ON DEFINITIONS OF DISCRETE TOPOLOGICAL CHAOS AND THEIR RELATIONS ON INTERVALS

SIMON FOUCART

Abstract. In this paper, we distinguish three groups of definition of chaos:

- the Devaney's type of chaos: we will introduce three definitions which are equivalent on Baire Hausdorff spaces with countable base, before noting a defect in Devaney's definition, leading us to give a new definition of chaos. We will also examine how the functor "Stone-Čech compactification" preserves and reflects chaos.
- the entropy type of chaos: we will extend the definition of entropy of a map to the case of completely regular Hausdorff spaces, using the Stone-Čech compactification again.
- the Li and Yorke's type of chaos.

A complete study of the relations between these three types will be held on intervals, introducing some other definitions of chaos.
Finally, we are going to give a non-trivial example of chaotic map.

In this paper, X is a topological space, most of the time Hausdorff and perfect, but those conditions will be stated when needed, and f is a continuous map from X into itself, the continuity of f being a essential hypothesis. When X is an interval, it will be denoted I.

1. Basic definitions

If (X, d) is a metric space, f has sensitive dependence on initial conditions if: $\exists \delta > 0 : \forall x \in X, \forall \epsilon > 0, \exists n \geq 0, \exists y \in X, 0 < d(x, y) < \epsilon : d(f^n(x), f^n(y)) > \delta$.
If A is a subset of X, the backward, respectively forward, orbit of A (under f) is defined by: $\mathcal{O}_f^-(A) := \bigcup_{n\geq 0} f^{-n}(A)$, respectively $\mathcal{O}_f^+(A) := \bigcup_{n\geq 0} f^n(A)$. When A is a singleton $\{x\}$, we write $\mathcal{O}_f^+(x)$ instead of $\mathcal{O}_f^+(\{x\})$.
We say that f is topologically transitive if for every non-empty open U, $\mathcal{O}_f^-(U)$ (which is open) is dense in X. Equivalently, for all non-empty open U and V, $\exists n \geq 0$ such that: $f^n(V) \cap U \neq \emptyset$. Equivalently, for every non-empty open V, $\mathcal{O}_f^+(V)$ (not necessarily open) is dense in X.
The ω-limit set of a point $x \in X$ by f is the set of limits of all convergent subsequences of the sequence $(f^n(x))_{n\geq 0}$, ie $\omega_f(x) := \bigcap_{N\geq 0} cl(\mathcal{O}_f^+(f^N(x)))$

Notations. $\Omega := \{x \in X : \omega_f(x) = X\}$ and for $N \geq 0$, $\Delta_N := \{x \in X : cl(\mathcal{O}_f^+(f^N(x))) = X\}$.

Lemma 1.1. $\Omega = \bigcap_{n\geq 0} \Delta_n \subseteq \cdots \subseteq \Delta_N \subseteq \cdots \subseteq \Delta_2 \subseteq \Delta_1 \subseteq \Delta_0$ and if X is perfect and Hausdorff, $\Omega = \cdots = \Delta_N = \cdots = \Delta_2 = \Delta_1 = \Delta_0$

2000 *Mathematics Subject Classification.* Primary 37B99; 37E05.
Key words and phrases. discrete dynamical systems, chaos, topological transitivity, ω-limit set, Devaney, Stone-Čech compactification, entropy, Li and Yorke, horseshoe.

Proof. ($x \in \Omega \Leftrightarrow x \in \bigcap_{n\geq 0} \Delta_n$) and ($x \in \Delta_{n+1} \Rightarrow x \in \Delta_n$) are obvious, keeping in mind that $\omega_f(x) := \bigcap_{N\geq 0} cl(\mathcal{O}_f^+(f^N(x)))$ and that $\mathcal{O}_f^+(f^{n+1}(x)) \subseteq \mathcal{O}_f^+(f^n(x))$. X is perfect means that for every $y \in X$ and every $U \in \mathcal{N}_y$ (U a neighbourhood of y), $U \setminus \{y\}$ is non-empty. Now if X is perfect Hausdorff, let us pick $x \in \Delta_0$. For $y \in X, U \in \mathcal{N}_y$ and $N \geq 0, V := U \setminus \{x, f(x), ..., f^{N-1}(x)\}$ is an open set (since X is Hausdorff) which can not be empty, otherwise U would be finite: $U =: \{x_1, x_2, ..., x_n\}$, then $V := U \setminus \{x_2, ..., x_n\} = \{x_1\}$ would be open, impossible because X is perfect. So by the density of $\mathcal{O}_f^+(x)$, $\mathcal{O}_f^+(x) \cap V \neq \emptyset$, ie $\exists n \geq N : f^n(x) \in U$, true for all y, U. Then: $\forall N \geq 0, cl(\mathcal{O}_f^+(f^N(x))) = X$. Hence $\omega_f(x) = X$, ie $x \in \Omega$. □

Requiring X to be perfect is not restrictive, since anyway if we want a map to have sensitive dependence on initials conditions, X has to be perfect. But if X is not, Ω can be stricly included in Δ_0, as shown by the following example: $X := \{1/2^n, n \geq 0\} \cup \{0\}$, with the absolute value as metric, and $f : x \in X \mapsto x/2 \in X$. $1 \in \Delta_0$, but $\omega_f(1) = \{0\} \neq X$. We see as well that f is not topologically transitive (no open visits an area to its left), hence ($\Delta_0 \neq \emptyset$)$\not\Rightarrow$(f is topologically transitive), unlike what can be often found in the literature. However, the following is true:

Lemma 1.2. ($x \in \Omega$) $\Rightarrow$ (f is topologically transitive and $x \in \Delta_0$), and if X is Hausdorff: ($x \in \Omega$) $\Leftrightarrow$ (f is topologically transitive and $x \in \Delta_0$).

Proof. $\Rightarrow$ We already know ($\Omega \subseteq \Delta_0$), and let us consider a non-empty open U: there exists $n \geq 0$ such that $f^n(x) \in U$, then $X = cl(\mathcal{O}_f^+(f^n(x))) \subseteq cl(\mathcal{O}_f^+(U))$, so f is topologically transitive.
$\Leftarrow$ Let $x \in \Delta_0$. Let us assume that $cl(\mathcal{O}_f^+(f(x))) \neq X$, so that there exists a non-empty open U such that $U \cap \mathcal{O}_f^+(f(x)) = \emptyset$. But $U \cap \mathcal{O}_f^+(x) \neq \emptyset$, since $x \in \Delta_0$, thus $x \in U$. If $U \setminus \{x\}$ was non-empty (it is open since X is Hausdorff), $(U \setminus \{x\}) \cap \mathcal{O}_f^+(x) \neq \emptyset$, contradicting $U \cap \mathcal{O}_f^+(f(x)) = \emptyset$. Thus, $\{x\} = U$ is open. $V := X \setminus \{x\}$ is also open, and non empty (the case $\#X = 1$ can be easily proved apart). Then, by the topological transitivity of f: $\exists n \geq 0, \exists y \in V : f^n(y) = x$ (necessarily $n \geq 1$). The open $f^{-n}(\{x\})$ is non-empty, so again by topological transitivity: $\exists m \geq 0 : f^m(x) \in f^{-n}(\{x\})$, which is $f^{n+m}(x) = x$. x is periodic, then $\mathcal{O}_f^+(f(x)) = \mathcal{O}_f^+(x)$, which is not. Consequently, $x \in \Delta_1$, and inductively $x \in \Delta_2$, ..., $x \in \Delta_n$, ... so that $x \in \Omega$. □

2. Devaney's type of chaos

2.1. **Three common definitions of chaos.** Let us state what we understand by Devaney's chaos (written D-chaos):

Definition 2.1. Assuming that X is a metric space, we say that f is D-chaotic, respectively D'-chaotic, if:

(D1) $\forall U$ non-empty open, $cl(\mathcal{O}_f^-(U)) = X$ (f is topologically transitive)	(D'1) $\Delta_0 \neq \emptyset$ (f has a dense orbit)
(D2) $cl(Per_f) = X$ (the periodic points are dense in X)	(D'2) = (D2)
(D3) f has sensitive dependence on initial conditions	(D'3) = (D3)

It would have been equivalent to define the D or D'-chaos by replacing every cl encountered in these definitions by der, where $der(A)$ is the set of accumulation points of $A \subseteq X$, since X is perfect, because of $(D3)$.

