Some inheritance properties for Chebyshev-type spaces

Simon Foucart

Department of Applied Mathematics and Theoretical Physics

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Markov interlacing property

Theorem 1. Let $p := (\cdot - t_1)\cdots(\cdot - t_n)$, $q := (\cdot - s_1)\cdots(\cdot - s_n)$ be two polynomials, with $t_1 \le \cdots \le t_n$ and $s_1 \le \cdots \le s_n$. Let $\eta_1 \le \cdots \le \eta_{n-1}$ be the zeros of p' and $\xi_1 \le \cdots \le \xi_{n-1}$ the zeros of q'.

Markov interlacing property

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If $t_1 < \cdots < t_n$, and

$$t_1 \le s_1 \le t_2 \le s_2 \le \cdots \le t_{n-1} \le s_{n-1} \le t_n \le s_n$$

then:

$$\eta_1 < \xi_1 < \eta_2 < \xi_2 < \cdots < \xi_{n-2} < \eta_{n-1} < \xi_{n-1}$$

unless $f = g$. ■

Bojanov-Rahman theorem

For P a polynomial of degree n, with n distinct zeros in $(-1, 1)$, we denote by $h_i(P)$, $i \in \{0, \ldots, n\}$ the values of the local extrema of P, including the values at -1 and 1.

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Theorem 2. Let p and q be two polynomials of degree n, having n distinct zeros in $(-1,1)$.

$$\left[\forall i \in \{0,\ldots,n\}, |h_i(p)| \leq |h_i(q)|\right] \Rightarrow \left[\forall i \in \{0,\ldots,n-1\}, |h_i(p')| \leq |h_i(q')|\right]$$

The different types of Chebyshev systems

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1. Defining them

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2. Zeros and sign changes

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4. Interpolation and its continuity

Let $g_1, \ldots, g_k$ be functions defined on I. The collocation determinant of:

- $t_1 < \cdots < t_k \in I$ is $D\begin{pmatrix} g_1 & \cdots & g_k \\ t_1 & \cdots & t_k \end{pmatrix} := \begin{vmatrix} g_1(t_1) & \cdots & g_k(t_1) \\ \vdots & \ddots & \vdots \\ g_1(t_k) & \cdots & g_k(t_k) \end{vmatrix}$

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  with, $\forall i \in \{1, \ldots, k\}$, $d_i := \max\{j : t_{i-j} = \cdots = t_i\}$, assuming that the g_j's are d_i times differentiable at t_i.

Definition 3. Let $g_1, \ldots, g_k$ be continuous (at least) functions on I. $(g_1, \ldots, g_k)$ is a:

- weak Chebyshev (WT) system on I if:

$$\forall t_1 < \cdots < t_k \in I, \quad D\begin{pmatrix} g_1 & \ldots & g_k \\ t_1 & \ldots & t_k \end{pmatrix} \geq 0$$

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Let now G_k be a k-dimensional subspace of $C(I)$ (resp. of $C^{k-1}(I)$). We say that G_k is a $\star$ space on I, where $\star$=WT, T (resp. ET, ECT, EOCT) if it admits a basis which is a $\star$ system on I

Zeros and sign changes of a function

Let f be a function defined on I. We write:

- $S^+(f) := \sup \left\{ \begin{array}{c} r \in \mathbb{N}: \quad \exists t_0 < \cdots < t_r \in I,\ \exists \varepsilon \in \{-1, 1\}, \\ \forall i \in \{0, \ldots, r\},\ \ \varepsilon(-1)^i f(t_i) \geq 0 \end{array} \right\}$
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- $z(f) := \sum_{t \in Z(f)} \zeta_f(t)$, where:

$$\zeta_g(t) := \max \left\{ r \in \mathbb{N} \cup \{+\infty\} : g(t) = 0, \ldots, g^{(r-1)}(t) = 0 \right\}$$

  $z(f)$ is the number of zeros of f, counting multiplicity, when f is sufficiently differentiable

Remark. The following inequalities are easy to obtain:

$$z(f) \underbrace{\geq}_{f \text{ suff. diff.}} \mathcal{Z}(f) \underbrace{\geq}_{f \text{ cont.}} S^{+}(f) \geq \# Z(f) \underbrace{\geq}_{f \text{ cont.}} S^{-}(f)$$

Theorem 4. Let $\underline{g} = (g_1, \ldots, g_k)$ be a linearly independent system of functions defined on I (with a sufficient degree of smoothness).

$$\underline{g} \text{ is a } \begin{array}{c} \text{WT} \\ \text{T} \\ \text{T} \\ \text{ET} \\ \text{EOCT} \end{array} \text{ sytem } \iff \forall \vec{a} \neq \vec{0}, \quad \begin{array}{c} S^-(g) \leq k-1 \\ \#Z(g) \leq k-1 \\ \mathcal{Z}(g) \leq k-1 \\ z(g) \leq k-1 \\ z(g) \leq S^-(a_1, \ldots, a_k) \end{array}$$

$$\text{where } g := \sum_{i=1}^{k} a_i g_i$$

Interpolation

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The extended Chebyshev spaces are exactly the ones in which the Lagrange-Hermite interpolation is always possible and unique.

