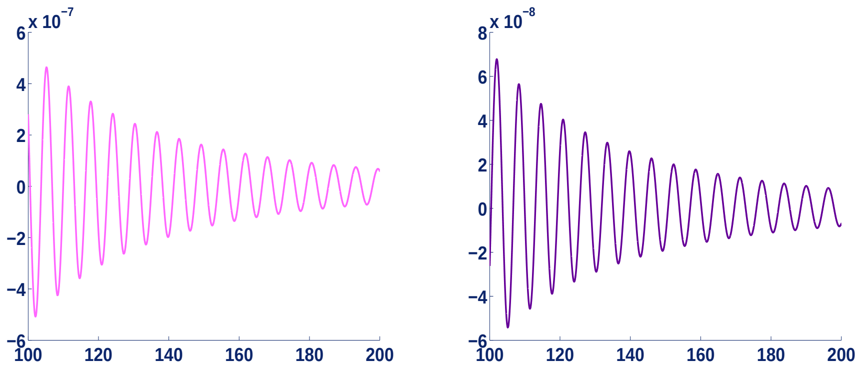

Figure 1: The error of the *asymptotic* method Q_2^A (right) and the *Filon*-type method Q_2^F (left) for $f(x) = \cos(x)$, $g(x) = x$, $\theta_1 = \theta_2 = 2$, $100 \le \omega \le 200$.

Example 1.1 Here we consider the *asymptotic* and the *Filon*-type methods with first derivatives for (3) over the interval $[0,1]$ for the case $g(x) = x$.

$$
\begin{aligned}
Q_2^A &= \frac{e^{i\omega} f(1) - f(0)}{i\omega} + \frac{e^{i\omega} f'(1) - f'(0)}{\omega^2}, \\
Q_2^F &= \left(-\frac{1}{i\omega} - 6\frac{1+e^{i\omega}}{i\omega^3} + 12\frac{1-e^{i\omega}}{\omega^4} \right) f(0) \\
&\quad + \left(-\frac{e^{i\omega}}{i\omega} + 6\frac{1+e^{i\omega}}{i\omega^3} - 12\frac{1-e^{i\omega}}{\omega^4} \right) f(1) \\
&\quad + \left(-\frac{1}{\omega^2} - 2\frac{2+e^{i\omega}}{i\omega^3} + 6\frac{1-e^{i\omega}}{\omega^4} \right) f'(0) \\
&\quad + \left(\frac{e^{i\omega}}{\omega^2} - 2\frac{1+e^{i\omega}}{i\omega^3} + 6\frac{1-e^{i\omega}}{\omega^4} \right) f'(1).
\end{aligned}
$$

In Figure 1 we present numerical results on the *asymptotic* and *Filon*-type methods, with function values and first derivatives at the end points only, $c_1 = 0, c_2 = 1$, for the integral

$$I[f] = \int_0^1 \cos(x) e^{i\omega x} dx, \qquad 100 \le \omega \le 200.$$

Both methods have the same asymptotic order and use exactly the same information. However, as we can see from Figure 1, the *Filon*-type method yields a greater measure of accuracy than the *asymptotic* method. Adding more internal interpolation points leads to the decay of the leading error constant, resulting in a marked improvement in the accuracy of approximation. However, interpolating function f at internal points does not contribute to the higher asymptotic order of the *Filon*-type method.

2 The Magnus method

In this section we focus on *Magnus* methods for approximation of a matrix-valued function X_ω in (1). There is a large list of publications available on the *Lie*-group methods, here we refer to some of them: [BCR00], [CG93], [Ise09], [Ise02a], [MK98], [BO97], [Zan96].

Currently, the most general theorem on the existence and convergence of the *Magnus* series is proved for a bounded linear operator $A(t)$ in a Hilbert space, [Cas07]. The following theorem gives us sufficient condition for convergence of the *Magnus* series, extending Theorem 3 from [MN08], where the same results are stated for real matrices.