Let us remind that in [2], Banks, Brooks, Cairns, Davis and Stacey have shown that if the set of periodic points of f, denoted Per_f, is dense in X and if f is topologically transitive, then f has sensitive dependence on initial conditions, provided that X is infinite (equivalently, provided that X is (metric) perfect, but it is easier to check that X is infinite than perfect). This not only tells us that the most intuitive hypothesis of Devaney's definition of chaos is redundant, but also that the Devaney's chaos is in fact a topological notion, since the metric structure of X was only required by the sensitive dependence on initial conditions. Hence, we introduce the following definitions:

Definition 2.2. f is B-chaotic, respectively B'-chaotic, if:

(B1) $\forall U$ non-empty open, $cl(\mathcal{O}_f^-(U)) = X$	(B'1)$\Delta_0 \neq \emptyset$
(B2) $cl(Per_f) = X$	(B'2) = (B2)
(B3) $\#X = +\infty$	(B'3) = (B3)

It is now clear that if X is a metric space: (f is D-chaotic)$\Longleftrightarrow$(f is B-chaotic), and by the following lemma, we have as well: (f is D'-chaotic)$\Longleftrightarrow$(f is B'-chaotic).

Lemma 2.3. If X is Hausdorff, $(cl(Per_f) = X, \Delta_0 \neq \emptyset) \Rightarrow$ (f is topologically transitive), and so if f is B'-chaotic, it is B-chaotic.

Proof. Let U, V be non-empty open of X, and let us take $x \in \Delta_0$. $\exists n \geq 0 : f^n(x) \in V$. X being Hausdorff, $W := U \setminus \{x, ..., f^n(x)\}$ is open.
First case: $W \neq \emptyset$, so $\mathcal{O}_f^+(x) \cap W \neq \emptyset$, implying that $f^n(x) \in V \cap \mathcal{O}_f^-(U)$.
Second case: $U \subseteq \{x, ..., f^n(x)\}$. $U \cap Per_f \neq \emptyset$, so $\exists k \in \{1, ..., n\} : f^k(x) \in Per_f$. Hence, $\mathcal{O}_f^+(x)$ is finite, then closed, so $X = \mathcal{O}_f^+(x)$ is finite, and therefore Per_f is finite, then closed, so $X = Per_f$, and finally $x \in Per_f$. Denoting q its period, for l such that $k + lq > n$, $f^{k+lq}(x) = f^k(x) \in U$, so: $f^n(x) \in V \cap f^{-(k+lq-n)}(U) \subseteq V \cap \mathcal{O}_f^-(U)$.
In each case, $V \cap \mathcal{O}_f^-(U) \neq \emptyset$, hence $cl(\mathcal{O}_f^-(U)) = X$, which means that f is topologically transitive. □

Let us now remark that if, for some $n \geq 1$, f^n is B-chaotic, or B'-chaotic, so is f (note that $Per_{f^n} = Per_f$). Besides, if f is a homeomorphism, f is B-chaotic if and only if f^{-1} is, thus if X is a metric space and if f is B-chaotic, both f and f^{-1} have sensitive dependence on initial conditions.

Let us remind that if X is Hausdorff, $(\Omega \neq \emptyset)\Leftrightarrow$(B1+B'1), and since we also have (B2+B'1)$\Rightarrow$(B1), we see that if X is Hausdorff, f is B'-chaotic if and only if $(\Omega \neq \emptyset)$, (B2) and (B3) are true. This could be another form of definition of chaos, that can even be strengthened, noticing that $(\Omega \neq \emptyset)\Rightarrow(cl(\Omega) = X)$. Indeed, for $x \in \Omega$, for all $N \geq 0$ and $n \geq 0$, $cl(\mathcal{O}_f^+(f^{N+n}(x))) = X$, then for all $n \geq 0$, $f^n(x) \in \Omega$, ie $\mathcal{O}_f^+(x) \subseteq \Omega$, which proves that Ω is dense in X. Hence:

Definition 2.4. f is B"-chaotic if:
(B"1) $cl(\Omega) = X$
(B"2) $cl(Per_f) = X$
(B"3) $\#X = +\infty$

Let us note that, for some $n \geq 1$, if f^n is B"-chaotic, so is f, as it is rather easy to see that $\Omega_{f^n} \subseteq \Omega_f$.

Theorem 2.5. In general, (B"-chaos) $\Rightarrow$ (B'chaos) and (B"-chaos) $\Rightarrow$ (B'chaos). If moreover X is Hausdorff, (B"-chaos) $\Leftrightarrow$ (B'-chaos) $\Rightarrow$ (B-chaos). If at last X is a

Baire space with countable base for the topology, (B1) $\Rightarrow$ (B'1), thus (B-chaos) $\Rightarrow$ (B'-chaos). Finally, if X is a Baire Hausdorff space with countable base, (B"-chaos) $\Leftrightarrow$ (B'-chaos) $\Leftrightarrow$ (B-chaos).

Proof. Everything has already been done, but the implication (B1)$\Rightarrow$(B'1). Let then X be a Baire space with a countable base $\{U_i \neq \emptyset, i \geq 0\}$, and let us assume that f is topologically transitive, so: $\forall i \geq 0, cl(\mathcal{O}_f^-(U_i) = X$, and since $\mathcal{O}_f^-(U_i)$ is open, $Y := \bigcap_{i \geq 0} \mathcal{O}_f^-(U_i)$ is dense in X, because X is a Baire space. Let us pick $y \in Y$, and let V be a non empty open, there exists $j \geq 0$ such that $U_j \subseteq V$, but $y \in \mathcal{O}_f^-(U_j) \subseteq \mathcal{O}_f^-(V)$, meaning that $\mathcal{O}_f^+(y) \cap V \neq \emptyset$. Hence, $y \in \Delta_0$. So $\Delta_0 \supseteq Y$ is dense in X. $\square$

2.2. **A new definition of chaos.** As already stated in [8], the Devaney's definition of chaos contains a defect in itself, namely a map making each point periodic can be B-chaotic!

Proposition 2.6. If X is Hausdorff and Per_f is dense in X, $\tilde{f}$ denoting the restriction of f to Per_f, (f: $X \longrightarrow X$ is B-chaotic) $\Leftrightarrow$ ($\tilde{f}$: $Per_f \longrightarrow Per_f$ is B-chaotic). In particular, if X is Hausdorff and f is B-chaotic, so is $\tilde{f}$.