ECC spaces

Theorem 5. G_k k-dimensional subspace of $C^{k-1}(I)$, $t \in I$. G_k is an extended complete Chebyshev on I if and only if there exists $w_0 \in C^{k-1}(I), \ldots, w_{k-1} \in C(I)$, $w_0 > 0, \ldots, w_{k-1} > 0$ on I, such that the following ECT system is a basis of G_k:

$$\begin{aligned}
u_0(\cdot,t) &:= w_0 \mathcal{I}_0(\cdot,t) = w_0 \\
u_1(\cdot,t) &:= w_0 \mathcal{I}_1(\cdot,t,w_1) = w_0 \int_t^{\cdot} w_1(x_1)dx_1 \\
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&= w_0 \int_t^{\cdot} w_1(x_1) \cdots \int_t^{x_{k-2}} w_{k-1}(x_{k-1})dx_{k-1} \ldots dx_1
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Remark. We write $G_k = ECT_t(w_0, \ldots, w_{k-1}) = ECT(w_0, \ldots, w_{k-1})$. This representation is not necessarily unique.

Extended complete Chebyshev spaces are the natural generalization of the spaces of polynomials ($w_0 = 1, w_1 = 1, \ldots, w_{k-1} = k-1$).

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where, in this context, the natural i-th differentiation, $i \in \{1, \ldots, k\}$, is $L_{w_{i-1},\ldots,w_0} := D\left(\frac{\cdot}{w_{i-1}}\right) \circ \cdots \circ D\left(\frac{\cdot}{w_0}\right)$.

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B-splines can be constructed for such spaces; we then talk about Chebyshevian B-splines.Only the recurrence relation is lost, because it uses the factorization property of polynomials.

Interpolation, again

$G_{k+1} =: ECT(w_0, \ldots, w_k)$ an extended complete Chebyshev space on $[a, b]$, $\overline{\Delta}_k := \{t_1 \leq \cdots \leq t_k \in [a, b]\}$.

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$$\forall i \in \{1, \ldots, k\}, \omega^{(d_i)}(t_i; t_1, \ldots, t_k) = 0$$

$$\text{and } \frac{1}{w_k} L_{w_{k-1},\ldots,w_0}(\omega(\cdot; t_1, \ldots, t_k)) = 1 \blacksquare$$

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Proposition 6.

$$\Omega : (t_1, \ldots, t_k) \in \overline{\Delta}_k \mapsto \omega(\cdot; t_1, \ldots, t_k) \in G_{k+1}$$

is a homeomorphism from $\overline{\Delta}_k$ onto its image.

Markov interlacing property

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2. A general result

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3. The case of extended complete Chebyshev spaces (2 proofs)

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5. An application: $N_{t_0,\ldots,t_k} \geq 0$

Easy proof for polynomials

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Theorem 7. Let $p := (\cdot - t_1) \cdots (\cdot - t_n)$ and $q := (\cdot - s_1) \cdots (\cdot - s_n)$.

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Easy proof for polynomials

Let f and g be two functions having $t_1 \leq \cdots \leq t_n$ and $s_1 \leq \cdots \leq s_n$, respectively, as zeros.We write:

$$f \prec g \text{ if : } t_1 < s_1 < t_2 < s_2 < \cdots < t_{n-1} < s_{n-1} < t_n < s_n$$

$$f \preceq g \text{ if : } t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq t_{n-1} \leq s_{n-1} \leq t_n \leq s_n$$

Theorem 7. Let $p := (\cdot - t_1) \cdots (\cdot - t_n)$ and $q := (\cdot - s_1) \cdots (\cdot - s_n)$.

