# On the least condition number of a basis of quadratic polynomials

Simon Foucart

University of Cambridge, UK
Department of Applied Mathematics and Theoretical Physics
Numerical Analysis Group

## Condition number of a basis

For $\underline{v}$ a basis of a $n$-dimensional normed space $V$, the $\ell_\infty$-condition number of $\underline{v}$ is:

$$\kappa_\infty(\underline{v}) := \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\|\sum_{i=1}^n \alpha_i v_i\|_V}{\|\alpha\|_\infty} \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\|\alpha\|_\infty}{\|\sum_{i=1}^n \alpha_i v_i\|_V}$$

## Condition number of a basis

For $\underline{v}$ a basis of a $n$-dimensional normed space $V$, the $\ell_\infty$-condition number of $\underline{v}$ is:

$$\kappa_\infty(\underline{v}) := \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\| \sum_{i=1}^n \alpha_i v_i \|_V}{\|\alpha\|_\infty} \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\|\alpha\|_\infty}{\| \sum_{i=1}^n \alpha_i v_i \|_V}$$

Particular interest in the $\ell_\infty$-condition number of the B-spline basis of order $k$, $\kappa_\infty(\underline{B}_k)$, and in its supremum over all knot sequences, $\kappa_{\infty,k}$ (whose smallness insures the stability of numerical computations).

## Condition number of a basis

For $\underline{v}$ a basis of a $n$-dimensional normed space $V$, the $\ell_\infty$-condition number of $\underline{v}$ is:

$$\kappa_\infty(\underline{v}) := \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\| \sum_{i=1}^n \alpha_i v_i \|_V}{\|\alpha\|_\infty} \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\|\alpha\|_\infty}{\| \sum_{i=1}^n \alpha_i v_i \|_V}$$

Particular interest in the $\ell_\infty$-condition number of the B-spline basis of order $k$, $\kappa_\infty(\underline{B}_k)$, and in its supremum over all knot sequences, $\kappa_{\infty,k}$ (whose smallness insures the stability of numerical computations).

$\kappa_{\infty,k} < k 2^k$ (Scherer, Shadrin, 99) and $\kappa_{\infty,k} \geq \dfrac{k-1}{k} 2^{k-\frac{3}{2}}$ (Lyche, 78)

## Condition number of a basis

For $\underline{v}$ a basis of a $n$-dimensional normed space $V$, the $\ell_\infty$-condition number of $\underline{v}$ is:

$$\kappa_\infty(\underline{v}) := \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\| \sum_{i=1}^n \alpha_i v_i \|_V}{\|\alpha\|_\infty} \sup_{\alpha \in \ell_\infty^n \setminus \{0\}} \frac{\|\alpha\|_\infty}{\| \sum_{i=1}^n \alpha_i v_i \|_V}$$

Particular interest in the $\ell_\infty$-condition number of the B-spline basis of order $k$, $\kappa_\infty(\underline{B}_k)$, and in its supremum over all knot sequences, $\kappa_{\infty,k}$ (whose smallness insures the stability of numerical computations).

$\kappa_{\infty,k} < k2^k$ (Scherer, Shadrin, 99) and $\kappa_{\infty,k} \geq \dfrac{k-1}{k} 2^{k-\frac{3}{2}}$ (Lyche, 78)

Computations suggest that $\kappa_{\infty,k} \underset{k\to\infty}{\sim} 2^{k-\frac{3}{2}}$ (de Boor, 76).

## Condition number: other formalism

$U$ and $V$ two normed spaces, $T : U \to V$ isomorphism, the condition number of $T$ is:

$$\kappa(T) := \|T\|\|T^{-1}\|$$

## Condition number: other formalism

$U$ and $V$ two normed spaces, $T : U \to V$ isomorphism, the condition number of $T$ is:

$$\kappa(T) := \|T\| \|T^{-1}\|$$

With $\underline{v}$ a basis of a $n$-dimensional normed space $V$ , $\underline{e}$ the canonical basis of $\ell_\infty^n$ and $T : \underline{e} \mapsto \underline{v}$, we have: $\kappa_\infty(\underline{v}) = \kappa(T)$.