Proof. $\Rightarrow$ Per_f can not be finite, otherwise, since X is Hausdorff, Per_f would be closed, then $X = cl(Per_f) = Per_f$ would be finite, which is not. $Per_{\tilde{f}} = Per_f$ is obviously dense in Per_f. Finally, if $\tilde{U}$ is a non-empty open of Per_f, there exists U non-empty open of X such that $\tilde{U} = U \cap Per_f$. Then, as one easily checks, $\mathcal{O}_{\tilde{f}}^-(\tilde{U}) = \mathcal{O}_f^-(U) \cap Per_f$. For $\tilde{V}$ non-empty open of Per_f, $\tilde{V} = V \cap Per_f$ for some V non-empty open of X. $V \cap \mathcal{O}_f^-(U) \neq \emptyset$, but this is an open of X, so $V \cap \mathcal{O}_f^- \cap Per_f \neq \emptyset$, which is: $\tilde{V} \cap \mathcal{O}_{\tilde{f}}^-(\tilde{U}) \neq \emptyset$, ie $\mathcal{O}_{\tilde{f}}^-(\tilde{U})$ is dense in Per_f.
$\Leftarrow$ X, containing the infinite set Per_f, is infinite. Now, let U be a non-empty open of X, $\tilde{U} := U \cap Per_f$ is a non-empty open of Per_f. For V non-empty open of X, $\tilde{V} := V \cap Per_f$ is a non-empty open of Per_f, then: $\mathcal{O}_{\tilde{f}}^-(\tilde{U}) \cap \tilde{V} \neq \emptyset$, ie $\mathcal{O}_f^-(U) \cap V \cap Per_f \neq \emptyset$, consequently: $\mathcal{O}_f^-(U) \cap V \neq \emptyset$, showing that $\mathcal{O}_f^-(U)$ is dense in X. $\square$

To prevent such a phenomenon to happen, we propose the following definition of chaos :

Definition 2.7. f is F-chaotic if:
(F1)=(B1) $\forall U$ non-empty open, $cl(\mathcal{O}_f^-(U)) = X$
(F2)=(B2) $cl(Per_f) = X$
(F3) $cl(X \setminus Per_f) = X$
Equivalently, f is F-chaotic if for every U non-empty open of X, $Per_f \cap \mathcal{O}_f^-(U)$ and $(X \setminus Per_f) \cap \mathcal{O}_f^-(U)$ are dense in X.

Here as well, if for some $n \geq 1$, f^n is F-chaotic, so is f and if f is a homeomorphism, f is F-chaotic if and only if f^{-1} is.

Theorem 2.8. If X is a Hausdorff space: (B'-chaos) $\Rightarrow$ (F-chaos) $\Rightarrow$ (B-chaos), hence if X is a Baire Hausdorff space with a coutable base for the topology: (B"-chaos) $\Leftrightarrow$ (B'-chaos) $\Leftrightarrow$ (F-chaos) $\Leftrightarrow$ (B-chaos).

Proof. X being Hausdorff, let us assume that f is B'-chaotic. If there exists $x \in \Delta_0 \cap Per_f$, $\mathcal{O}_f^+(x)$ would be finite, then closed, so $X = cl(\mathcal{O}_f^+(x))$ would be finite, contradicting (B'3). Consequently, $X \setminus Per_f \supseteq \Delta_0$ is dense in X, and f is F-chaotic. Now, if X is Hausdorff and f is F-chaotic, we have to show that $\#X = +\infty$. But if X was finite, Per_f and $X \setminus Per_f$, which are dense in X, would be finite, then closed, so $X = Per_f$ and $X = X \setminus Per_f$, which is of course impossible (provided that $X \neq \emptyset$). □

2.3. **Basic properties of the Stone-Čech compactification.**

Proposition 2.9. Let X be a completely regular Hausdorff space. The Stone-Čech compactification βX of X is a compact Hausdorff space such that X is homeomorphic to a dense subset of βX, via a homoemorphism δ. If X is itself compact, $\beta X = \delta(X)$. Furthermore, if X and Y are two completely regular Hausdorff spaces and if h is a continuous map from X to Y, there is a unique continuous map βh from βX to βY such that, for all $x \in X, (\beta h \circ \delta)(x) = (\delta \circ h)(x)$.

Let us remark that if X is compact Hausdorff (hence completely regular), f and βf are topologically conjugate, then f is chaotic if and only if βf is, with respect to every previous meaning of the word chaotic (indeed, it is rather easy to see that the B"-chaos, B'-chaos, F-chaos and B-chaos are preserved under topological conjugacy). What about the general case? Let us first state, without proof, the following lemma, before giving a general result.

Lemma 2.10. $\delta(X)$ is open in βX if and only if X is locally compact.

Lemma 2.11. If X is a completely regular Hausdorff space, for $A \subseteq \beta X$, if $A \cap \delta(X)$ is dense in $\delta(X)$, then A is dense in βX; the converse being true if X is locally compact.

Proof. Let V be a non-empty open of βX, $\tilde{V} := V \cap \delta(X)$ is a non-empty open of $\delta(X)$, so $\tilde{V} \cap A \cap \delta(X) \neq \emptyset$, then $V \cap A \neq \emptyset$, and A is dense in βX. Now, if X is locally compact, ie if $\delta(X)$ is open in βX, and if A is dense in βX, let $\tilde{V}$ be a non-empty open of $\delta(X)$. There exists V non-empty open of βX such that $\tilde{V} = V \cap \delta(X)$. $\tilde{V}$ is then an open of βX, so $\tilde{V} \cap A \neq \emptyset$, ie $\tilde{V} \cap A \cap \delta(X) \neq \emptyset$, showing that $A \cap \delta(X)$ is dense in $\delta(X)$. □

Theorem 2.12. If X is a completely regular Hausdorff space, (f is B-chaotic) $\Rightarrow$ (βf is B-chaotic), (f is F-chaotic) $\Rightarrow$ (βf is F-chaotic) and (f is B'-chaotic) $\Rightarrow$ (βf is B'-chaotic). If X is moreover locally compact, the converses are true: (f is B-chaotic) $\Leftrightarrow$ (βf is B-chaotic), (f is F-chaotic) $\Leftrightarrow$ (βf is F-chaotic) and (f is B'-chaotic) $\Leftrightarrow$ (βf is B'-chaotic).

Proof. $\beta f(\delta(X)) \subseteq \delta(X)$, because for all $x \in X$, $(\beta f \circ \delta)(x) = (\delta \circ f)(x)$. We then denote $\widetilde{\beta f}$ the restriction of βf to $\delta(X)$: f and $\widetilde{\beta f}$ are topologically conjugate, consequently, f is B-chaotic, or F-chaotic, or B'-chaotic, if and only if $\widetilde{\beta f}$ is.
Showing that $\delta(X)$ is infinite if and only if βX is does not present any difficulties. Assuming that $\widetilde{\beta f}$ is B-chaotic, for U non-empty open of βX, $\tilde{U} := U \cap \delta(X)$ is a non-empty open of $\delta(X)$, so $\mathcal{O}^-_{\widetilde{\beta f}}(\tilde{U}) = \mathcal{O}^-_{\beta f}(U) \cap \delta(X)$ is dense in $\delta(X)$, then, by the previous lemma, $\mathcal{O}^-_{\beta f}(U)$ is dense in βX. Furthermore, since $Per_{\widetilde{\beta f}} = Per_{\beta f} \cap \delta(X)$, $Per_{\beta f}$ is dense in βX. This proves that βf is B-chaotic. Now, if X is moreover

