THE MAGNUS METHOD FOR SOLVING
OSCILLATORY LIE-TYPE ORDINARY
DIFFERENTIAL EQUATIONS

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Abstract

This work presents a new method for solving Lie-type equations \( X'_\omega = A_\omega(t)X_\omega, \ X_\omega(0) = I \). The solution to the equations of this type have the following representation \( X_\omega = \exp(\Omega) \). Here \( \Omega \) stands for an infinite Magnus series of integral commutators. We assume that the matrix \( A_\omega \) has large imaginary spectrum and \( \omega \) is a real parameter describing the frequency of oscillation. The novel method, called the FM method, combines the Magnus method, which is based on iterative techniques to approximate \( \Omega \), and the Filon-type method, an efficient quadrature rule for solving oscillatory integrals. We show that the FM method has superior performance compared to the classical Magnus method when applied to oscillatory differential equations.

1 Introduction

We proceed with the matrix ordinary differential equation

\[
X'_\omega = A_\omega X_\omega, \quad X_\omega(0) = I, \quad X_\omega = e^{\Omega},
\]  

(1)
where $\Omega$ represents the Magnus expansion, an infinite recursive series,

$$\Omega(t) = \int_0^t A_\omega(t) dt + \frac{1}{2} \int_0^t [A_\omega(\tau), \int_0^\tau A_\omega(\xi) d\xi] d\tau$$

$$+ \frac{1}{4} \int_0^t [A_\omega(\tau), \int_0^\tau [A_\omega(\xi), \int_0^\xi A_\omega(\zeta) d\zeta] d\xi] d\tau$$

$$+ \frac{1}{12} \int_0^t [[A_\omega(\tau), \int_0^\tau A_\omega(\xi) d\xi], \int_0^\xi A_\omega(\zeta) d\zeta] d\tau$$

$$+ \cdots .$$

We make the following assumptions: $A_\omega(t)$ is a smooth matrix-valued function, the spectrum $\sigma(A_\omega)$ of the matrix $A_\omega$ has large imaginary eigenvalues and that $\omega \gg 1$ is a real parameter describing frequency of oscillation.

It has been shown by Hausdorff, [Hau06], that the solution of the linear matrix equation in (1) is a matrix exponential $X_\omega(t) = e^{\Omega(t)}$, where $\Omega(t)$ satisfies the following nonlinear differential equation,

$$\Omega' = \sum_{j=0}^{\infty} \frac{B_j}{j!} \text{ad}^j_{\Omega}(A_\omega) = \text{dexp}_{\Omega}^{-1}(A_\omega(t)), \quad \Omega_0 = 0. \quad (2)$$

Here $B_j$ are Bernoulli numbers and the adjoint operator is defined as

$$\text{ad}_0^0(A_\omega) = A_\omega$$

$$\text{ad}_\Omega(A_\omega) = [\Omega, A_\omega] = \Omega A_\omega - A_\omega \Omega$$

$$\text{ad}_\Omega^{j+1}(A_\omega) = [\Omega, \text{ad}_\Omega^j A_\omega].$$

Later it was observed by W. Magnus, [Mag54], that solving equation (2) with Picard iteration it is possible to obtain an infinite recursive series for $\Omega$ and then the truncated expansion can be used in approximation of $\Omega$.

Magnus methods preserve peculiar geometric properties of the solution, ensuring that if $X_\omega$ is in matricial Lie group $G$ and $A_\omega$ is in associated Lie algebra $g$ of $G$, then the numerical solution after discretization stays on the manifold. Moreover, Magnus methods preserve time-symmetric properties of the solution after discretization, for any $a > b$, $X_\omega(a,b)^{-1} = X_\omega(b,a)$ yields $\Omega(a,b) = -\Omega(b,a)$. These properties appear to be valuable in applications.