THEOREM 2.1 *(F. Casas, [Cas07]) Consider the differential equation $X' = A(t)X$ defined in a Hilbert space $\mathscr{H}$ with $X(0) = I$, and let $A(t)$ be a bounded linear operator on $\mathscr{H}$. Then, the Magnus series $\Omega(t)$ in (2) converges in the interval $t \in [0,T)$ such that*

$$\int_0^T \|A(\tau)\| \mathrm{d}\tau < \pi$$

and the sum $\Omega(t)$ satisfies $\exp \Omega(t) = X(t)$.

Given the representation for $\Omega(t)$ in (2), the numerical task on evaluating the commutator brackets is fairly simple, [BCR00], [Ise09], [Ise02a]. For example, choosing symmetric grid $c_1, c_2, \ldots, c_\nu$, suppose taking Gaussian points with respect to $\frac{1}{2}$, consider set $\{A_1, A_2, \ldots, A_\nu\}$, with $A_k = hA(t_0 + c_k h), k = 1, 2, \ldots, \nu$. Linear combinations of this basis form an adjoint basis $\{B_1, B_2, \ldots, B_\nu\}$, with

$$A_k = \sum_{l=1}^{\nu} (c_k - \frac{1}{2})^{l-1} B_l, \quad k = 1, 2, \ldots, \nu.$$

In this basis the six-order method, with Gaussian points $c_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}, c_2 = \frac{1}{2}, c_3 = \frac{1}{2} + \frac{\sqrt{15}}{10}$, $A_k = hA(t_0 + c_k h)$, is

$$\begin{aligned}\Omega(t_0+h) &\approx B_1 + \frac{1}{12}B_3 - \frac{1}{12}[B_1,B_2] + \frac{1}{240}[B_2,B_3] \\ &+\frac{1}{360}[B_1,[B_1,B_3]] - \frac{1}{240}[B_2,[B_1,B_2]] + \frac{1}{720}[B_1[B_1,[B_1,B_2]]],\end{aligned}$$

where

$$B_1 = A_2, \quad B_2 = \frac{\sqrt{15}}{3}(A_3 - A_1), \quad B_3 = \frac{10}{3}(A_3 - 2A_2 + A_1).$$

This can be reduced further and written in a more compact manner [Ise02a], [Ise09],

$$\Omega(t_0+h) \approx B_1 + \frac{1}{12}B_3 + P_1 + P_2 + P_3,$$

where

$$\begin{aligned}P_1 &= [B_2, \frac{1}{12}B_1 + \frac{1}{240}B_3], \\ P_2 &= [B_1,[B_1, \frac{1}{360}B_3 - \frac{1}{60}P_1]], \\ P_3 &= \frac{1}{20}[B_2,P_1].\end{aligned}$$

A more profound approach taking Taylor expansion of $A(t)$ around the point $t_{1/2} = t_0 + \frac{h}{2}$ was introduced in [BCR00],

$$A(t) = \sum_{i=0}^{\infty} a_i (t - t_{1/2})^i, \quad \text{with} \quad a_i = \frac{1}{i!}\frac{\mathrm{d}^n A(t)}{\mathrm{d}t^i}|_{t=t_{1/2}}.$$

This can be substituted in the univariate integrals of the form

$$B^{(i)} = \frac{1}{h^{i+1}} \int_{-h/2}^{h/2} t^i A\left(t + \frac{h}{2}\right) \mathrm{d}t, \quad i = 0,1,2,\ldots$$

to obtain a new basis

$$\begin{aligned}B^{(0)} &= a_0 + \frac{1}{12}h^2 a_2 + \frac{1}{80}h^4 a_4 + + \frac{1}{448}h^6 a_6 \ldots \\ B^{(1)} &= \frac{1}{12}h a_1 + \frac{1}{80}h^3 a_3 + \frac{1}{448}h^5 a_5 + \ldots \\ B^{(2)} &= \frac{1}{12}a_0 + \frac{1}{80}h^2 a_2 + \frac{1}{448}h^4 a_4 + \frac{1}{2304}h^6 a_6 + \ldots\end{aligned}$$