## Condition number: other formalism

$U$ and $V$ two normed spaces, $T : U \to V$ isomorphism, the condition number of $T$ is:

$$\kappa(T) := \|T\|\|T^{-1}\|$$

With $\underline{v}$ a basis of a $n$-dimensional normed space $V$ , $\underline{e}$ the canonical basis of $\ell_\infty^n$ and $T : \underline{e} \mapsto \underline{v}$, we have: $\kappa_\infty(\underline{v}) = \kappa(T)$.
The Banach-Mazur distance between $U$ and $V$ is:

$$d(U,V) := \inf\{\kappa(T), T : U \to V \text{ isomorphism}\}$$

## Condition number: other formalism

$U$ and $V$ two normed spaces, $T : U \to V$ isomorphism, the condition number of $T$ is:

$$\kappa(T) := \|T\|\|T^{-1}\|$$

With $\underline{v}$ a basis of a $n$-dimensional normed space $V$ , $\underline{e}$ the canonical basis of $\ell_\infty^n$ and $T : \underline{e} \mapsto \underline{v}$, we have: $\kappa_\infty(\underline{v}) = \kappa(T)$.
The Banach-Mazur distance between $U$ and $V$ is:

$$d(U, V) := \inf\{\kappa(T), T : U \to V \text{ isomorphism}\}$$

Although they play a prominent role in the geometry of Banach spaces, few of those distances have been calculated.

## Condition number: other formalism

$U$ and $V$ two normed spaces, $T : U \to V$ isomorphism, the condition number of $T$ is:

$$\kappa(T) := \|T\| \|T^{-1}\|$$

With $\underline{v}$ a basis of a $n$-dimensional normed space $V$ , $\underline{e}$ the canonical basis of $\ell_\infty^n$ and $T : \underline{e} \mapsto \underline{v}$, we have: $\kappa_\infty(\underline{v}) = \kappa(T)$.
The Banach-Mazur distance between $U$ and $V$ is:

$$d(U, V) := \inf\{\kappa(T), T : U \to V \text{ isomorphism}\}$$

Although they play a prominent role in the geometry of Banach spaces, few of those distances have been calculated.
We will focus on $d(\mathcal{P}_2, \ell_\infty^3)$, with in mind its close connection with minimal projections onto $\mathcal{P}_2$.

## Projection constants

Let $X$ be a normed space, and $U$ one of its subspaces.

## Projection constants

Let $X$ be a normed space, and $U$ one of its subspaces. The projection constant of $U$ (with respect to $X$) is:

$$p(U) := \inf\{\|P\|, P : X \twoheadrightarrow U \text{ continuous projection}\} \geq 1$$

## Projection constants

Let $X$ be a normed space, and $U$ one of its subspaces. The projection constant of $U$ (with respect to $X$) is:

$$p(U) := \inf\{\|P\|, P : X \twoheadrightarrow U \text{ continuous projection}\} \geq 1$$

If $X = C([-1,1])$ is equipped with the supremum norm, the interpolating projection constant of $U$ (with respect to $C([-1,1])$) is:

$$p_{int}(U) := \inf\{\|P\|, P : C([-1,1]) \twoheadrightarrow U \text{ interpolating projection}\} \geq 1$$

## Fundamental inequalities: part 1

$X$ a normed space, $U$ a $n$-dimensional subspace of $X$, we have:

$$p(U) \le d(U, \ell_\infty^n) =: \kappa_\infty(U)$$

## Fundamental inequalities: part 1

$X$ a normed space, $U$ a $n$-dimensional subspace of $X$, we have:

$$p(U) \le d(U, \ell_\infty^n) =: \kappa_\infty(U)$$

*Proof.* There exists $T : U \to \ell_\infty^n$ such that $\kappa(T) = d(U, \ell_\infty^n)$.