locally compact, and if βf is B-chaotic, again by lemma 2.11, $Per_{\widetilde{\beta f}}$ is dense in $\delta(X)$, and for $\tilde{U}$ non-empty open of $\delta(X)$, $\tilde{U} = U \cap \delta(X)$ for some U non-empty open of βX, so $\mathcal{O}^-_{\widetilde{\beta f}}(\tilde{U})$ is dense in $\delta(X)$. Hence $\widetilde{\beta f}$ is B-chaotic. Since $Per_{\widetilde{\beta f}} = Per_{\beta f} \cap \delta(X)$, we have: $\delta(X) \setminus Per_{\widetilde{\beta f}} = (\beta X \setminus Per_{\beta f}) \cap \delta(X)$, and the result for the F-chaos follows from the previous lemma with $A = \beta X \setminus Per_{\beta f}$. Now, if $\widetilde{\beta f}$ is B'-chaotic, it has a dense orbit, ie there is a $x \in X$ such that $\mathcal{O}^+_{\widetilde{\beta f}}(\delta(x)) = \mathcal{O}^+_{\beta f}(\delta(x)) \cap \delta(X)$ is dense in $\delta(X)$, then $\mathcal{O}^+_{\beta f}(\delta(x))$ is dense in βX, ie βf has a dense orbit. With what has just been done, we can conclude that βf is B'-chaotic. Supposing that X is locally compact, let us assume conversly that βf is B'-chaotic. Since βX is Hausdorff, $\Omega_{\beta f}$ is dense in βX, so $\Omega_{\beta f} \cap \delta(X)$ is dense in $\delta(X)$, then $\Delta_0^{\beta f} \cap \delta(X)$ is dense in $\delta(X)$. In particular, it is non-empty, ie there exists $x \in X$ such that $\mathcal{O}^+_{\beta f}(\delta(x))$ is dense in βX, so $\mathcal{O}^+_{\beta f}(\delta(x)) \cap \delta(X) = \mathcal{O}^+_{\widetilde{\beta f}}(\delta(x))$ is dense in $\delta(X)$, and $\Delta_0^{\widetilde{\beta f}} \neq \emptyset$. We can then conclude that $\widetilde{\beta f}$ is B'-chaotic. □

3. Chaos via entropy

For the very definition of the topological entropy (using open covers) of a continuous map f from a compact space X into itself, we will consult [1]. If we deal with a compact metric space, we can define the entropy in a more intuitive way, using (n, ϵ)-spanning subsets: roughly speaking, the entropy of f represents the exponential growth rate for the number of orbit segments distinguishable with arbitrarily fine but finite precision. For details, we will consult [7], where it is also shown that these two approches are equivalent. When X is compact, the topological entropy of f is written $h_{top}(f)$. Here is an overview of its properties: for $n \geq 1$, $h_{top}(f^n) = n h_{top}(f)$; if f is a homeomorphism, $h_{top}(f^{-1}) = h_{top}(f)$; and if X and Y are two compact spaces, with $g : Y \longrightarrow Y$ and $f : X \longrightarrow X$ topologically conjugate, $h_{top}(g) = h_{top}(f)$.

As already mentioned, if X is compact Hausdorff, f and βf are topologically conjugate, so that $h_{top}(f) = h_{top}(\beta f)$. This in fact allows us to define the topological entropy of f when X is just a completely regular Hausdorff space by $h_{top}(f) := h_{top}(\beta f)$, which verifies, as one easily checks:

- for $n \geq 1$, $h_{top}(f^n) = n h_{top}(f)$
- if f is a homeomorphism, $h_{top}(f^{-1}) = h_{top}(f)$
- if X and Y are two completely regular Hausdorff spaces, with $g : Y \longrightarrow Y$ and $f : X \longrightarrow X$ topologically conjugate, $h_{top}(g) = h_{top}(f)$.

Now, we can extend a very common definition of chaos :

Definition 3.1. If X is a completely regular Hausdorff space, we say that f is E-chaotic if $h_{top}(f) > 0$.

Here again, if for some $n \geq 1$, f^n is E-chaotic, so is f, the converse being also true, and if f is a homeomorphism, f is E-chaotic if and only if f^{-1} is. Let us remark that this latest situation is impossible on intervals, as we will see later.

4. Li and Yorke's type of chaos

Definition 4.1. If (X, d) is a metric space, an uncountable susbet S of X is called a scrambled set of f if for all $x, y \in S$, $x \neq y$, $\limsup_{n \to \infty} d(f^n(x), f^n(y))$ exists and is positive and $\liminf_{n\to\infty} d(f^n(x), f^n(y))$ exists and is null. f is called L-Y-chaotic if there exists a scrambled set of f.

Let us note that in this case again, if f^n is L-Y-chaotic for some $n \geq 1$, f is L-Y-chaotic as well.

Some authors, following Li and Yorke, define a scrambled set with the additional property: $\forall x \in S$, $\forall p \in Per_f$, $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$. We do not, because a scrambled set contains at most one point which does not satisfy this latest property. Indeed, if $x, y \in S$, $x \neq y$, are such that there exist periodic points p and q with $\limsup_{n\to\infty} d(f^n(x), f^n(p)) = 0$ and $\limsup_{n\to\infty} d(f^n(y), f^n(q)) = 0$, we have: $d(f^n(x), f^n(p)) \longrightarrow 0$ as $n \to \infty$, and $d(f^n(y), f^n(q)) \longrightarrow 0$ as $n \to \infty$. For $n \geq 0$, $d(f^n(p), f^n(q)) \leq d(f^n(p), f^n(x)) + d(f^n(x), f^n(y)) + d(f^n(y), f^n(q))$, consequently: $\liminf_{n\to\infty} d(f^n(p), f^n(q)) \leq 0 + \liminf_{n\to\infty} d(f^n(x), f^n(y)) + 0 = 0$. But, since p and q are periodic, $\{d(f^n(p), f^n(q)), n \geq 0\}$ is finite, so there exists $k \geq 0$ such that: $d(f^k(p), f^k(q)) = \liminf_{n\to\infty} d(f^n(p), f^n(q)) = 0$, so $f^i(p) = f^i(q)$ for all $i \geq k$. Then, if l and m are the periods of p and q respectively, $p = f^{lmk}(p) = f^{lmk}(q) = q$. Consequently, $d(f^n(x), f^n(y)) \leq d(f^n(x), f^n(p)) + d(f^n(q), f^n(y)) \longrightarrow 0$ as $n \to \infty$, which contradicts: $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.

To finish our brief description of the L-Y-chaos, we will just check that it is preserved under topological conjugacy. Strangely enough, I have not found a proof of this result in the litterature, and the proof I propose, deal, unfortunately, only with compact metric spaces.

Proposition 4.2. If (X, d) and (Y, d') are two compact metric spaces and $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ are two continuous, topologically conjugate maps, (f is L-Y-chaotic) $\Leftrightarrow$ (g is L-Y-chaotic).

Proof. Let $h : X \longrightarrow Y$ be the topological conjugacy between f and g and let $S \subseteq X$ be a scrambled set of f. $S' := h(S)$ is an uncountable subset of Y. Let $h(x), h(y) \in S'$, $h(x) \neq h(y)$, ie $x \neq y$. There exists an increasing sequence of positive integers $(n_k)_{k\geq 0}$ such that $d(f^{n_k}(x), f^{n_k}(y)) \longrightarrow 0$ as $k \to \infty$. h is uniformely continuous, since continuous on a compact, then $d'(h(f^{n_k}(x)), h(f^{n_k}(y))) \longrightarrow 0$ as $k \to \infty$. But for all $i \geq 0$, $h \circ f^i = g^i \circ h$, so $d'(g^{n_k}(h(x)), g^{n_k}(h(y))) \longrightarrow 0$ as $k \to \infty$, so: $\liminf_{n\to\infty} d'(g^n(h(x)), g^n(h(y))) = 0$. Let us now suppose that $\limsup_{n\to\infty} d'(g^n(h(x)), g^n(h(y))) = 0$ (it exists because Y is compact). Thus: $d'(g^n(h(x)), g^n(h(y))) = d'(h(f^n(x)), h(f^n(y))) \longrightarrow 0$ as $n \to \infty$, and because of the uniform continuity of h^{-1}, we then have: $d(f^n(x), f^n(y)) \longrightarrow 0$ as $n \to \infty$, contradicting the fact that S is a scrambled set of f. Hence: $\limsup_{n\to\infty} d'(g^n(h(x)), g^n(h(y))) > 0$. Finally, S' is a scrambled set of g, so (f is L-Y-chaotic)$\Rightarrow$(g is L-Y-chaotic). The converse is obtained by exchanging f and g. □

5. Chaos on intervals

5.1. **Devaney's-type-of-chaos block.** Let us denote I an interval, with the absolute value as a metric, and f a continuous self-map of I. I being a Baire Hausdorff space with countable base, on I: B"-chaos=B'-chaos=F-chaos=B-chaos. Now, let us recall the result given in [10]:

Proposition 5.1. On intervals, if f is topologically transitive, f is B-chaotic.