(i) if $p \prec q$, then $p' \prec q'$

(ii) if $p \preceq q$, then $p' \preceq q'$

(iii) if $t_1 < \cdots < t_n$, $s_1 < \cdots < s_n$ and $p \preceq q$, then $p' \prec q'$, unless $p = q$

Proof. Let $p := (\cdot - x_1)\cdots(\cdot - x_n)$ and let η a zero of p':

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Remark. Let $w > 0$ be C^1,

$$\frac{(p/w)'}{(p/w)} = \frac{p'}{p} - \frac{w'}{w} = \sum_{i=1}^{n} \frac{1}{\cdot - x_i} - \frac{w'}{w}$$

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and, if η is a zero of $(p/w)'$, we obtain as well $\frac{\partial \eta}{\partial x_j} > 0$.

A general result

Let $f, g \in C^1(I)$ having exactly n zeros, $t_1 < \cdots < t_n$ and $s_1 < \cdots < s_n$, say. Let, for $(\lambda, \mu) \neq (0, 0)$, $q_{\lambda,\mu} := \lambda f + \mu g$. Let us assume that one of the following conditions is fulfilled:

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Claim: For all $(\lambda, \mu) \neq (0,0)$, $\#Z(q'_{\lambda,\mu}) \leq n-1$ and $\#Z(q_{\lambda,\mu}) \leq n$. If $q_{\lambda,\mu}$ has exactly n zeros, it changes its sign there, and, $q'_{\lambda,\mu}$ has exactly $n-1$ zeros, strictly inside the zeros of $q_{\lambda,\mu}$, where it changes its sign.

Lemma 8. If $f \prec g$, then, for any $(\lambda, \mu) \neq (0,0)$, with $q_{\lambda,\mu} := \lambda f + \mu g$, there is no $t \in I$ such that $q_{\lambda,\mu}(t) = 0$ and $q'_{\lambda,\mu}(t) = 0$. ■

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Case of ECT spaces

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Proof 2. $\blacksquare$ $\square$

Some spline spaces

Working on $[a, b]$, we consider a sequence of n distinct breakpoints $\underline{t} := a < t_1 < \cdots < t_n < b$. We let $t_0 := a$ and $t_{n+1} := b$.

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Notations 11.

$\mathcal{S}_k(\underline{t})$: splines of degree $\leq k$ with breakpoints $t_1 < \cdots < t_n$

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\mathcal{S}_k(\underline{t}) & : & \text{splines of degree} \le k \text{ with breakpoints } t_1 < \cdots < t_n \\
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Fixed breakpoints

Proposition 12. Let $k \geq 2$ (resp $k \geq 3$). If the $n+k$ zeros of some $f \in \mathcal{S}_k(\underline{t})$ (resp $\Omega_k(\underline{t})$) interlace strictly with the $n+k$ zeros of some $g \in \mathcal{S}_k(\underline{t})$ (resp $\Omega_k(\underline{t})$), then the $n+k-1$ zeros of f' interlace strictly with the $n+k-1$ zeros of g'.

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Proof. We first show that, for any $(\lambda, \mu) \neq (0,0)$, $S^-(\lambda f^{(k)} + \mu g^{(k)}) \leq n$ (resp $S^-(\lambda f^{(k-1)} + \mu g^{(k-1)}) \leq n+1$) $\blacksquare$

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Remark. A $s \in \mathcal{S}_k(\underline{t})$ (resp $s \in \Omega_k(\underline{t})$) having exactly $n+k$ given zeros is unique, up to a multiplicative constant.

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Proposition 13. Let $k \geq 2$. If the $n+k$ zeros of some $f \in \mathcal{P}_{k,n} := \bigcup_{t_1<\cdots<t_n} \mathcal{P}_k(\underline{t})$ interlace strictly with the $n+k$ zeros of some $g \in \mathcal{P}_{k,n}$, then the $n+k-1$ zeros of f' interlace strictly with the $n+k-1$ zeros of g'.

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Remark. It is well known that, given $x_1 < \cdots < x_{n+k}$, there exists a unique (up to multiplicative constant) non-trivial perfect spline $p \in \mathcal{P}_{k,\leq n}$ vanishing at the x_i's. Moreover, p has no other zeros, and $p \in \mathcal{P}_{k,n}$.

Free breakpoints, continued

The zeros of the derivatives of two functions in $\mathcal{S}_{k,n} := \bigcup_{t_1<\cdots<t_n} \mathcal{S}_k(\underline{t})$ whose zeros interlace do not necessarily interlace.

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Nevertheless, there is hope to prove such a result with the additional hypothesis of interlacing breakpoints.

However, this is not an immediate consequence of the previous considerations.

An application: $N_{t_0,\ldots,t_k} \geq 0$

We can use the Markov interlacing property for splines to show this classical result. ■

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We can use the Markov interlacing property for splines to show this classical result. The interest of it is that it (certainly) works for Chebyshevian splines, for which we do not have the usual recurrence relation.