## Fundamental inequalities: part 1

$X$ a normed space, $U$ a $n$-dimensional subspace of $X$, we have:

$$p(U) \leq d(U, \ell_\infty^n) =: \kappa_\infty(U)$$

*Proof.* There exists $T : U \to \ell_\infty^n$ such that $\kappa(T) = d(U, \ell_\infty^n)$. There exists $\tilde{T} : X \to V$ extending $T$ with $\|\tilde{T}\| = \|T\|$.

**Fundamental inequalities: part 1**

$X$ a normed space, $U$ a $n$-dimensional subspace of $X$, we have:

$$p(U) \leq d(U, \ell_\infty^n) =: \kappa_\infty(U)$$

*Proof.* There exists $T : U \to \ell_\infty^n$ such that $\kappa(T) = d(U, \ell_\infty^n)$. There exists $\tilde{T} : X \to V$ extending $T$ with $\|\tilde{T}\| = \|T\|$. Then, $p(U) \leq \|T^{-1}\tilde{T}\|$

## Fundamental inequalities: part 1

$X$ a normed space, $U$ a $n$-dimensional subspace of $X$, we have:

$$p(U) \leq d(U, \ell_\infty^n) =: \kappa_\infty(U)$$

*Proof.* There exists $T : U \to \ell_\infty^n$ such that $\kappa(T) = d(U, \ell_\infty^n)$. There exists $\tilde{T} : X \to V$ extending $T$ with $\|\tilde{T}\| = \|T\|$. Then, $p(U) \leq \|T^{-1}\tilde{T}\| \leq \|T^{-1}\|\|\tilde{T}\|$

## Fundamental inequalities: part 1

$X$ a normed space, $U$ a $n$-dimensional subspace of $X$, we have:

$$p(U) \leq d(U, \ell_\infty^n) =: \kappa_\infty(U)$$

*Proof.* There exists $T : U \to \ell_\infty^n$ such that $\kappa(T) = d(U, \ell_\infty^n)$. There exists $\tilde{T} : X \to V$ extending $T$ with $\|\tilde{T}\| = \|T\|$. Then, $p(U) \leq \|T^{-1}\tilde{T}\| \leq \|T^{-1}\|\|\tilde{T}\| = \kappa_\infty(T)$. □

**Fundamental inequalities: part 2**

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \leq p_{int}(U)$$

**Fundamental inequalities: part 2**

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \le p_{int}(U)$$

*Proof.* For $P = \sum_{i=1}^n \bullet(x_i) l_i \ : C([-1,1]) \twoheadrightarrow U$ an interpolating projection,

## Fundamental inequalities: part 2

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \leq p_{int}(U)$$

*Proof.* For $P = \sum_{i=1}^n \bullet(x_i) l_i \; : C([-1,1]) \twoheadrightarrow U$ an interpolating projection, $\|P\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$ (Lebesgue function).

**Fundamental inequalities: part 2**

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \le p_{int}(U)$$

*Proof.* For $P = \sum_{i=1}^n \bullet(x_i) l_i \ : C([-1,1]) \twoheadrightarrow U$ an interpolating projection, $\|P\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$ (Lebesgue function). If $T : \ell_\infty^n \to U$ maps $\underline{e}$ to $\underline{l}$,

**Fundamental inequalities: part 2**

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \leq p_{int}(U)$$

*Proof.* For $P = \sum_{i=1}^n \bullet(x_i) l_i \ : C([-1,1]) \twoheadrightarrow U$ an interpolating projection, $\|P\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$ (Lebesgue function). If $T : \ell_\infty^n \to U$ maps $\underline{e}$ to $\underline{l}$, $\|T\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$

## Fundamental inequalities: part 2

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \le p_{int}(U)$$

*Proof.* For $P = \sum_{i=1}^n \bullet(x_i) l_i \ : C([-1,1]) \twoheadrightarrow U$ an interpolating projection, $\|P\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$ (Lebesgue function). If $T : \ell_\infty^n \to U$ maps $\underline{e}$ to $\underline{l}$, $\|T\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$ , and $\|T^{-1}\| \le 1$

## Fundamental inequalities: part 2

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \le p_{int}(U)$$

*Proof.* For $P = \sum_{i=1}^n \bullet(x_i) l_i \; : C([-1,1]) \twoheadrightarrow U$ an interpolating projection, $\|P\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$ (Lebesgue function). If $T : \ell_\infty^n \to U$ maps $\underline{e}$ to $\underline{l}$, $\|T\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$, and $\|T^{-1}\| \le 1$ (equality if $U$ contains the constants).