Employing the Magnus method to solve highly oscillatory differential equations we develop it in combination with appropriate quadrature rules, such as
Filon-type methods, proved to be more efficient for oscillatory equations than for example classical Gaussian quadrature. Below we provide a brief background on the Filon-type method and the asymptotic method. The asymptotic method provides theoretical background for the Filon-type method. At this point of discussion we find it appropriate to state the two important theorems from [IN05], describing the asymptotic and the Filon-type methods used to approximate highly oscillatory integrals of the form

\[ I[f] = \int_a^b f(x)e^{i\omega g(x)}dx. \quad (3) \]

Suppose \( f, g \in C^\infty \) are smooth, \( g \) is strictly monotone in \([a, b]\), \( a \leq x \leq b \) and the frequency is \( \omega \gg 1 \).

**Lemma 1.1** [A. Iserles & S. Nørsett][IN05] Let \( f, g \in C^\infty \), \( g'(x) \neq 0 \) on \([a, b]\) and

\[
\begin{align*}
\sigma_0[f](x) &= f(x), \\
\sigma_{k+1}[f](x) &= \frac{d}{dx} \frac{\sigma_k[f](x)}{g'(x)}, \quad k = 0, 1, \ldots
\end{align*}
\]

Then, for \( \omega \gg 1 \),

\[ I[f] \sim -\sum_{m=1}^\infty \frac{1}{(i\omega)^m} \left[ e^{i\omega g(1)} \sigma_{m-1}[f](1) - e^{i\omega g(0)} \sigma_{m-1}[f](0) \right]. \]

The asymptotic method is defined as follows,

\[ Q_s^A[f] = -\sum_{m=1}^s \frac{1}{(-i\omega)^m} \left[ e^{i\omega g(1)} \sigma_{m-1}[f](1) - e^{i\omega g(0)} \sigma_{m-1}[f](0) \right]. \]

**Theorem 1.2** [A. Iserles & S. Nørsett][IN05] For every smooth \( f \) and \( g \), such that \( g'(x) \neq 0 \) on \([a, b]\), it is true that

\[ Q_s^A[f] - I[f] \sim O(\omega^{-s-1}), \quad \omega \to \infty. \]

The Filon-type method was first pioneered in the work of L. N. G. Filon in 1928 and later developed by A. Iserles and S. Nørsett in 2004, [IN05]. The method
works as follows. First interpolate function $f$ in (3) at chosen node points. For instance, take Hermite interpolation
\[
\tilde{v}(x) = \sum_{l=1}^{\nu} \sum_{j=0}^{\theta_l-1} \alpha_{l,j}(x) f^{(j)}(c_l),
\]
which satisfies $\tilde{v}^{(j)}(c_l) = f^{(j)}(c_l)$, at node points $a = c_1 < c_2 < \ldots < c_\nu = b$, with $\theta_1, \theta_2, \ldots, \theta_\nu \geq 1$ associated multiplicities, $j = 0, 1, \ldots, \theta_l-1, l = 1, 2, \ldots, \nu$ and $r = \sum_{l=1}^{\nu} \theta_l - 1$ being the order of approximation polynomial. Then, for the Filon-type method, by definition,
\[
Q_s^F[f] = I[\tilde{v}] = \int_a^b \tilde{v}(x)e^{i\omega g(x)}dx = \sum_{l=1}^{\nu} \sum_{j=0}^{\theta_l-1} b_{l,j}(w) f^{(j)}(c_l),
\]
where
\[
b_{l,j} = \int_0^1 \alpha_{l,j}(x)e^{i\omega g(x)}dx, \quad j = 0, 1, \ldots, \theta_l - 1, \quad l = 1, 2, \ldots, \nu.
\]

**Theorem 1.3** [A. Iserles & S. Nørsett][IN05] Suppose that $\theta_1, \theta_\nu \geq s$. For every smooth $f$ and $g, g'(x) \neq 0$ on $[a, b]$, it is true that
\[
I[f] - Q_s^F[f] \sim O(\omega^{-s-1}), \quad \omega \rightarrow \infty.
\]

The Filon-type method has the same asymptotic order as the asymptotic method. For formal proof of the asymptotic order of the Filon-type method for univariate integrals, we refer the reader to [IN05] and for matrix and vector-valued functions we refer to [Kha08]. The following theorem from [Kha08] is on the numerical order of the Filon-type method. It was also shown in [Kha08] that the numerical solution obtained after discretisation of the integral with Filon-type method is convergent.