In these terms a second order method will look as follows, $e^{\Omega} = e^{hB^{(0)}} + \mathscr{O}(h^3)$. Whereas for a six-order method, $\Omega = \sum_{i=1}^{4} \tilde{\Omega}_i + \mathscr{O}(h^7)$, one needs to evaluate only four commutators,

$$\begin{aligned}
\tilde{\Omega}_1 &= hB^{(0)} \\
\tilde{\Omega}_2 &= h^2[B^{(1)}, \frac{3}{2}B^{(0)} - 6B^{(2)}] \\
\tilde{\Omega}_3 + \tilde{\Omega}_4 &= h^2[B^{(0)}, [B^{(0)}, \frac{1}{2}hB^{(2)} - \frac{1}{60}\tilde{\Omega}_2]] + \frac{3}{5}h[B^{(1)}, \tilde{\Omega}_2].
\end{aligned}$$

Numerical behavior of the fourth and six order classical *Magnus* method is illustrated in Figures 2, 3 and 4, and 5, 6 and 7, respectively. The method is applied to solve Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $t \in [0,1000]$, for varies step-sizes, $h = \frac{1}{4}$, $h = \frac{1}{10}$ and $h = \frac{1}{25}$. Comparison shows that for a bigger interval steps h both fourth and six order methods give similar results, as illustrated in Figures 2, 3, 5 and 6. However, for smaller steps six order *Magnus* method has a more rapid improvement in approximation compared to a fourth order method, Figures 4 and 7.

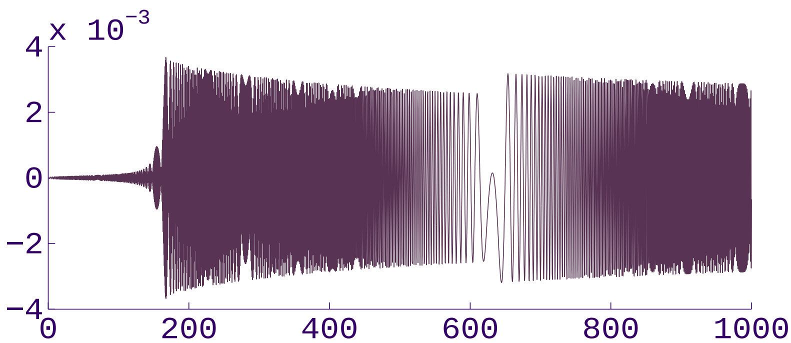

Figure 2: Global error of the fourth order *Magnus* method for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \le t \le 1000$ and step-size $h = \frac{1}{4}$.

In current work we present an alternative method to solve equations of the kind (1). We apply the *Filon* quadrature to evaluate integral commutators for Ω and then solve the matrix exponential X_ω with the *Magnus* method or the *modified Magnus* method. The combination of the *Filon*-type methods and the *Magnus*

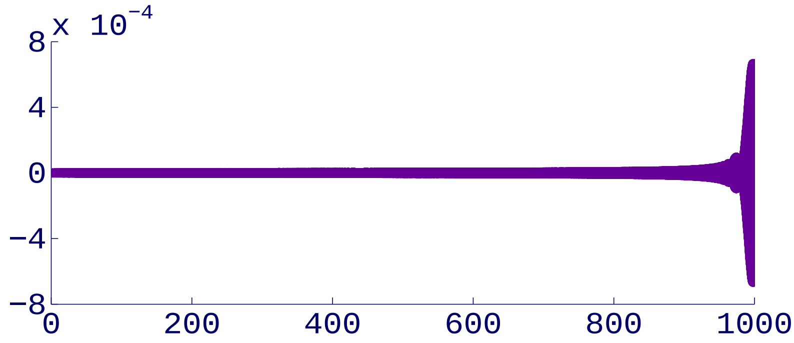

Figure 3: Global error of the fourth order *Magnus* method for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\top$ initial conditions, $0 \le t \le 1000$ and step-size $h = \frac{1}{10}$.

methods forms the *FM* method, presented in the last section. Application of the *FM* method to solve systems of highly oscillatory ordinary differential equation can be found in [Kha09].

3 The modified Magnus method

In this section we continue our discussion on the solution of the matrix differential equation

$$X'_\omega = A_\omega(t)X_\omega, \quad X_\omega(0) = I, \quad X_\omega = \exp(\Omega). \tag{5}$$

It was shown in [Ise02c] that one can achieve better accuracy in approximation of the fundamental matrix X_ω, if one solves (5) at each time step locally for a constant matrix A_ω.