**Fundamental inequalities: part 2**

$U$ a $n$-dimensional subspace of $C([-1,1])$, we have:

$$\kappa_\infty(U) := d(U, \ell_\infty^n) \leq p_{int}(U)$$

*Proof.* For $P = \sum_{i=1}^n \bullet(x_i) l_i \ : C([-1,1]) \twoheadrightarrow U$ an interpolating projection, $\|P\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$ (Lebesgue function). If $T : \ell_\infty^n \to U$ maps $\underline{e}$ to $\underline{l}$, $\|T\| = \left\| \sum_{i=1}^n |l_i| \right\|_{[-1,1]}$, and $\|T^{-1}\| \leq 1$ (equality if $U$ contains the constants). Thus, $\kappa(\underline{l}) \leq \|P\|$. □

## Fundamental inequalities: conclusion and problem raised

$U$ a $n$-dimensional subspace of $C([-1,1])$:

$$p(U) \leq \kappa_\infty(U) \leq p_{int}(U)$$

## Fundamental inequalities: conclusion and problem raised

$U$ a $n$-dimensional subspace of $C([-1,1])$:

$$p(U) \leq \kappa_\infty(U) \leq p_{int}(U)$$

**Problem.** Is one of these inequalities an equality, in particular in the case $U = \mathcal{P}_n$, the space of real algebraic polynomials of degree at most $n$ on $[-1,1]$?

## Fundamental inequalities: conclusion and problem raised

$U$ a $n$-dimensional subspace of $C([-1,1])$:

$$p(U) \leq \kappa_\infty(U) \leq p_{int}(U)$$

**Problem.** Is one of these inequalities an equality, in particular in the case $U = \mathcal{P}_n$, the space of real algebraic polynomials of degree at most $n$ on $[-1,1]$?
Yes, if $n = 0$ and $n = 1$.

## Fundamental inequalities: conclusion and problem raised

$U$ a $n$-dimensional subspace of $C([-1,1])$:

$$p(U) \leq \kappa_\infty(U) \leq p_{int}(U)$$

**Problem.** Is one of these inequalities an equality, in particular in the case $U = \mathcal{P}_n$, the space of real algebraic polynomials of degree at most $n$ on $[-1,1]$?

Yes, if $n = 0$ and $n = 1$. For $n = 2$:

$p(\mathcal{P}_2) \approx 1.220\ldots$ (Chalmers, Metcalf, 90) and $p_{int}(\mathcal{P}_2) = \dfrac{5}{4} = 1.25$

**Fundamental inequalities: conclusion and problem raised**

$U$ a $n$-dimensional subspace of $C([-1,1])$:

$$p(U) \leq \kappa_\infty(U) \leq p_{int}(U)$$

**Problem.** Is one of these inequalities an equality, in particular in the case $U = \mathcal{P}_n$, the space of real algebraic polynomials of degree at most $n$ on $[-1,1]$?

Yes, if $n = 0$ and $n = 1$. For $n = 2$:

$p(\mathcal{P}_2) \approx 1.220\ldots$ (Chalmers, Metcalf, 90) and $p_{int}(\mathcal{P}_2) = \dfrac{5}{4} = 1.25$

We will prove: $\kappa_\infty(\mathcal{P}_2) \lessapprox 1.248\ldots < p_{int}(\mathcal{P}_2)$.

