Best conditioned bases and minimal projections

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Preliminaries

where the concepts are introduced and the problems raised.

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A minimal projection from X onto V is a projection $P : X \twoheadrightarrow V$ such that

$$\|P\| = p(V, X).$$

Best conditioned basis

The ℓ_p-condition number of a basis $\underline{v} = (v_1, \ldots, v_n)$ of a normed space $(V, \|\bullet\|)$ is

$$\kappa_p(\underline{v}) \quad := \quad \sup_{a \in \ell_p^n \setminus \{0\}} \frac{\left\|\sum_{i=1}^n a_i v_i\right\|}{\|a\|_p} \quad \times \quad \sup_{a \in \ell_p^n \setminus \{0\}} \frac{\|a\|_p}{\left\|\sum_{i=1}^n a_i v_i\right\|}$$

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In the terminology of Banach space geometry, the Banach-Mazur distance between X and Y is defined by

$$d(X, Y) := \inf\{\|T\| \times \|T^{-1}\|,\ T : X \to Y \text{ isomorphism}\}.$$

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Therefore, $\kappa_\infty(\underline{v^N}) \le \kappa_\infty(\underline{v})$.

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Proposition. Let V be a subspace of a normed space X, and consider an embedding $i : X \hookrightarrow C(K)$, where K is a compact Hausdorff space, then

$$p(V, X) \leq p(i(V), C(K)).$$

Interlude

where our liking for extreme points is revealed.

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$$\mathrm{Ex}(B_{V^*}) \subseteq \mathrm{Ex}(B_{X^*})_{|V} := \{\lambda_{|V},\ \lambda \in \mathrm{Ex}(B_{X^*})\}.$$

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Lemma (Auerbach). V be a n-dimensional space, there exist $(v_1, \ldots, v_n)$ basis of V and $(\lambda_1, \ldots, \lambda_n)$ basis of V^* such that

$$\forall\, i, j \in \{1, \ldots, n\}\,,\ \|v_i\| = 1,\ \lambda_i \in \mathrm{cl}(\mathrm{Ex}(B_{V^*}))\ \text{(hence } \|\lambda_i\| = 1\text{) and }\ \lambda_i(v_j) = \delta_{i,j}.$$

Transition

where we return to our original aim.

The case $p(V, C(K)) = \kappa_\infty(V) = 1$

For V n-dimensional subspace of $C(K)$, there exist $t_1, \dots, t_n \in K$ (the Fekete points) and a basis $(v_1, \dots, v_n)$ of V such that

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If $V = \mathcal{P}_{n-1}([-1, 1])$, we have $\sum_{i=1}^n v_i^2 \leq 1$, which implies that $\mathcal{P}_{2n-2}$ contains an isometric copy of ℓ_∞^n.

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For any n, there exists a space V_n such that $p(V_n, C([-1, 1])) = \kappa_\infty(V_n)$ $(= 1)$.

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Proposition (Wulbert). If V is a n-dimensional WS-subspace of $C(K)$ such that $p(V, C(K)) = 1$, then $\kappa_\infty(V) = 1$

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Proposition (Wulbert). If V is a n-dimensional WS-subspace of $C(K)$ such that $p(V, C(K)) = 1$, then $\kappa_\infty(V) = 1$ (and also $p_{\text{int}}(V, C(K)) = 1$).

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$$\forall\, i,j \in \{1, \dots, n\}\,,\ \|v_i\| = 1, \qquad \text{and} \qquad v_j(t_i) = \delta_{i,j}.$$

If $V = \mathcal{P}_{n-1}([-1,1])$, we have $\sum_{i=1}^n v_i^2 \leq 1$, which implies that $\mathcal{P}_{2n-2}$ contains an isometric copy of ℓ_∞^n.

For any n, there exists a space V_n such that $p(V_n, C([-1,1])) = \kappa_\infty(V_n)$ $(= 1)$.

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It is possible to find projections of norm 1 onto WS-subspaces, and those must be unique.

V and X	minimal projection	best ℓ_∞-conditioned basis
	$p(V,X)$	$\kappa_\infty(V)$
	uniqueness	uniqueness
ℓ_2^n and ℓ_2	orthogonal	orthonormal
	1	$\sqrt{n}$
	yes	yes
$\mathcal{T}_n$ and $L_p(\mathbb{T})$	Fourier projection	?
	$\frac{1}{\pi}\int_{\mathbb{T}}\lvert D_n\rvert$	?
	yes	?
$\mathcal{T}_1$ and $C(\mathbb{T})$	Fourier projection	equidistant Lagrange
	$1.435991...$	$\frac{5}{3}$
	yes	yes
$\mathcal{P}_1$ and $L_1([-1,1])$	...	$(x \mapsto \frac{1}{2}, x \mapsto x)$
	$1.220404...$	$\frac{5}{4}$
	yes	yes
$\mathcal{P}_2$ and $C([-1,1])$	...	...
	$1.220173...$	$1.248394...?$
	?	yes?

The heart of the matter

where the problems of existence, uniqueness and characterization are approached.

Existence and Uniqueness

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No contradiction: if V is a 1-dimensional subspace of X, then $p(X, V) = 1$.