**Theorem 1.4** [MK][Kha08] Let $\theta_1, \theta_\nu \geq s$. Then the numerical order of the Filon-type method is equal to $r = \sum_{l=1}^{\nu} \theta_l - 1$. 
Figure 1: The error of the asymptotic method $Q^A_2$ (right) and the Filon-type method $Q^F_2$ (left) for $f(x) = \cos(x)$, $g(x) = x$, $\theta_1 = \theta_2 = 2$, $100 \leq \omega \leq 200$.

**EXAMPLE 1.1** Here we consider the asymptotic and the Filon-type methods with first derivatives for (3) over the interval $[0, 1]$ for the case $g(x) = x$.

\[
Q^A_2 = \frac{e^{i\omega} f(1) - f(0)}{i\omega} + \frac{e^{i\omega} f'(1) - f'(0)}{\omega^2},
\]

\[
Q^F_2 = \left( -\frac{1}{i\omega} - 6 \frac{1 + e^{i\omega}}{i\omega^3} + 12 \frac{1 - e^{i\omega}}{\omega^4} \right) f(0) + \left( -\frac{e^{i\omega}}{i\omega} + 6 \frac{1 + e^{i\omega}}{i\omega^3} - 12 \frac{1 - e^{i\omega}}{\omega^4} \right) f(1) + \left( -\frac{1}{\omega^2} - 2 \frac{1 + e^{i\omega}}{i\omega^3} + 6 \frac{1 - e^{i\omega}}{\omega^4} \right) f'(0) + \left( \frac{e^{i\omega}}{\omega^2} - 2 \frac{1 + e^{i\omega}}{i\omega^3} + 6 \frac{1 - e^{i\omega}}{\omega^4} \right) f'(1).
\]

In Figure 1 we present numerical results on the asymptotic and Filon-type methods, with function values and first derivatives at the end points only, $c_1 = 0, c_2 = 1$, for the integral

\[
I[f] = \int_0^1 \cos(x)e^{i\omega x} dx, \quad 100 \leq \omega \leq 200.
\]

Both methods have the same asymptotic order and use exactly the same information. However, as we can see from Figure 1, the Filon-type method yields a
greater measure of accuracy than the asymptotic method. Adding more internal interpolation points leads to the decay of the leading error constant, resulting in a marked improvement in the accuracy of approximation. However, interpolating function \( f \) at internal points does not contribute to the higher asymptotic order of the Filon-type method.

2 The Magnus method

In this section we focus on Magnus methods for approximation of a matrix-valued function \( X_\omega \) in (1). There is a large list of publications available on the Lie-group methods, here we refer to some of them: [BCR00], [CG93], [Ise09], [Ise02a], [MK98], [BO97], [Zan96].

Currently, the most general theorem on the existence and convergence of the Magnus series is proved for a bounded linear operator \( A(t) \) in a Hilbert space, [Cas07]. The following theorem gives us sufficient condition for convergence of the Magnus series, extending Theorem 3 from [MN08], where the same results are stated for real matrices.

**Theorem 2.1** (F. Casas, [Cas07]) Consider the differential equation \( X' = A(t)X \) defined in a Hilbert space \( \mathcal{H} \) with \( X(0) = I \), and let \( A(t) \) be a bounded linear operator on \( \mathcal{H} \). Then, the Magnus series \( \Omega(t) \) in (2) converges in the interval \( t \in [0, T) \) such that

\[
\int_0^T \|A(\tau)\|d\tau < \pi
\]

and the sum \( \Omega(t) \) satisfies \( \exp \Omega(t) = X(t) \).

