On the value of the max-norm of the orthogonal spline projection

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$$\mathcal{S}_{k,m}(\Delta) := \left\{ s \in \mathcal{C}^{m-1}[-1,1] : s_{|(t_{i-1},t_i)} \text{ is a polynomial of order } k \right\}.$$

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Theorem 1 (Shadrin,01).

$$\left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty = \sup_{\|f\|_\infty \le 1} \left\| P_{\mathcal{S}_{k,m}(\Delta)}(f) \right\|_\infty \le c_k.$$

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Problem. Estimation of the quantity $\Lambda_{k,m} := \sup_\Delta \|P_{\mathcal{S}_{k,m}(\Delta)}\|_\infty$.

The need for a new method

Shadrin's proof is based on the bound

$$\|P_{\mathcal{S}_{k,k-1}(\Delta)}\|_\infty \le \|G_\Delta^{-1}\|_\infty, \qquad G_\Delta := \left[\langle M_i, N_j \rangle\right].$$

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We are making the guess

$$\Lambda_{k,m} \stackrel{?}{\asymp} \frac{k}{\sqrt{k-m}}.$$

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$$\rho_{k,m+1} \overset{?}{\leq} \rho_{k,m} \overset{?}{\leq} \frac{k-m-1/2}{k-m-1}\rho_{k,m+1}.$$

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$$\mathcal{S}_{k,m}(\Delta \cup \{t\}) \ = \ \mathcal{S}_{k,m}(\Delta) \ \oplus \ \text{span}\left\{(\bullet - t)_+^m, \ldots, (\bullet - t)_+^{k-1}\right\}.$$

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- $P_t = (I - PQ_t)^{-1}P(I - Q_t) + (I - Q_t P)^{-1} Q_t (I - P).$

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One deduces that

$$\sup_{\|f\|_\infty \le 1} |P_t(f)(1)| \ge \sup_{\|f\|_\infty \le 1} |[1 - Q_t(\mathbf{1})(1)]\, P(f)(1) + Q_t(f)(1)| - o(t).$$

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$$= \frac{m}{k} \sup_{\|f\|_\infty \le 1} |P(f)(1)| + \rho_{k,m} - o(t) \rightsquigarrow \frac{m}{k} K_N + \rho_{k,m}. \quad \square$$

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$$\mathcal{R}_{k,m}(\Delta) = \left\{ F^{(k)},\ F \in \mathcal{S}_{2k,k}(\Delta),\ \begin{array}{l} F \equiv 0 \;\; k\text{-fold at } t_0, \\ F \equiv 0 \;\; (k-m)\text{-fold at } t_1, \dots, t_{N-1}, \\ F \equiv 0 \;\; k\text{-fold at } t_N. \end{array} \right\}$$

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$$=: \mathcal{R}^1_{k,m}(\Delta) \oplus \mathcal{R}^2_{k,m}(\Delta) \oplus \cdots \oplus \mathcal{R}^{N-1}_{k,m}(\Delta),$$

each space $\mathcal{R}^i_{k,m}(\Delta)$ supported on $[t_{i-1}, t_{i+1}]$ and of dimension m.

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Lemma 3. Consider bases $(\varphi_i)_{i=1}^{m(N-1)}$ and $(\widehat{\varphi}_j)_{j=1}^{m(N-1)}$ of $\mathcal{R}_{k,m}(\Delta)$ and introduce the Gramian matrix $M := \left[\langle \varphi_i, \widehat{\varphi}_j \rangle\right]_{i,j=1}^{m(N-1)}$. If

(i) $\|M^{-1}\|_\infty \le \kappa$, (ii) $\|\varphi_i\|_1 \le \gamma_1$, (iii) $\left\|\sum a_j \widehat{\varphi}_j\right\|_\infty \le \gamma_\infty \|a\|_\infty$,

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$$\|P_{\mathcal{R}_{k,1}(\Delta)}\|_\infty \le \frac{2k(k+1)}{(k-1)^2}\,\sigma_{k,0}, \qquad \|P_{\mathcal{S}_{k,1}(\Delta)}\|_\infty \le \frac{3k^2+1}{(k-1)^2}\,\sigma_{k,0}.$$

The space $\mathcal{R}^i_{k,1}(\Delta)$ is spanned by the function

$$f_i(x) := \begin{cases} 2\delta_i\, F^{(k)}\left(\dfrac{2x-t_{i-1}-t_i}{h_i}\right), & x \in (t_{i-1}, t_i), \\ -2\delta_{i+1} F^{(k)}\left(\dfrac{t_i+t_{i+1}-2x}{h_{i+1}}\right), & x \in (t_i, t_{i+1}), \end{cases} \qquad h_i := t_i - t_{i-1}, \quad \delta_i := \frac{1}{h_i},$$

$$F^{(k)}(x) = \frac{(-1)^{k-1}}{2^{k-1}\,k!}\,\frac{d^k}{dx^k}\left[(1-x)^{k-1}(1+x)^k\right] = P^{(1,0)}_{k-1}(x).$$

The case of continuous splines, ctd

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We obtain $\|f_i\|_1 = \dfrac{4}{k}\sigma_{k,0}$. With the renormalization $\widehat{f}_i := \frac{1}{4(\delta_i+\delta_{i+1})} f_i$, we get $\|\sum a_j \widehat{f}_j\|_\infty \le \dfrac{k+1}{2}\|a\|_\infty$. Finally, we derive $\|M^{-1}\|_\infty \le \frac{k^2}{(k-1)^2}$ from the expression

$$
M \;=\; \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccc} \widehat{f}_1 & \widehat{f}_2 & \widehat{f}_3 & \widehat{f}_4 & \cdots \end{array} \\
\begin{bmatrix}
1 & \frac{(-1)^k}{k}\alpha_2 & 0 & 0 & \cdots \\
\frac{(-1)^k}{k}\beta_1 & 1 & \frac{(-1)^k}{k}\alpha_3 & 0 & \cdots \\
0 & \frac{(-1)^k}{k}\beta_2 & 1 & \ddots & \\
0 & 0 & \frac{(-1)^k}{k}\beta_3 & 1 & \\
\vdots & \vdots & 0 & \ddots & \ddots
\end{bmatrix}
\end{array},
\qquad
\begin{aligned}
\alpha_i &:= \frac{\delta_i}{\delta_i + \delta_{i+1}}, \\
\beta_i &:= \frac{\delta_{i+1}}{\delta_i + \delta_{i+1}}.
\end{aligned}
$$

The case of differentiable splines (m=2)

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Our bound for $\|M^{-1}\|_\infty$ was only valid for $k > 4$. Another choice of bases of $\mathcal{R}_{k,2}(\Delta)$ might give better results. The basis of $\mathcal{R}^i_{k,2}(\Delta)$ we used consists of the previous function f_i and its orthogonal function.