We commence from a linear oscillator studied in [Ise02c],

$$y' = A_\omega(t)y, \quad y(0) = y_0.$$

We also assume that the spectrum of the matrix $A_\omega(t)$ has large imaginary values. Introducing local change of variables at each mesh point, we write the solution in the form,

$$y(t) = \mathrm{e}^{(t-t_n)A_\omega(t_n+\frac{1}{2}h)}x(t-t_n), \qquad t \ge t_n,$$

and observe that

$$x' = B(t)x, \qquad x(0) = y_n,$$

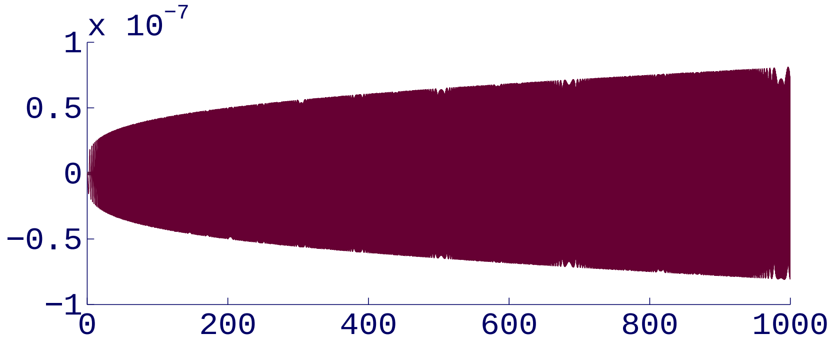

Figure 4: Global error of the fourth order *Magnus* method for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \le t \le 1000$ and step-size $h = \frac{1}{25}$.

with

$$B(t) = \mathrm{e}^{-tA_\omega(t_n+\frac{1}{2}h)}[A_\omega(t_n+t) - A_\omega(t_{n+\frac{1}{2}})]\mathrm{e}^{tA_\omega(t_n+\frac{1}{2}h)}, \quad \text{where} \quad t_{n+\frac{1}{2}} = t_n + \frac{h}{2}.$$

The *modified Magnus* method is defined as a local approximation of the solution vector y by solving

$$x' = B(t)x, \qquad x(0) = y_n,$$

with classical *Magnus* method. This approximation results in the following algorithm,

$$\begin{aligned} y_{n+1} &= \mathrm{e}^{hA_\omega(t_n+h/2)}x_n, \\ x_n &= \mathrm{e}^{\tilde{\Omega}_n}y_n. \end{aligned}$$

Performance of the *modified Magnus* method is better than that of the classical *Magnus* method due to a number of reasons. Firstly, the fact that the matrix B is small, $B(t) = \mathscr{O}(t - t_{n+\frac{1}{2}})$, contributes to higher order correction to the solution. Secondly, the order of the *modified Magnus* method increases from $p = 2s$ to $p = 3s + 1$, [Ise02b].

4 The FM method

Consider the *Airy*-type equation

$$y''(t) + g(t)y(t) = 0, \quad g(t) > 0 \quad \text{for} \quad t > 0, \quad \text{and} \quad \lim_{t\to\infty} g(t) = +\infty.$$

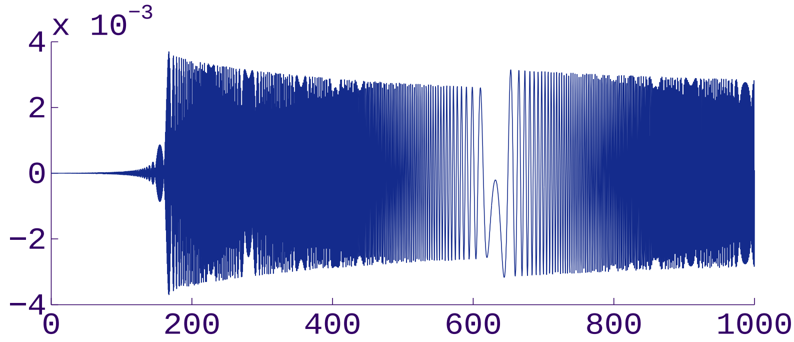

Figure 5: Global error of the six order *Magnus* method for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\top$ initial conditions, $0 \le t \le 1000$ and step-size $h = \frac{1}{4}$.