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when C and K are convex, K compact, the function $f : C \times K \longrightarrow \mathbb{R}$ is convex with respect to both variables, $f(x,\bullet)$ is continuous for any $x \in C$ and $(f(\bullet,\lambda))_{\lambda \in D}$ is equicontinuous at x^*.

Applications

Proposition (Cheney). If V is a Haar subspace of X, and if $P : X \twoheadrightarrow V$ is a minimal projection with $\|P\| > 1$, then there exists $\lambda_1, \ldots, \lambda_{n+1} \in \mathrm{Ex}(B_{X^*})$ ($\lambda_i \neq \pm\lambda_j$, for $i \neq j$) such that $\|P^*(\lambda_i)\| = \|P\|$.

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Generalized interpolating projections

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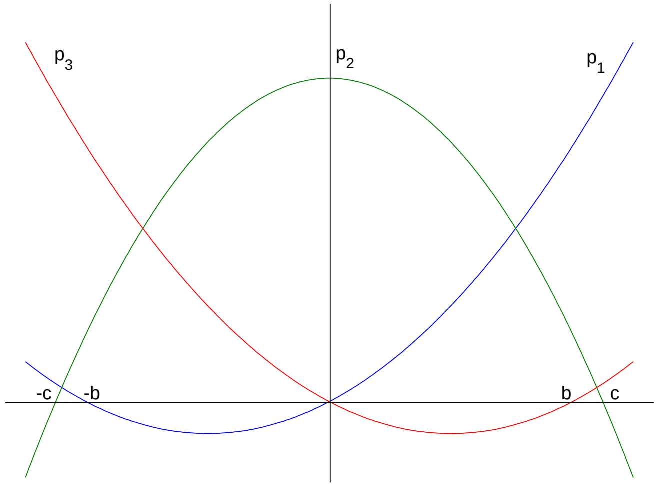
Tailpiece

where we briefly explain how to obtain $\kappa_\infty(\mathcal{P}_2)$.

Let $\underline{p} := \underline{p}(b, c, d)$ be the following symmetric basis of $\mathcal{P}_2$, where $b, c, d \in (0, +\infty)$:

$$p_1(x) := \frac{x(x+b)}{2d}, \quad p_2(x) := c^2 - x^2, \quad p_3(x) := \frac{x(x-b)}{2d}, \qquad x \in [-1, 1].$$



Theorem. $\min_{b,c,d>0} \kappa_\infty(\underline{p}(b,c,d)) \approx 1.248394563 < \frac{5}{4} = p_{\text{int}}(\mathcal{P}_2, C([-1,1]))$.

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The dual basis of $\underline{p}$ has the expression, for $f \in \mathcal{P}_2$, with $C := c^2$ and $t^* := \frac{C(1+b)}{b+C}$,

$$\begin{aligned}
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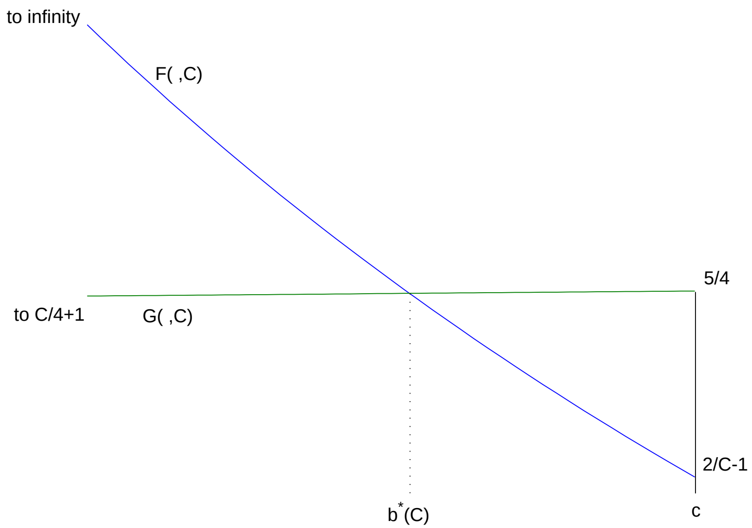
Obtaining $\kappa_\infty(\underline{p}) < \frac{5}{4}$ is only possible if $b \le c \le 1$. In this case, the optimal normalization of $\underline{p}$ leads to the choice

$$d = \frac{1}{\lambda} = \frac{(b+C)^2 - C(C-b^2)}{(b+C)^2 + C(C-b^2)}.$$

We then have to minimize $\max(F(b,C), G(b,C))$, where

$$F(b,C) := \frac{\lambda+1}{C} - 1 \quad \text{and} \quad G(b,C) := \frac{b^2\lambda^2}{4C} + 1.$$



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$$\begin{cases} F(b,C) - G(b,C) & = & 0, \\ \left[\frac{\partial F}{\partial b}\frac{\partial G}{\partial C} - \frac{\partial F}{\partial C}\frac{\partial G}{\partial b}\right](b,C) & = & 0. \end{cases}$$

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Using the Groebner package from Maple, this system is equivalent to

$$144C^8+6498C^7+25839C^6-25108C^5+9827C^4-17192C^3+2336C^2+1088C-192 = 0,$$
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In the prescribed domain, this has the unique solution

$$C \approx 0.9402938300 \quad \text{and} \quad b \approx 0.8675381234.$$

THE END