Given the representation for \( \Omega(t) \) in (2), the numerical task on evaluating the commutator brackets is fairly simple, [BCR00], [Ise09], [Ise02a]. For example, choosing symmetric grid \( c_1, c_2, ..., c_\nu \), suppose taking Gaussian points with respect to \( \frac{1}{2} \), consider set \( \{A_1, A_2, ..., A_\nu \} \), with \( A_k = hA(t_0 + c_k h), k = 1, 2, ..., \nu \). Linear combinations of this basis form an adjoint basis \( \{B_1, B_2, ..., B_\nu \} \), with

\[
A_k = \sum_{l=1}^{\nu} (c_k - \frac{1}{2})^{l-1}B_l, \quad k = 1, 2, ..., \nu.
\]
In this basis the six-order method, with Gaussian points \( c_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}, c_2 = \frac{1}{2}, c_3 = \frac{1}{2} + \frac{\sqrt{15}}{10} \), \( A_k = hA(t_0 + c_k h) \), is

\[
\Omega(t_0 + h) \approx B_1 + \frac{1}{12} B_3 - \frac{1}{12} [B_1, B_2] + \frac{1}{240} [B_2, B_3] \\
+ \frac{1}{360} [B_1, B_3] - \frac{1}{240} [B_2, [B_1, B_2]] + \frac{1}{720} [B_1 [B_1, [B_1, B_2]]],
\]

where

\[
B_1 = A_2, \quad B_2 = \frac{\sqrt{15}}{3} (A_3 - A_1), \quad B_3 = \frac{10}{3} (A_3 - 2A_2 + A_1).
\]

This can be reduced further and written in a more compact manner \([Ise02a],[Ise09]\),

\[
\Omega(t_0 + h) \approx B_1 + \frac{1}{12} B_3 + P_1 + P_2 + P_3,
\]

where

\[
P_1 = [B_2, \frac{1}{12} B_1 + \frac{1}{240} B_3], \\
P_2 = [B_1, [B_1, \frac{1}{360} B_3 - \frac{1}{60} P_1]], \\
P_3 = \frac{1}{20} [B_2, P_1].
\]

A more profound approach taking Taylor expansion of \( A(t) \) around the point \( t_{1/2} = t_0 + \frac{h}{2} \) was introduced in \([BCR00]\),

\[
A(t) = \sum_{i=0}^{\infty} a_i (t - t_{1/2})^i, \quad \text{with} \quad a_i = \frac{1}{i!} \frac{d^i A(t)}{dt^i} |_{t = t_{1/2}}.
\]

This can be substituted in the univariate integrals of the form

\[
B^{(i)} = \frac{1}{h^{i+1}} \int_{-h/2}^{h/2} t^i A \left( t + \frac{h}{2} \right) dt, \quad i = 0, 1, 2, ...
\]

to obtain a new basis

\[
B^{(0)} = a_0 + \frac{1}{12} h^2 a_2 + \frac{1}{80} h^4 a_4 + + \frac{1}{448} h^6 a_6 ... \\
B^{(1)} = \frac{1}{12} h a_1 + \frac{1}{80} h^3 a_3 + \frac{1}{448} h^5 a_5 + ... \\
B^{(2)} = \frac{1}{12} a_0 + \frac{1}{80} h^2 a_2 + \frac{1}{448} h^4 a_4 + \frac{1}{2304} h^6 a_6 + ...
\]
In these terms a second order method will look as follows, $e^\Omega = e^{hB(0)} + O(h^3)$. Whereas for a six-order method, $\Omega = \sum_{i=1}^{4} \tilde{\Omega}_i + O(h^7)$, one needs to evaluate only four commutators,

$$\tilde{\Omega}_1 = hB(0)$$
$$\tilde{\Omega}_2 = h^2[B^{(1)}, \frac{3}{2}B^{(0)} - 6B^{(2)}]$$
$$\tilde{\Omega}_3 + \tilde{\Omega}_4 = h^2[B^{(0)}, [B^{(0)}, \frac{1}{2}hB^{(2)} - \frac{1}{60}\tilde{\Omega}_2]] + \frac{3}{5}h[B^{(1)}, \tilde{\Omega}_2].$$