Replacing the second order differential equation by the first order system, we obtain $y' = A_\omega(t)y$, where

$$A_\omega(t) = \begin{pmatrix} 0 & 1 \\ -g(t) & 0 \end{pmatrix}.$$

Due to a large imaginary spectrum of the matrix A_ω, this *Airy*-type equation is rapidly oscillating. We apply the *modified Magnus* method to solve the system $y' = A_\omega(t)y$,

$$y_{n+1} = \mathrm{e}^{hA_\omega(t_n+h/2)}\mathrm{e}^{\tilde{\Omega}_n}y_n.$$

Here the integral commutators in $\tilde{\Omega}_n$ are computed according to the rules of the *Filon* quadrature. This results in highly accurate numerical method, the *FM* method. Taking into account that the entries of the matrix $B(t)$ are likely to be oscillatory, the advantage of the *Filon*-type method is evident.

In the example below we provide a more detailed evaluation of the *FM* method applied to solve the *Airy*-type equation $y' = A_\omega(t)y$.

EXAMPLE 4.1 *Once we have obtained the representation for $\tilde{\Omega}_n$, the commutator brackets are now formed by the matrix $B(t)$. It is possible to reduce cost of evaluation of the matrix $B(t)$ by simplifying it, [Ise04]. Denote $q = \sqrt{g(t_n + \frac{1}{2}h)}$*

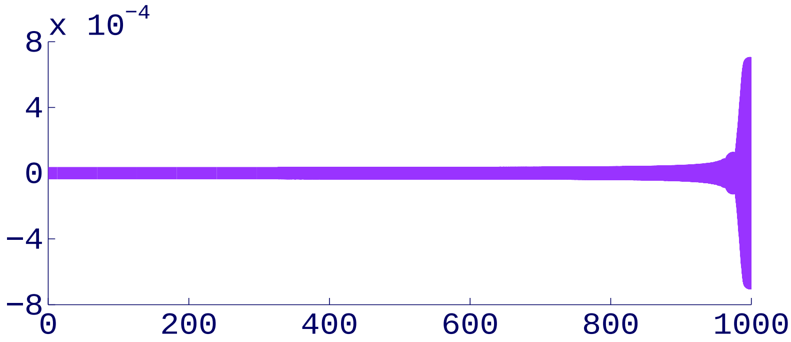

Figure 6: Global error of the six order *Magnus* method for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \le t \le 1000$ and step-size $h = \frac{1}{10}$.

and $v(t) = g(t_n + t) - g(t_n + \frac{1}{2}h)$. *Then,*

$$\begin{aligned} B(t) &= v(t) \begin{bmatrix} (2q)^{-1}\sin 2qt & q^{-2}\sin^2 qt \\ -\cos^2 qt & -(2q)^{-1}\sin 2qt \end{bmatrix} \\ &= v(t) \begin{bmatrix} q^{-1}\sin qt \\ -\cos qt \end{bmatrix} \begin{bmatrix} \cos qt & q^{-1}\sin qt \end{bmatrix}, \end{aligned}$$

and for the product

$$B(t)B(s) = v(t)v(s)\frac{\sin q(s-t)}{q} \begin{bmatrix} q^{-1}\sin qt \\ -\cos qt \end{bmatrix} \begin{bmatrix} \cos qs & q^{-1}\sin qs \end{bmatrix}.$$

It was shown in [Ise04] that

$$\|B(t)\| = \cos^2 wt + \frac{\sin^2 wt}{w^2} \quad \text{and} \quad B(t) = \mathcal{O}(t - t_{n+\frac{1}{2}}).$$

Given the compact representation of the matrix $B(t)$ with oscillatory entries, we solve $\Omega(t)$ in (2) with a *Filon*-type method, approximating functions in $B(t)$ by a polynomial $\tilde{v}(t)$, for example Hermite polynomial (4), as in classical *Filon*-type method. Furthermore, in our approximation we use end points only, although the method is general and more nodes of approximation can be required.