**Determining $\|T\|$**

For $\underline{p} = (p_1, p_2, p_3)$ basis of $\mathcal{P}_2$, $T$ the isomorphism mapping $\underline{e}$ to $\underline{p}$:

**Determining $\|T\|$**

For $\underline{p} = (p_1, p_2, p_3)$ basis of $\mathcal{P}_2$, $T$ the isomorphism mapping $\underline{e}$ to $\underline{p}$:

$$\|T\| = \sup_{|\alpha_i| \le 1} \left\| \sum \alpha_i p_i \right\|_{[-1,1]}$$

## Determining $\|T\|$

For $\underline{p} = (p_1, p_2, p_3)$ basis of $\mathcal{P}_2$, $T$ the isomorphism mapping $\underline{e}$ to $\underline{p}$:

$$\|T\| = \sup_{|\alpha_i| \leq 1} \left\| \sum \alpha_i p_i \right\|_{[-1,1]} = \| |p_1| + |p_2| + |p_3| \|_{[-1,1]}$$

**Determining $\|T\|$**

For $\underline{p} = (p_1, p_2, p_3)$ basis of $\mathcal{P}_2$, $T$ the isomorphism mapping $\underline{e}$ to $\underline{p}$:

$$\|T\| = \sup_{|\alpha_i| \leq 1} \left\| \sum \alpha_i p_i \right\|_{[-1,1]} = \| |p_1| + |p_2| + |p_3| \|_{[-1,1]} =: \|g\|_{[-1,1]}$$

**Determining $\|T\|$**

For $\underline{p} = (p_1, p_2, p_3)$ basis of $\mathcal{P}_2$, $T$ the isomorphism mapping $\underline{e}$ to $\underline{p}$:

$$\|T\| = \sup_{|\alpha_i| \le 1} \left\| \sum \alpha_i p_i \right\|_{[-1,1]} = \| |p_1| + |p_2| + |p_3| \|_{[-1,1]} =: \|g\|_{[-1,1]}$$

Our basis of interest is $\underline{p}(b, c, d)$, the symmetric basis defined by:

$$p_1(x) := \frac{x(x+b)}{2d} \;\; ; \;\; p_2(x) := C - x^2 \;\; ; \;\; p_3(x) := \frac{x(x-b)}{2d}$$

with $b, c \in (0, 1]$, $d > 0$, and $C := c^2$.

**Determining $\|T\|$**

For $\underline{p} = (p_1, p_2, p_3)$ basis of $\mathcal{P}_2$, $T$ the isomorphism mapping $\underline{e}$ to $\underline{p}$:

$$\|T\| = \sup_{|\alpha_i| \leq 1} \left\| \sum \alpha_i p_i \right\|_{[-1,1]} = \| |p_1| + |p_2| + |p_3| \|_{[-1,1]} =: \|g\|_{[-1,1]}$$

Our basis of interest is $\underline{p}(b, c, d)$, the symmetric basis defined by:

$$p_1(x) := \frac{x(x+b)}{2d} \ ; \ p_2(x) := C - x^2 \ ; \ p_3(x) := \frac{x(x-b)}{2d}$$

with $b, c \in (0, 1]$, $d > 0$, and $C := c^2$.

$$\|T\| = \max\left(g(x^*), g(1)\right)$$

**Determining $\|T^{-1}\|$: starting point**

**Lemma.** $U$ and $V$ be finite-dimensional normed spaces, $F \in L(U, V)$. There exists $u \in Ex(B_U)$, the set of extreme points of the unit ball of $U$, such that: $\|F(u)\| = \|F\|$

## Determining $\|T^{-1}\|$: starting point

**Lemma.** $U$ and $V$ be finite-dimensional normed spaces, $F \in L(U, V)$. There exists $u \in Ex(B_U)$, the set of extreme points of the unit ball of $U$, such that: $\|F(u)\| = \|F\|$

*Proof.* $\mathcal{E}(F) := \{u \in B_U : \|F(u)\| = \|F\|\}$ is a non-empty compact subset of $U$,

## Determining $\|T^{-1}\|$: starting point

**Lemma.** $U$ and $V$ be finite-dimensional normed spaces, $F \in L(U, V)$. There exists $u \in Ex(B_U)$, the set of extreme points of the unit ball of $U$, such that: $\|F(u)\| = \|F\|$

*Proof.* $\mathcal{E}(F) := \{u \in B_U : \|F(u)\| = \|F\|\}$ is a non-empty compact subset of $U$, therefore, by the Krein-Milman theorem: $Ex(\mathcal{E}(F)) \neq \emptyset$.