Thus, on intervals, we have what I call the Devaney's-type-of-chaos block:

topological transitivity = B-chaos = F-chaos = B'-chaos=B"-chaos

5.2. **Horseshoes and B-C-chaos.** Following [6], we now introduce an other definition of chaos, only valid on intervals:

Definition 5.2. We say that a continuous map $g : I \longrightarrow I$ has a horseshoe if there exist $a < c < b$ in I such that $[a, b] \subseteq g([a, c]) \cap g([c, b])$. f is said to be H-chaotic if f^m has a horseshoe for some $m \geq 1$.

Again, if f^n is H-chaotic for some $n \geq 1$, f is also H-chaotic. By induction, we remark that if g has a horseshoe, so has g^n, for all $n \geq 1$, and using this property, we see that if f is H-chaotic, so is f^n, for all $n \geq 1$. Moreover, we note that a homeomorphism can not be H-chaotic.

The definition of L-Y-chaos has followed the famous article by Li and Yorke, called "period three implies chaos". The following proposition, generalizing such a result, will explain what it means:

Proposition 5.3. On intervals, if f has a periodic point whose least period is not a power of 2, then f is H-chaotic.

Proof. First case: there exists a point of least period 3, let $x_1 < x_2 < x_3$ be the three distinct points of the considered orbit.
First subcase: $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$. $f(x_2) > x_2$ and $f(x_3) < x_3$, then, by the intermediate value theorem, there exists $z \in (x_2, x_3)$ such that $f(z) = z$. Next, $f(x_1) < z$ and $f(x_2) > z$, so there exists $y \in (x_1, x_2)$ with $f(y) = z$. But then: $f^2(y) = z > y$, $f^2(x_2) = x_1 < y$ and $f^2(z) = z > y$, so there exist $s \in (x_2, z)$, $r \in (y, x_2)$ such that $f^2(s) = y$ and $f^2(r) = y$. Thus: $[y, z] \subseteq f^2([y, r]) \subseteq f^2([y, x_2])$ and $[y, z] \subseteq f^2([s, z]) \subseteq f^2([x_2, z])$, showing that f^2 has a horseshoe.
Second subcase: $f(x_1) = x_3$, $f(x_3) = x_2$, $f(x_2) = x_1$. This is treated as before, exchanging $>$ and $<$, x_1 and x_3.
Second case: there is a point of least period $(2k + 1).2^n$, with $k \geq 1$, $n \geq 0$. By the Sharkovsky's theorem, there exists a point of least period 3.2^{n+1}. Thus $f^{2^{n+1}}$ has a periodic point of period 3, so $f^{2^{n+2}}$ has a horseshoe, and f is H-chaotic. □

This latest property can stand for a definition of chaos, in fact, it is the one used on intervals in [4]; it is clear that it can be extended to any topological space:

Definition 5.4. f is B-C-chaotic if f has a periodic point whose least period is not a power of 2.

We see that if f is a homeomorphism, f is B-C-chaotic if and only if f^{-1} is. Besides:

Proposition 5.5. If f^n is B-C-chaotic for some $n \geq 1$, then f is also B-C-chaotic, and on intervals, if f is B-C-chaotic, then for all $n \geq 1$, f^n is also B-C-chaotic.

Proof. Let us assume that f is not B-C-chaotic. Let p' denote the period, by f^n, of a $x \in Per_{f^n}$, and let p, which is a power of 2, denote its period by f. With $d := gcd(p, n)$, $d.p' = p$. Indeed, writing $d.m = p$ and $d.i = n$ with m and i relatively prime, we have $m.n = i.p$, so $f^{nm}(x) = x$, and if $f^{nk}(x) = x$, $p = d.m$ divises $n.k = d.i.k$, ie m divises $i.k$, so m divises k, proving that $p' = m$, as announced. This implies that p' is a power of 2, since p is, thus f^n is not B-C-chaotic.
Now, considering the situation on intervals, we assume that f is B-C-chaotic, ie that there exists a periodic point by f of period $p = 2^k.(2i + 1)$, with $k \geq 0$, $i \geq 1$.

Let us write $n = 2^l.(2j+1)$, with $l \geq 0$, $j \geq 0$. By the Sharkovsky's theorem, there exists a point of period $n.p = 2^{k+l}.(2i+1)(2j+1)$ by f. This point is a periodic point by f^n of period $n.p/(gcd(n.p,n)) = p$, which is not a period of 2, thus f^n is B-C-chaotic. □

5.3. **Entropy-type-of-chaos block.** We have noticed that the E-chaos, the H-chaos and the B-C-chaos have in common the fact that, on intervals, if f is chaotic, so is f^n, for all $n \geq 1$. This is not surprising, since, as we are going to see, they are the same. First we need a preliminary lemma:

Lemma 5.6. If there exists $x \in I$ such that $f^3(x) \leq x \leq f(x) < f^2(x)$ (or $f^3(x) \geq x \geq f(x) > f^2(x)$), there is a point of period 3 by f.

Proof. The following statements are easy to check:
Statement 1: for every continuous function $g : [a,b] \longrightarrow \mathbb{R}$, if $[a,b] \subseteq g([a,b])$, there exists a fixed point by g in $[a,b]$.
Statement 2: for every continuous function $g : I \longrightarrow \mathbb{R}$, I an interval, if $J \subseteq I$ and K are two compact intervals with $K \subseteq f(J)$, there exists a compact interval $L \subseteq J$ such that $K = f(L)$.
Now, let us write $K := [x, f(x)]$, we have $f([f(x), f^2(x)]) \supseteq [f^3(x), f^2(x)] \supseteq K$, so by statement 2, there exists a compact interval $L \subseteq [f(x), f^2(x)]$ such that $f(L) = K$. Then: $f^3(L) = f^2(K) \supseteq [f^3(x), f^2(x)] \supseteq L$, so by statement 1, there exists $y \in L$ such that $f^3(y) = y$. Assuming that 3 is not the least period of y, one must have: $f(y) = y$. In this case: $y \in L \cap f(L) \subseteq [f(x), f^2(x)] \cap [x, f(x)]$, ie $y = f(x)$, hence $f(x) = f^2(x)$, which is absurd. □

Theorem 5.7. On intervals, f is B-C-chaotic if and only if it is H-chaotic.