Bojanov-Rahman theorem

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1. Interpolation at extremal points

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2. Connections to Markov's inequality

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3. Generalizations

Interpolation at extremal points

Notations 14. Consider $G_{k+1} =: ECT(1, w_1, \ldots, w_k)$ an extended complete Chebyshev space on $[-1,1]$, and $f \in G_{k+1}$ having k distinct zeros in $(-1,1)$. Define the $k+1$ extremal values of f by:

$$\forall i \in \{0, \ldots, k\},\ h_i(f) = f(t_i)$$

where $t_1 < \cdots < t_{k-1}$ are the zeros of f' and $t_0 := -1$, $t_k := 1$

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Theorem 15.

$$\forall \alpha_0, \ldots, \alpha_k > 0, \exists f \in G_{k+1} : \forall i \in \{0, \ldots, k\},\ h_i(f) = (-1)^{k+i} \alpha_i$$

Such a f is unique.■

Remark. Let $\alpha_0, \ldots, \alpha_k > 0$, and let $g_\alpha \in G_{k+1} := ECT(1, w_1, \ldots, w_k)$ be defined by: $\forall i \in \{0, \ldots, k\}$, $h_i(g_\alpha) = (-1)^{k+1}\alpha_i$.

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g_α is extremal for the linear functional:

$$g \in \left(G_{k+1}, \| \cdot \|_{\alpha,t} \right) \mapsto \frac{1}{w_k} L_{w_{k-1},\ldots,w_1,1}(g) \in \mathbb{R}$$

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Connections to Markov's inequality

Theorem 16. Let p and q be two polynomials of degree n, having n distinct zeros in $(-1, 1)$.

$$\left[\forall i \in \{0, \ldots, n\}, |h_i(p)| \leq |h_i(q)|\right] \Rightarrow \left[\forall i \in \{0, \ldots, n-1\}, |h_i(p')| \leq |h_i(q')|\right]$$

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Remark. This implies Markov's inequality for oscillating polynomials. Indeed, let p be a polynomial of degree n, with n distinct zeros in $(-1, 1)$, and whose uniform norm on $[-1, 1]$ does not exceed 1. We have, for all $i \in \{0, \dots, n\}$, $|h_i(p)| \le 1 = |h_i(T_n)|$, thus, for all $k \in \{1, \dots, n\}$, and for all $i \in \{0, \dots, n-k\}$, $\left|h_i(p^{(k)})\right| \le \left|h_i(T_n^{(k)})\right|$, ie $\|p^{(k)}\|_\infty \le \|T_n^{(k)}\|_\infty$.

Remark. We have used the fact that $\|p^{(k)}\|_\infty$ is an increasing function of the $|h_i(p)|$'s. But so are $\|p^{(k)}\|_q$, for $1 \leq q < +\infty$, and $l(p)$, the arc length of the p.

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- the arc length

- the L_q-norm of any k-th derivative, $k \in \{1, \ldots, n\}$, $1 \le q \le +\infty$

Generalizations

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My interest here is different, it consists in the generalization of the result for extended complete Chebyshev spaces, namely:

Problem 17. Consider $G_{k+1} =: ECT(1, w_1, \ldots, w_k)$ an extended complete Chebyshev space on $[-1, 1]$, and $f, g \in G_{k+1}$ having k distinct zeros in $(-1, 1)$:

$$\left[\forall i \in \{0, \ldots, k\}, |h_i(f)| \leq |h_i(g)|\right] \Longrightarrow \left[\forall i \in \{0, \ldots, k-1\}, \left|h_i\left(\frac{f'}{w_1}\right)\right| \leq \left|h_i\left(\frac{g'}{w_1}\right)\right|\right]$$

Markov's inequality for ECT spaces

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Definition 18 (The snake). There is a unique pair (T, S) satisfying:

$$T \in G_{k+1}, \ \ |T| \leq w_0, \ \ S = (-1 \leq s_0 < s_1 < \cdots < s_k \leq 1),$$
$$\text{and, for all } i \in \{0, \ldots, k\}, T(s_i) = (-1)^{k+i} w_0(s_i)$$

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Problem 19. For $m \in \{1, \dots, k\}$, T is maximizing $\left\| \frac{1}{w_m} L_{w_{m-1,\dots,w_0}}(g) \right\|_\infty$ over the set of $g \in G_{k+1}$ having k distinct zeros in $(-1, 1)$ and satisfying $|g| \leq w_0$.