## Determining $\|T^{-1}\|$: starting point

**Lemma.** $U$ and $V$ be finite-dimensional normed spaces, $F \in L(U,V)$. There exists $u \in Ex(B_U)$, the set of extreme points of the unit ball of $U$, such that: $\|F(u)\| = \|F\|$

*Proof.* $\mathcal{E}(F) := \{u \in B_U : \|F(u)\| = \|F\|\}$ is a non-empty compact subset of $U$, therefore, by the Krein-Milman theorem: $Ex(\mathcal{E}(F)) \neq \emptyset$. But $\mathcal{E}(F)$ is an extreme set of $B_U$, thus: $Ex(\mathcal{E}(F)) = \mathcal{E}(F) \bigcap Ex(B_U)$. □

## Determining $\|T^{-1}\|$: next step

**Lemma (Konheim, Rivlin, 66).** A polynomial $p \in B_{\mathcal{P}_n}$, the unit ball of $\mathcal{P}_n$, is an extreme point of $B_{\mathcal{P}_n}$ if and only if it achieves the values $-1$ and 1 (counting multiplicity) at least $n+1$ times.

## Determining $\|T^{-1}\|$: next step

**Lemma (Konheim, Rivlin, 66).** A polynomial $p \in B_{\mathcal{P}_n}$, the unit ball of $\mathcal{P}_n$, is an extreme point of $B_{\mathcal{P}_n}$ if and only if it achieves the values $-1$ and 1 (counting multiplicity) at least $n+1$ times.
(Generalizable to extended complete Chebyshev spaces)

**Determining $\|T^{-1}\|$: next step**

**Lemma (Konheim, Rivlin, 66).** A polynomial $p \in B_{\mathcal{P}_n}$, the unit ball of $\mathcal{P}_n$, is an extreme point of $B_{\mathcal{P}_n}$ if and only if it achieves the values $-1$ and 1 (counting multiplicity) at least $n+1$ times.
(Generalizable to extended complete Chebyshev spaces)

Consequently, $Ex(B_{\mathcal{P}_2})$ is the following one-parameter family:

## Determining $\|T^{-1}\|$: next step

**Lemma (Konheim, Rivlin, 66).** A polynomial $p \in B_{\mathcal{P}_n}$, the unit ball of $\mathcal{P}_n$, is an extreme point of $B_{\mathcal{P}_n}$ if and only if it achieves the values $-1$ and $1$ (counting multiplicity) at least $n+1$ times.
(Generalizable to extended complete Chebyshev spaces)

Consequently, $Ex(B_{\mathcal{P}_2})$ is the following one-parameter family:

$$Ex(B_{\mathcal{P}_2}) = \{\pm q_t, t \in [-1,1]\} \bigcup \{-1, 1\}$$
$$\text{with } q_t := \min\left(\frac{2}{(1+t)^2}, \frac{2}{(1-t)^2}\right)(\bullet - t)^2 - 1$$

**Determining $\|T^{-1}\|$: last step**

With $h := \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} := T^{-1}(1)$, and for $t \in [-1, 1]$, $f(t) := \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} := T^{-1}(q_t)$ (ie $1 = \sum h_i p_i$, $q_t = \sum f_i(t) p_i$):

## Determining $\|T^{-1}\|$: last step

With $h := \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} := T^{-1}(1)$, and for $t \in [-1,1]$, $f(t) := \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} := T^{-1}(q_t)$ (ie $1 = \sum h_i p_i$, $q_t = \sum f_i(t) p_i$):

$$\|T^{-1}\| \;=\; \max\left(\|f_1\|_{[-1,1]}, \|f_2\|_{[-1,1]}, \|f_3\|_{[-1,1]}, \|h\|_\infty\right)$$