Proof. The $\Rightarrow$ part has already been shown. Now let us assume that there exists $n \geq 1$ such that $f^n =: g$ has a horseshoe, ie $[a,b] \subseteq g([a,c]) \cap g([c,b])$ for some $a < c < b$. $a \in g([c,b])$, so there exists $b_0 \in [c,b]$ such that $g(b_0) = a$.
First case: there exists $c_0 \in [c, b_0]$ such that $g(c_0) = b$, then $[a,b] \subseteq g([c_0, b_0])$. So there exists $y \in [c_0, b_0]$ such that $g(y) = b_0$. Then $y \in [a,b] \subseteq g([a,c])$, so there is $x \in [a,c]$ such that $g(x) = y$. We have: $x \leq c \leq c_0 \leq y = g(x)$, $g(x) = y \leq b_0 = g(y) = g^2(x)$. If $g(x) = g^2(x)$, $y = b_0$, so $g(y) = g(b_0)$, ie $b_0 = a$, contradicting $b_0 \in [c,b]$. Finally, $g^3(x) = g(b_0) = a \leq x$. In brief: $g^3(x) \leq x \leq g(x) < g^2(x)$.
Second case: $\forall u \in [c, b_0]$, $g(u) \neq b$, so there exists $d \in [b_0, b]$ such that $g(d) = b$. Since $[a,b] \subseteq g([a,c])$, there also exists $a_0 \in [a,c]$ such that $g(a_0) = b$. Thus, $[a,b] \subseteq g([b_0, d])$ and $[a,b] \subseteq g([a_0, b_0])$. Therefore, there exists $y \in [a_0, b_0]$ such that $g(y) = a_0$, and then there exists $x \in [b_0, d]$ such that $g(x) = y$. We have: $x \geq b_0 \geq y = g(x)$, $g(x) = y \geq a_0 = g(y) = g^2(x)$. If $g(x) = g^2(x)$, $y = a_0$, so $g(y) = g(a_0)$, ie $a_0 = b$, contradicting $a_0 \in [a,c]$. Finally, $g^3(x) = g(a_0) = b \geq x$. In brief: $g^3(x) \geq x \geq g(x) > g^2(x)$.
In each case, by the previous lemma, $g = f^n$ has a periodic point of least period 3, so f^n, and then f, is B-C-chaotic. □

Theorem 5.8. On intervals, f is H-chaotic if and only if it is E-chaotic.

In fact, this is the reason why H-chaos is the definition of chaos adopted in [6]; a proof of this result can be found in [7], p489-496 (as well as in [4], p208-218.)

Therefore, as we considered a Devaney's-type-of-chaos block, we can consider, on intervals, an entropy-type-of-chaos block:

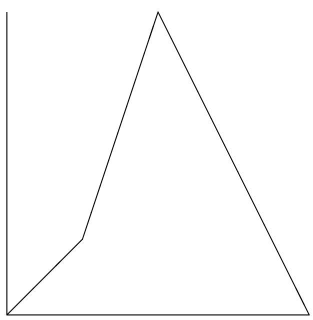
E-chaos = H-chaos = B-C-chaos

Let us not forget the L-Y-chaos block.

5.4. **The relations between the three blocks.** We claim that, on intervals, the Devaney's chaos is the strongest, whereas the Li and Yorke's chaos is the weakest.

Theorem 5.9. On intervals, the Devaney's type of chaos implies the entropy type of chaos, and the converse is not true.

Proof. Let us assume that Ω is dense in I but f is not H-chaotic. First, there exists $z \in int(I)$ such that $f(z) = z$. Indeed, assuming the contrary: $(\forall x \in int(I), f(x) > x)$ or $(\forall x \in int(I), f(x) < x)$, then: $(\forall x \in I, f(x) \geq x)$ or $(\forall x \in I, f(x) \leq x)$, therefore, no point of I could have a dense orbit. If there exists $y \in int(I)$, $y \neq z$, for example $y < z$, with $f(y) = z$, we can find a point $x \in \Omega \cap (y, z)$. Hence, there exists $n \geq 0$ such that $f^n(x) < y$ (necessarily $n \geq 1$). Then: $f^n([y, x]) \supseteq [f^n(x), f^n(y)] \supseteq [y, z]$ and $f^n([x, z]) \supseteq [f^n(x), f^n(z)] \supseteq [y, z]$. We obtain a horseshoe for f^n, which is impossible. Consequently, there is no $y \in int(I)$, $y \neq z$, such that $f(y) = z$. Let us now consider the two following open subintervals of I: $I_1 := int(I) \cap (-\infty, z)$; $I_2 := int(I) \cap (z, +\infty)$. $f(I_1) \subseteq I_1$ is uncompatible with the density of Ω in I, so there exists $x \in I_1$ such that $f(x) \geq z$. For $u \in I_1$, if $f(u) \leq z$, there exists y between x and u, so $y \in int(I) \setminus \{z\}$, such that $f(y) = z$, which is absurd. Hence: $f(I_1) \subseteq I_2$. Similarly, $f(I_2) \subseteq I_1$. Let us consider $g := f^2 : I_1 \longrightarrow I_1$. Taking U and V two non-empty open of I_1, they are also non-empty open of I, so there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$. But this can not occur if n is odd, because in this case, $f^n(U) \subseteq I_2$. So, $n =: 2m$ and $g^m(U) \cap V \neq \emptyset$. This means that g is topologically transitive. Applying the previous reasoning to g instead of f, we obtain a $t \in I_1$ with $f^2(t) = t$. We then write $I_{1,1} := I_1 \cap (-\infty, t)$ and $I_{1,2} := I_1 \cap (t, +\infty)$, and we have $g(I_{1,2}) \subseteq I_{1,1}$. But $z \in cl(I_{1,2})$, so $g(z) = z \in cl(I_{1,1})$, implying that $z \leq t$, which is absurd. This achieves the first part of the proof. (We have adapted here the arguments given in [1], p259-260, where it is stated that if f is topologically transitive, f^2 has an horseshoe, so that $h_{top}(f) \geq \log 2/2$.)
Now, the following map, for which there is a forward invariant open set, is not topologically transitive, still it is H-chaotic, showing the second part of the proof.



□

Theorem 5.10. On intervals, the entropy type of chaos implies the Li and Yorke chaos, and the converse is not true.