In Figure 8 we present the global error of the fourth order *modified Magnus* method with exact integrals for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial

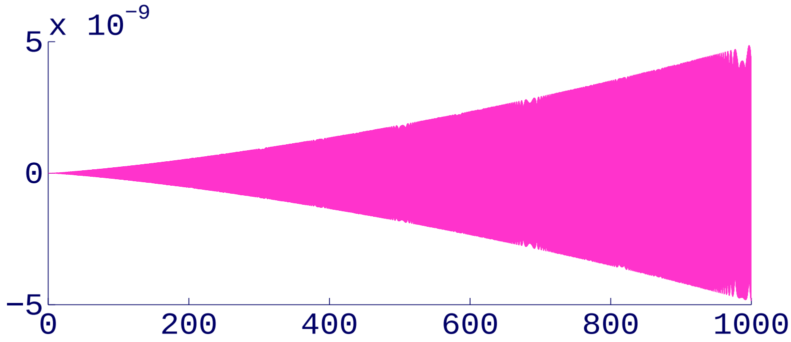

Figure 7: Global error of the six order *Magnus* method for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \le t \le 1000$ and step-size $h = \frac{1}{25}$.

conditions, $0 \le t \le 2000$ time interval and step-size $h = \frac{1}{5}$. This can be compared with the global error of the *FM* method applied to the same equation with exactly the same conditions and step-size, Figure 9. In Figures 10 and 11 we compare the fourth order classical *Magnus* method with a remarkable performance of the fourth order *FM* method, applied to the *Airy* equation with a large step-size equal to $h = \frac{1}{2}$. While in Figures 12 and 13 we solve the *Airy* equation with the fourth order *FM* method with step-sizes $h = \frac{1}{4}$ and $h = \frac{1}{10}$ respectively.

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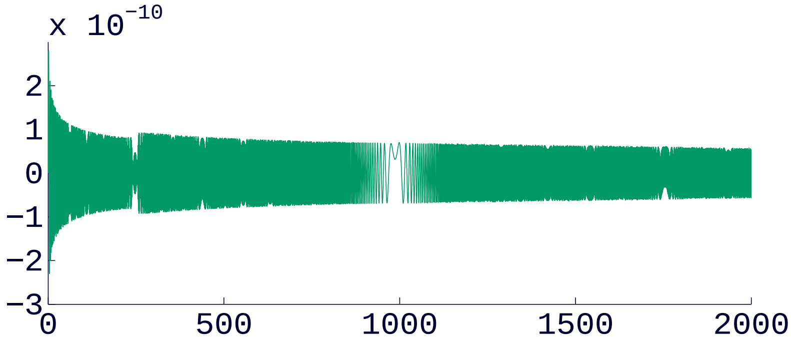

Figure 8: Global error of the fourth order *Modified Magnus* method with exact evaluation of integral commutators for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\mathsf{T}$ initial conditions, $0 \le t \le 2000$ and step-size $h = \frac{1}{5}$.

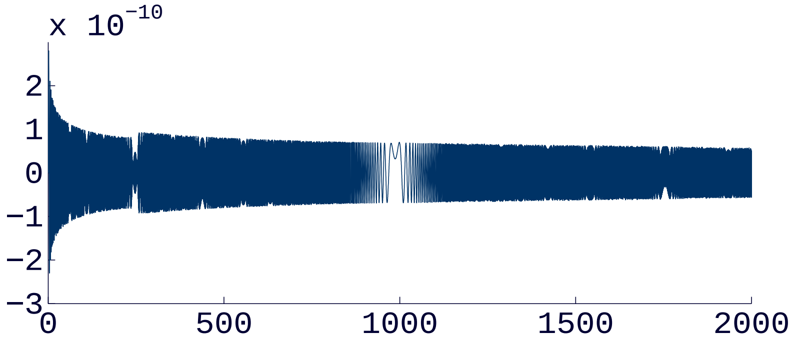

Figure 9: Global error of the fourth order *FM* method with end points only and multiplicities all 2 for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\mathsf{T}$ initial conditions, $0 \le t \le 2000$ and step-size $h = \frac{1}{5}$.