Numerical behavior of the fourth and six order classical Magnus method is illustrated in Figures 2, 3 and 4, and 5, 6 and 7, respectively. The method is applied to solve Airy equation $y''(t) = -ty(t)$ with $[1, 0]^T$ initial conditions, $t \in [0, 1000]$, for varies step-sizes, $h = \frac{1}{4}$, $h = \frac{1}{10}$ and $h = \frac{1}{25}$. Comparison shows that for a bigger interval steps $h$ both fourth and six order methods give similar results, as illustrated in Figures 2, 3, 5 and 6. However, for smaller steps six order Magnus method has a more rapid improvement in approximation compared to a fourth order method, Figures 4 and 7.

![Figure 2: Global error of the fourth order Magnus method for the Airy equation $y''(t) = -ty(t)$ with $[1, 0]^T$ initial conditions, $0 \leq t \leq 1000$ and step-size $h = \frac{1}{4}$.](image)

In current work we present an alternative method to solve equations of the kind (1). We apply the Filon quadrature to evaluate integral commutators for $\Omega$ and then solve the matrix exponential $X_\omega$ with the Magnus method or the modified Magnus method. The combination of the Filon-type methods and the Magnus
methods forms the \textit{FM} method, presented in the last section. Application of the \textit{FM} method to solve systems of highly oscillatory ordinary differential equation can be found in [Kha09].

3 The modified Magnus method

In this section we continue our discussion on the solution of the matrix differential equation

$$X'_\omega = A_\omega(t)X_\omega, \quad X_\omega(0) = I, \quad X_\omega = \exp(\Omega). \quad (5)$$

It was shown in [Ise02c] that one can achieve better accuracy in approximation of the fundamental matrix $X_\omega$, if one solves (5) at each time step locally for a constant matrix $A_\omega$.

We commence from a linear oscillator studied in [Ise02c],

$$y' = A_\omega(t)y, \quad y(0) = y_0.$$ 

We also assume that the spectrum of the matrix $A_\omega(t)$ has large imaginary values. Introducing local change of variables at each mesh point, we write the solution in the form,

$$y(t) = e^{(t-t_n)A_\omega(t_n + \frac{h}{2})}x(t - t_n), \quad t \geq t_n,$$

and observe that

$$x' = B(t)x, \quad x(0) = y_n,$$
Consider the Airy-type equation
\[ y''(t) + g(t)y(t) = 0, \quad g(t) > 0 \quad \text{for} \quad t > 0, \quad \text{and} \quad \lim_{t \to \infty} g(t) = +\infty. \]
Figure 5: Global error of the six order Magnus method for the Airy equation \( y''(t) = -ty(t) \) with \([1, 0]^T\) initial conditions, \(0 \leq t \leq 1000\) and step-size \(h = \frac{1}{4}\).

Replacing the second order differential equation by the first order system, we obtain \( y' = A\omega(t)y \), where

\[
A\omega(t) = \begin{pmatrix} 0 & 1 \\ -g(t) & 0 \end{pmatrix}.
\]

Due to a large imaginary spectrum of the matrix \( A\omega \), this Airy-type equation is rapidly oscillating. We apply the modified Magnus method to solve the system \( y' = A\omega(t)y \),

\[
y_{n+1} = e^{hA\omega(t_{n}+h/2)} e^{\hat{\Omega}_{n}} y_{n}.
\]

Here the integral commutators in \( \hat{\Omega}_{n} \) are computed according to the rules of the Filon quadrature. This results in highly accurate numerical method, the FM method. Taking into account that the entries of the matrix \( B(t) \) are likely to be oscillatory, the advantage of the Filon-type method is evident.

In the example below we provide a more detailed evaluation of the FM method applied to solve the Airy-type equation \( y' = A\omega(t)y \).