Proof. Let us assume that f^m has a horseshoe for some $m \geq 1$, so f^m has a point of period 3 (see the proof of theorem 5.7), thus there are some points $y < r < s < z$ such that: $[y,z] \subseteq f^{2m}([y,r])$ and $[y,z] \subseteq f^{2m}([s,z])$ (see the proof of proposition 5.3). We write $I_0 := [y,r]$, $I_1 := [s,z]$ and $g := f^{2m}$. We claim: $\forall n \geq 1$, $\forall u_0, u_1, ..., u_n \in \{0,1\}$, there exists a non-empty compact interval $I_{u_0 u_1 ... u_n} \subseteq I_{u_0 ... u_{n-1}}$ such that: $g(I_{u_0 ... u_n}) = I_{u_1 ... u_n}$. We prove this claim by induction on n. For $n = 1$, we have: $I_{u_1} \subseteq g(I_{u_0})$, so, by statement 2 in lemma 5.6, there exists a compact interval $I_{u_0 u_1} \subseteq I_{u_0}$ such that $g(I_{u_0 u_1}) = I_{u_1}$, and then necessarily $I_{u_0 u_1}$ is non-empty. Let us then assume that our claim is true untill an integer $n \geq 1$. By the induction hypothesis: $I_{u_1 ... u_{n+1}} \subseteq I_{u_1 ... u_n} = g(I_{u_0 ... u_n})$, so there exists a compact subinterval $I_{u_0 ... u_{n+1}} \subseteq I_{u_0 ... u_n}$ such that $g(I_{u_0 ... u_{n+1}}) = I_{u_1 ... u_{n+1}}$, and necessarily $I_{u_0 ... u_{n+1}}$ is non-empty, which shows that the claim is true for $n+1$, and achieves the proof by induction. We then set, for $u \in \{0,1\}^{\mathbb{N}}$, $I_u := \bigcap_{n \geq 0} I_{u_0 ... u_n}$. I_u is non-empty, otherwise, because of the compacity of $[y,z]$, there would exist $N \geq 0$ such that $\bigcap_{n=0}^{N} I_{u_0 ... u_n} = I_{u_0 ... u_N} = \emptyset$, which is not. Let us remark that, for all $u \in \{0,1\}^{\mathbb{N}}$, and for all $n \geq 0$,$(x \in I_{u_0 ... u_n}) \Rightarrow (\forall k \in \{0,...,n\}, g^k(x) \in I_{u_k})$. Indeed, for $k \in \{0,...,n\}$, $g^k(x) \in g^k(I_{u_0 ... u_n}) = I_{u_k ... u_n} \subseteq I_{u_k}$. We then easily see that if $x \in I_u$, for all $n \geq 0$, $g^n(x) \in I_{u_n}$. This implies in particular that for $u, v \in \{0,1\}^{\mathbb{N}}$, $(u \neq v) \Rightarrow (I_u \cap I_v = \emptyset)$. Moreover, I_u, as intersection of compact intervals, is a single point or an interval $[c,d]$, $c < d$. Let $C := \{u \in \{0,1\}^{\mathbb{N}} : l(I_u) := length(I_u) > 0\}$, $C = \bigcup_{n \geq 0} C_n$, where $C_n := \{u \in \{0,1\}^{\mathbb{N}} : l(I_u) > 1/(n+1)\}$. Taking $u^1, u^2, ..., u^k$ distinct elements of C_n, since $I_{u^1} \cup ... \cup I_{u^k} \subseteq [y,z]$, we have: $z - y \geq l(I_{u^1} \cup ... \cup I_{u^k}) = l(I_{u_1}) + ... + l(I_{u_k}) > k/(n+1)$, ie: $k < (n+1)(z-y)$, meaning that C_n is finite, and then C is countable. For $u \in \{0,1\}^{\mathbb{N}} \setminus C$, we then write $I_u =: \{x_u\}$, and fixing $a \in \{0,1\}^{\mathbb{N}} \setminus C$, we set: $S := \{x_u, u := u(b) := a_0 b_0 a_0 a_1 b_0 b_1 a_0 a_1 a_2 b_0 b_1 b_2 ... \notin C, b \in \{0,1\}^{\mathbb{N}}\}$. We will show that S is a scrambled set of f. Since $\{0,1\}^{\mathbb{N}}$ is uncoutable, since $b \in \{0,1\}^{\mathbb{N}} \mapsto u(b) \in \{0,1\}^{\mathbb{N}}$ is injective, since C is countable, and since $u \in \{0,1\}^{\mathbb{N}} \setminus C \mapsto x_u \in [y,z]$ is injective, S is uncountable. Now let $x_u, x_v \in S$, $x_u \neq x_v$, ie there exists $i \geq 0$ such that $b_i \neq b'_i$, where $u =: a_0 b_0 a_0 a_1 b_0 b_1 a_0 a_1 a_2 b_0 b_1 b_2 ...$ and $v =: a_0 b'_0 a_0 a_1 b'_0 b'_1 a_0 a_1 a_2 b'_0 b'_1 b'_2 ...$. Since for all $n \geq 0$ and $k \in \{0,...,n\}$, $u_{2(1+2+...+n)+k} = u_{n(n+1)+k} = a_k = v_{n(n+1)+k}$, $u_{2(1+2+...+n)+n+1+k} = u_{(n+1)^2+k} = b_k$ and $v_{(n+1)^2+k} = b'_k$, we have, for all $n \geq i$, $g^{(n+1)^2+i}(x_u) \in I_{b_i}$ and $g^{(n+1)^2+i}(x_v) \in I_{b'_i}$, so $|g^{(n+1)^2+i}(x_u) - g^{(n+1)^2+i}(x_v)| \geq s - r > 0$. From $(|g^{(n+1)^2+i}(x_u) - g^{(n+1)^2+i}(x_v)|)_{n \geq 0}$, we now can extract a convergent subsequence, showing that: $\limsup_{n \to \infty} |f^n(x_u) - f^n(x_v)| \geq s - r > 0$. Let us now assume that $\alpha := \liminf_{n \to \infty} |f^n(x_u) - f^n(x_v)| > 0$ (α exists because the sequence is bounded). Since $a \notin C$, I_a is a point, then there exists $k \geq 0$ such that $l(I_{a_0 ... a_k}) =: \beta < \alpha$. We have, for all $n \geq k$, since $x_u \in I_{u_0 ... u_{n(n+1)+n}}$, $g^{n(n+1)}(x_u) \in I_{u_{n(n+1)} ... u_{n(n+1)+n}} = I_{a_0 ... a_n} \subseteq I_{a_0 ... a_k}$, and similarly: $g^{n(n+1)}(x_v) \in I_{a_0 ... a_k}$, hence: $|g^{n(n+1)}(x_u) - g^{n(n+1)}(x_v)| \leq \beta$. Now, from the sequence $(|g^{n(n+1)}(x_u) - g^{n(n+1)}(x_v)|)_{n \geq 0}$, we can extract a subsequence converging to γ, say. We have: $\gamma \leq \beta < \alpha$, which is absurd. Finally, $\liminf_{n \to \infty} |f^n(x_u) - f^n(x_v)| = 0$, and this achieves the first part of the proof.

For the second part, we can find an example of a L-Y-chaotic map which is not E-chaotic in [5]. □

The situation on intervals is summed up by the following diagram:

Devaney's type of chaos

$\Downarrow\not\Uparrow$

Entropy type of chaos

$\Downarrow\not\Uparrow$

Li and Yorke chaos

6. A non-trivial chaotic map

If (X,d) is a metric space, let $(\mathcal{K}(X),\Delta)$ be the metric space of all compact subsets of X, where Δ is the Hausdorff metric: $\forall A,B \in \mathcal{K}(X)$, $\Delta(A,B) := sup\{d(x,B), x \in A\} + sup\{d(y,A), y \in B\}$. Let us consider the map $\mathcal{F}$ from $\mathcal{K}(X)$ to $\mathcal{K}(X)$ defined by: $\forall A \in \mathcal{K}(X)$, $\mathcal{F}(A) := f(A)$. We know that if (X,d) is compact, $(\mathcal{K}(X),\Delta)$ is also compact, so, for f and $\mathcal{F}$ (which is continuous, as we are going to prove), D-chaos, B-chaos, F-chaos, D'-chaos, B'-chaos and B"-chaos are all the same.

Let us remind that f is said to be topologically mixing if for every U and V non-empty open of X, there exists $m \geq 0$ such that: $\forall n \geq m$, $U \cap f^n(V) \neq \emptyset$. On intervals, a topologically mixing map is B-chaotic, since it is topologically transitive. For example, if f is the tent map or the logistic map, f is topologically mixing, then, as a result of the next proposition, $\mathcal{F}$ is topologically mixing and B-chaotic, it is also E-chaotic, B-C-chaotic and L-Y-chaotic.