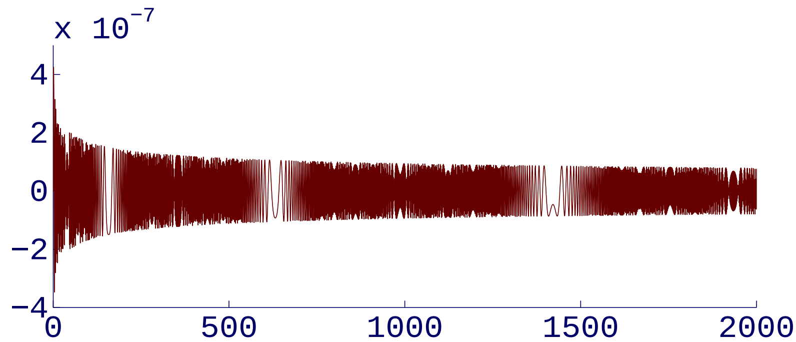

Figure 10: Global error of the fourth order *FM* method with end points only and multiplicities all 2 for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \le t \le 2000$ and step-size $h = \frac{1}{2}$.

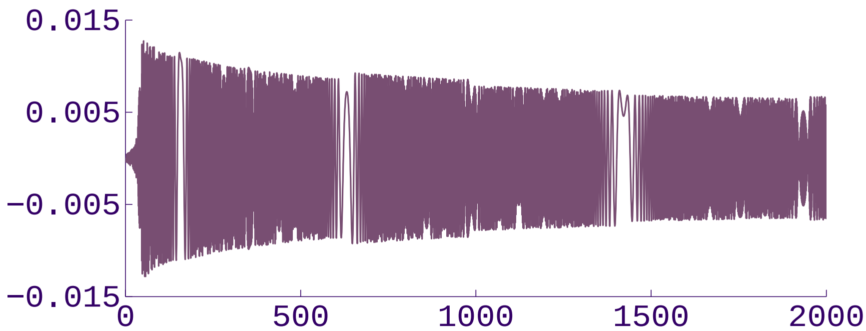

Figure 11: Global error of the fourth order *Magnus* method for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \le t \le 2000$ and step-size $h = \frac{1}{2}$.

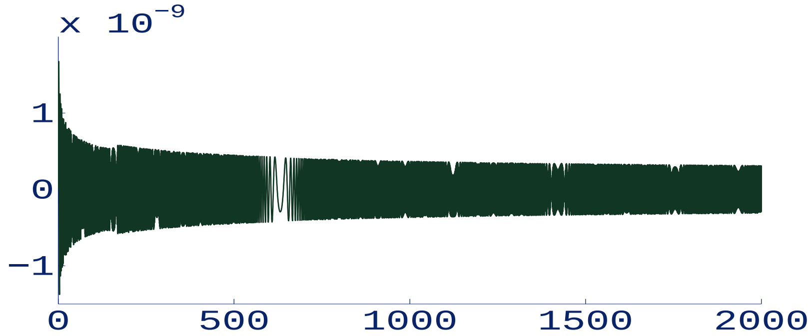

Figure 12: Global error of the fourth order *FM* method with end points only and multiplicities all 2, for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \leq t \leq 2000$ and step-size $h = \frac{1}{4}$.

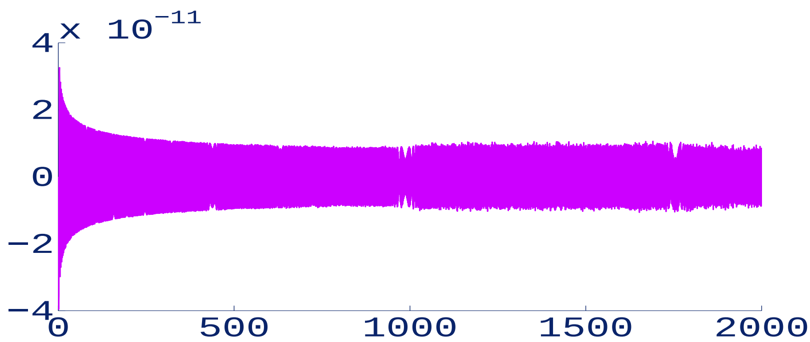

Figure 13: Global error of the fourth order *FM* method with end points only and multiplicities all 2, for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\intercal$ initial conditions, $0 \leq t \leq 2000$ and step-size $h = \frac{1}{10}$.