## Determining $\|T^{-1}\|$: last step

With $h := \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} := T^{-1}(1)$, and for $t \in [-1, 1]$, $f(t) := \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} :=$
$T^{-1}(q_t)$ (ie $1 = \sum h_i p_i$, $q_t = \sum f_i(t) p_i$):

$$\begin{aligned} \|T^{-1}\| &= \max\Big(\|f_1\|_{[-1,1]}, \|f_2\|_{[-1,1]}, \|f_3\|_{[-1,1]}, \|h\|_\infty\Big) \\ &= \max\Big(\|f_1\|_{[0,1]}, |f_2(0)|, \|f_3\|_{[0,1]}, \|h\|_\infty\Big) \text{ if } \underline{p} \text{ is symmetric} \end{aligned}$$

## Determining $\|T^{-1}\|$: last step

With $h := \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} := T^{-1}(1)$, and for $t \in [-1, 1]$, $f(t) := \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} := T^{-1}(q_t)$ (ie $1 = \sum h_i p_i$, $q_t = \sum f_i(t) p_i$):

$$\begin{aligned} \|T^{-1}\| &= \max\left(\|f_1\|_{[-1,1]}, \|f_2\|_{[-1,1]}, \|f_3\|_{[-1,1]}, \|h\|_\infty\right) \\ &= \max\left(\|f_1\|_{[0,1]}, |f_2(0)|, \|f_3\|_{[0,1]}, \|h\|_\infty\right) \text{ if } \underline{p} \text{ is symmetric} \end{aligned}$$

For our basis of interest:

$$\|T^{-1}\| = \frac{1}{C} \max(1, d, d\lambda) \quad \text{, with } \lambda := \lambda(b, C) := \frac{(b+C)^2 + C(C-b^2)}{(b+C)^2 - C(C-b^2)}$$

## Minimizing $\boldsymbol{\kappa_\infty(\underline{p}(b, c, d))}$

We need to minimize, for $b \le c$:

$$\kappa_\infty\left(\underline{p}\left(b, c, \frac{1}{\lambda}\right)\right) = \max\left(\underbrace{\frac{\lambda + 1 - C}{C}}_{=:F(b,C)}, \underbrace{\frac{\frac{b^2\lambda^2}{4} + C}{C}}_{=:G(b,C)}\right)$$

## Minimizing $\boldsymbol{\kappa_\infty(\underline{p}(b,c,d))}$

We need to minimize, for $b \leq c$:

$$\kappa_\infty\left(\underline{p}\left(b,c,\frac{1}{\lambda}\right)\right) = \max\left(\underbrace{\frac{\lambda+1-C}{C}}_{=:F(b,C)}, \underbrace{\frac{\frac{b^2\lambda^2}{4}+C}{C}}_{=:G(b,C)}\right)$$

$F(b,C)$ decreases with $b$, whereas $G(b,C)$ increases with $b$.

## Minimizing $\boldsymbol{\kappa_\infty(\underline{p}(b,c,d))}$

We need to minimize, for $b \leq c$:

$$\kappa_\infty\left(\underline{p}\left(b,c,\frac{1}{\lambda}\right)\right) = \max\left(\underbrace{\frac{\lambda+1-C}{C}}_{=:F(b,C)}, \underbrace{\frac{\frac{b^2\lambda^2}{4}+C}{C}}_{=:G(b,C)}\right)$$

$F(b,C)$ decreases with $b$, whereas $G(b,C)$ increases with $b$. Thus, for each $C$ (in $\left[\frac{8}{9},1\right]$), there exists a unique $b^*(C) \in [0,c]$ such that:

$$F(b^*(C),C) = G(b^*(C),C) =: H(C)$$

.

## Minimizing $\kappa_\infty(\underline{p}(b, c, d))$(continued)

Note that $H\left(\frac{8}{9}\right) = H(1) = \frac{5}{4}$ and that, for all $C \in \left(\frac{8}{9}, 1\right)$, $H(C) < \frac{5}{4}$.