**Example 4.1** Once we have obtained the representation for \( \hat{\Omega}_{n} \), the commutator brackets are now formed by the matrix \( B(t) \). It is possible to reduce cost of evaluation of the matrix \( B(t) \) by simplifying it, [Ise04]. Denote \( q = \sqrt{g(t_{n} + \frac{1}{2}h)} \).
Figure 6: Global error of the six order Magnus method for the Airy equation $y''(t) = -ty(t)$ with $[1, 0]^T$ initial conditions, $0 \leq t \leq 1000$ and step-size $h = \frac{1}{10}$.

and $v(t) = g(t_n + t) - g(t_n + \frac{1}{2}h)$. Then,

$$B(t) = \begin{bmatrix}
(2q)^{-1} \sin 2qt & q^{-2} \sin^2 q t \\
-\cos^2 q t & -(2q)^{-1} \sin 2qt
\end{bmatrix}$$

$$= \begin{bmatrix}
q^{-1} \sin qt & q^{-1} \sin qt \\
-\cos qt & -\cos qt
\end{bmatrix},$$

and for the product

$$B(t)B(s) = v(t)v(s) \frac{\sin q(s-t)}{q} \begin{bmatrix}
q^{-1} \sin qt & q^{-1} \sin qs \\
-\cos qt & -\cos qt
\end{bmatrix} .$$

It was shown in [Ise04] that

$$\|B(t)\| = \cos^2 wt + \frac{\sin^2 wt}{w^2} \quad \text{and} \quad B(t) = \mathcal{O}(t - t_n + \frac{1}{2}).$$

Given the compact representation of the matrix $B(t)$ with oscillatory entries, we solve $\Omega(t)$ in (2) with a Filon-type method, approximating functions in $B(t)$ by a polynomial $v(t)$, for example Hermite polynomial (4), as in classical Filon-type method. Furthermore, in our approximation we use end points only, although the method is general and more nodes of approximation can be required.

In Figure 8 we present the global error of the fourth order modified Magnus method with exact integrals for the Airy equation $y''(t) = -ty(t)$ with $[1, 0]^T$ initial
conditions, $0 \leq t \leq 2000$ time interval and step-size $h = \frac{1}{5}$. This can be compared with the global error of the $FM$ method applied to the same equation with exactly the same conditions and step-size, Figure 9. In Figures 10 and 11 we compare the fourth order classical $Magnus$ method with a remarkable performance of the fourth order $FM$ method, applied to the Airy equation with a large step-size equal to $h = \frac{1}{2}$. While in Figures 12 and 13 we solve the Airy equation with the fourth order $FM$ method with step-sizes $h = \frac{1}{4}$ and $h = \frac{1}{10}$ respectively.

References


Figure 8: Global error of the fourth order Modified Magnus method with exact evaluation of integral commutators for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\top$ initial conditions, $0 \leq t \leq 2000$ and step-size $h = \frac{1}{5}$.

Figure 9: Global error of the fourth order FM method with end points only and multiplicities all 2 for the Airy equation $y''(t) = -ty(t)$ with $[1,0]^\top$ initial conditions, $0 \leq t \leq 2000$ and step-size $h = \frac{1}{5}$. 
Figure 10: Global error of the fourth order FM method with end points only and multiplicities all 2 for the Airy equation \( y''(t) = -ty(t) \) with \([1,0]^T\) initial conditions, \(0 \leq t \leq 2000\) and step-size \(h = \frac{1}{2}\).

Figure 11: Global error of the fourth order Magnus method for the Airy equation \( y''(t) = -ty(t) \) with \([1,0]^T\) initial conditions, \(0 \leq t \leq 2000\) and step-size \(h = \frac{1}{2}\).
Figure 12: Global error of the fourth order FM method with end points only and multiplicities all 2, for the Airy equation \( y''(t) = -ty(t) \) with \([1, 0]^T\) initial conditions, \(0 \leq t \leq 2000\) and step-size \(h = \frac{1}{4}\).

Figure 13: Global error of the fourth order FM method with end points only and multiplicities all 2, for the Airy equation \( y''(t) = -ty(t) \) with \([1, 0]^T\) initial conditions, \(0 \leq t \leq 2000\) and step-size \(h = \frac{1}{10}\).