Proposition 6.1. If (X,d) is a compact metric space, (f is D-chaotic and topologically mixing) $\Rightarrow$ ($\mathcal{F}$ is D-chaotic and topologically mixing), (f is E-chaotic) $\Rightarrow$ ($\mathcal{F}$ is E-chaotic), (f is B-C-chaotic) $\Rightarrow$ ($\mathcal{F}$ is B-C-chaotic) and (f is L-Y-chaotic) $\Rightarrow$ ($\mathcal{F}$ is L-Y-chaotic)

Proof. First of all, let us check that $\mathcal{F}$ is continuous. Let $\epsilon > 0$, by the uniform continuity of f: $\exists \alpha > 0$: $\forall x,y \in X$, $(d(x,y) < \alpha) \Rightarrow (d(f(x),f(y)) < \epsilon/2)$. Let $A,B \in \mathcal{K}(X)$ with $\Delta(A,B) < \alpha$. For all $x \in A$, $d(x,B) \leq sup\{d(t,B), t \in A\} \leq \Delta(A,B) < \alpha$. But, there exists $b \in B$ such that $d(x,b) = d(x,B)$ (since $d(x,-)$ is continuous on the compact B). $d(x,b) < \alpha$, then $d(f(x),f(b)) < \epsilon/2$. We have: $d(f(x),\mathcal{F}(B)) < \epsilon/2$, holding for all $x \in A$, so: $sup\{d(f(x),\mathcal{F}(B)), x \in A\} \leq \epsilon/2$. Likewise, we obtain: $sup\{d(f(y),\mathcal{F}(A)), y \in B\} \leq \epsilon/2$. Finally, $\Delta(A,B) \leq \epsilon$, proving that $\mathcal{F}$ is uniformely continuous. Next, $\#\mathcal{K}(X) = +\infty$, since $\#X = +\infty$ and for all $x \in X$, $\{x\} \in \mathcal{K}(X)$. Let us now show that $Per_{\mathcal{F}}$ is dense in $\mathcal{K}(X)$. Let then $A \in \mathcal{K}(X)$ and $\epsilon > 0$. A being compact, there exist $N \geq 0$, $x_1, ..., x_N \in A$ such that: $A \subseteq \bigcup_{i=1}^{N} B(x_i,\epsilon/3)$, where $B(x_i,\epsilon/3)$ is the open ball (in X) of center x_i and of radius $\epsilon/3$. But Per_f is dense in X, so for all $i \in \{1,...,N\}$, we can take a point $p_i \in (Per_f \cap B(x_i,\epsilon/3))$. We then set $B := \{p_1,...,p_N\}$. Clearly, $B \in \mathcal{K}(X)$ and $B \in Per_{\mathcal{F}}$. For $i \in \{1,...,N\}$, $d(p_i,A) \leq d(p_i,x_i) < \epsilon/3$, then: $sup\{d(y,A), y \in B\} < \epsilon/3$. For $x \in A$, there exists $i \in \{1,...,N\}$ such that $d(x,x_i) < \epsilon/3$. Consequently: $d(x,B) \leq d(x,p_i) \leq d(x,x_i) + d(x_i,p_i) < \epsilon/3 + \epsilon/3 = 2\epsilon/3$, hence: $sup\{d(x,B), x \in A\} \leq 2\epsilon/3$. It follows that $\Delta(A,B) < \epsilon$, which proves the density of $Per_{\mathcal{F}}$ in $\mathcal{K}(X)$. It remains to show that $\mathcal{F}$ is topologically mixing (and therefore topologically transitive). It is easy to see that for a metric space (Y,δ), a map g from Y to Y is topologically mixing if and only if for all $x,y \in Y$, for all $\epsilon > 0$, there exists $m \geq 0$ such that for all $n \geq m$, there exists $z_n \in Y$

such that $\delta(x, z_n) < \epsilon$ and $\delta(y, f^n(z_n)) < \epsilon$. Let then $A, B \in \mathcal{K}(X)$, and let $\epsilon > 0$. Since A and B are compact, there exist $N \geq 0$, $x_1, ..., x_N \in A$, $y_1, ..., y_N \in B$ such that: $A \subseteq \bigcup_{i=1}^{N} B(x_i, \epsilon/3)$, $B \subseteq \bigcup_{i=1}^{N} B(y_i, \epsilon/3)$. But f being topologically mixing, for all $i \in \{1, ..., N\}$, $\exists m_i \geq 0$: $\forall n \geq m_i$, $\exists z_{i,n} \in X$: $d(z_{i,n}, x_i) < \epsilon/3$ and $d(y_i, f^n(z_{i,n})) < \epsilon/3$. Let $m := max\{m_i, i \in \{1, ..., N\}\}$, and let $n \geq m$. We write $C_n := \{z_{1,n}, ..., z_{N,n}\}$. Obviously, $C_n \in \mathcal{K}(X)$. On the one hand, for $i \in \{1, ..., N\}$, $d(z_{i,n}, A) \leq d(z_{i,n}, x_i) < \epsilon/3$, so: $sup\{d(y, A), y \in C_n\} < \epsilon/3$, and for $x \in A$, there exists $i \in \{1, ..., N\}$ such that $d(x, x_i) < \epsilon/3$, so: $d(x, C_n) \leq d(x, z_{i,n}) \leq d(x, x_i) + d(x_i, z_{i,n}) < \epsilon/3 + \epsilon/3 = 2\epsilon/3$, hence: $sup\{d(x, C_n), x \in A\} \leq 2\epsilon/3$. Finally: $\Delta(A, C_n) < \epsilon$. On the other hand, for $i \in \{1, ..., N\}$, $d(f^n(z_{i,n}), B) \leq d(f^n(z_{i,n}), yi) < \epsilon/3$, so: $sup\{d(x, B), x \in \mathcal{F}^n(C_n)\} < \epsilon/3$, and for $y \in B$, there exists $i \in \{1, ..., N\}$ such that $d(y, y_i) < \epsilon/3$, so: $d(y, \mathcal{F}^n(C_n)) \leq d(y, f^n(z_{i,n})) \leq d(y, y_i) + d(y_i, f^n(z_{i,n})) < \epsilon/3 + \epsilon/3 = 2\epsilon/3$, hence: $sup\{d(y, \mathcal{F}^n(C_n)), y \in B\} \leq 2\epsilon/3$. Finally: $\Delta(B, \mathcal{F}^n(C_n)) < \epsilon$. This proves that $\mathcal{F}$ is topologically mixing.
Now, let us assume that f is E-chaotic. The map $h : x \in X \mapsto \{x\} \in \mathcal{K}(X)$ is injective, and verifies: $h \circ f = \mathcal{F} \circ h$, so $h_{top}(F) \geq h_{top}(f)$ (as proved in [ALlM], p 192), then $\mathcal{F}$ is E-chaotic.
If f is B-C-chaotic, so is $\mathcal{F}$, because if $x \in Per_f$, $\{x\} \in Per_{\mathcal{F}}$, with the same least period.
At last, it is easy to see that if S is a scrambled set of f, $\{\{s\} \in \mathcal{K}(X), s \in S\}$ is a scrambled set of $\mathcal{F}$, remarking that for $a, b \in X$, $\Delta(\{a\}, \{b\}) = 2d(a, b)$. □

We note that this process can be iterated again and again.

References

[1] Ll.Alseda, J.Llibre and M.Misiurewicz, "Combinatorial dynamics and entropy in dimension one", World Scientific, 1993.
[2] J.Banks, J.Brooks, G.Cairns, G.Davis and P.Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly, April 1992, 332-334.
[3] R.L.Devaney, "An introduction to chaotic dynamical systems", Addison–Wesley, 1989.
[4] L.S.Block and W.A.Coppel, "Dynamics in one dimension", Springer–Verlag, 1992.
[5] B.S.Du, *Smooth weakly chaotic interval map with zero topological entropy*, in "Dynamical systems and related topics" (ed. K.Shiraiwa), World Scientific, 1991.
[6] P.Glendinning, "Stability, unstability and chaos: an introduction to the theory of nonlinear differential equations", Cambridge texts in Applied Mathematics, 1994.
[7] A.Katok and B.Hasselblat, "Introduction to the modern theory of dynamical systems", Cambridge University Press, 1995.
[8] C.Knudsen, *Chaos without nonperiodicity*, Amer. Math. Monthly, June-July 1994, 563-565.
[9] H.Lehning, *Ensemble limite d'une suite chaotique*, Revue des Mathématiques Spéciales, October 1997.
[10] M.Vellekop and R.Berglund, *On intervals, transitivity=chaos*, Amer. Math. Monthly, April 1994, 353-355.

University of Cambridge, DAMTP, Silver Street, Cambridge CB3 9EW, United Kingdom
E-mail address: sf259@maths.cam.ac.uk