## Minimizing $\kappa_\infty(\underline{p}(b,c,d))$(continued)

Note that $H\left(\frac{8}{9}\right) = H(1) = \frac{5}{4}$ and that, for all $C \in \left(\frac{8}{9}, 1\right)$, $H(C) < \frac{5}{4}$. We now need to minimize $H(C)$ over $C \in \left[\frac{8}{9}, 1\right]$, which reduces to the resolution of the following (polynomial) system with unknown $(b, C)$:

$$\begin{cases} F(b,C) - G(b,C) & = & 0 \\ \left[\frac{\partial F}{\partial b}\frac{\partial G}{\partial C} - \frac{\partial F}{\partial C}\frac{\partial G}{\partial b}\right](b,C) & = & 0 \end{cases}$$

## Minimizing $\kappa_\infty(\underline{p}(b,c,d))$(continued)

Note that $H\left(\frac{8}{9}\right) = H(1) = \frac{5}{4}$ and that, for all $C \in \left(\frac{8}{9}, 1\right)$, $H(C) < \frac{5}{4}$. We now need to minimize $H(C)$ over $C \in \left[\frac{8}{9}, 1\right]$, which reduces to the resolution of the following (polynomial) system with unknown $(b, C)$:

$$\begin{cases} F(b,C) - G(b,C) & = & 0 \\ \left[\frac{\partial F}{\partial b}\frac{\partial G}{\partial C} - \frac{\partial F}{\partial C}\frac{\partial G}{\partial b}\right](b,C) & = & 0 \end{cases}$$

A numerical resolution, using Maple, provides us with the solution:

$$b = 0.867\ldots, C = 0.940\ldots$$

giving the following value of the $\ell_\infty$-condition number:

$$F(b,C) = 1.248\ldots$$

## Concluding questions

To a (symmetric) basis with $\ell_\infty$-condition number less than $\frac{5}{4}$, we can associate, theoretically, a (symmetric?) projection form $C([-1,1])$ onto $\mathcal{P}_2$ with norm less than $\frac{5}{4}$,

## Concluding questions

To a (symmetric) basis with $\ell_\infty$-condition number less than $\frac{5}{4}$, we can associate, theoretically, a (symmetric?) projection form $C([-1,1])$ onto $\mathcal{P}_2$ with norm less than $\frac{5}{4}$, does it have an easy expression?

## Concluding questions

To a (symmetric) basis with $\ell_\infty$-condition number less than $\frac{5}{4}$, we can associate, theoretically, a (symmetric?) projection form $C([-1,1])$ onto $\mathcal{P}_2$ with norm less than $\frac{5}{4}$, does it have an easy expression?

Is a (the?) basis of $\mathcal{P}_2$ which minimizes the $\ell_\infty$-condition number symmetric?

**Concluding questions**

To a (symmetric) basis with $\ell_\infty$-condition number less than $\frac{5}{4}$, we can associate, theoretically, a (symmetric?) projection form $C([-1,1])$ onto $\mathcal{P}_2$ with norm less than $\frac{5}{4}$, does it have an easy expression?

Is a (the?) basis of $\mathcal{P}_2$ which minimizes the $\ell_\infty$-condition number symmetric?

In the affirmative, is it of the type we have considered?

## Concluding questions

To a (symmetric) basis with $\ell_\infty$-condition number less than $\frac{5}{4}$, we can associate, theoretically, a (symmetric?) projection form $C([-1,1])$ onto $\mathcal{P}_2$ with norm less than $\frac{5}{4}$, does it have an easy expression?

Is a (the?) basis of $\mathcal{P}_2$ which minimizes the $\ell_\infty$-condition number symmetric?

In the affirmative, is it of the type we have considered? If yes, we would have found the Banach-Mazur distance between $\mathcal{P}_2$ and $\ell_\infty^3$,

## Concluding questions

To a (symmetric) basis with $\ell_\infty$-condition number less than $\frac{5}{4}$, we can associate, theoretically, a (symmetric?) projection form $C([-1,1])$ onto $\mathcal{P}_2$ with norm less than $\frac{5}{4}$, does it have an easy expression?

Is a (the?) basis of $\mathcal{P}_2$ which minimizes the $\ell_\infty$-condition number symmetric?

In the affirmative, is it of the type we have considered? If yes, we would have found the Banach-Mazur distance between $\mathcal{P}_2$ and $\ell_\infty^3$, and none of the inequalities considered at the beginning would be an equality.