On the best conditioned bases of quadratic polynomials

Simon Foucart

University of Cambridge, Department of Applied Mathematics and Theoretical Physics, Wilberforce Road, Cambridge, CB3 0WA, UK

Abstract

It is known that for $\mathcal{P}_n$, the subspace of $\mathcal{C}([-1,1])$ of all polynomials of degree at most n, the least basis condition number $\kappa_\infty(\mathcal{P}_n)$ (also called the Banach-Mazur distance between $\mathcal{P}_n$ and ℓ_∞^{n+1}) is bounded from below by the projection constant of $\mathcal{P}_n$ in $\mathcal{C}([-1,1])$. We show that $\kappa_\infty(\mathcal{P}_n)$ is in fact the generalized interpolating projection constant of $\mathcal{P}_n$ in $\mathcal{C}([-1,1])$, and is consequently bounded from above by the interpolating projection constant of $\mathcal{P}_n$ in $\mathcal{C}([-1,1])$. Hence the condition number of the Lagrange basis (say, at the Chebyshev extrema), which coincides with the norm of the corresponding interpolating projection and thus grows like $\mathcal{O}(\ln n)$, is of optimal order, and for $n=2$,

$$1.2201\ldots \leq \kappa_\infty(\mathcal{P}_2) \leq 1.25.$$

We prove that there is a basis $\underline{u}$ of $\mathcal{P}_2$ such that

$$\kappa_\infty(\underline{u}) \approx 1.24839.$$

This result means that no Lagrange basis of $\mathcal{P}_2$ is best conditioned. It also seems likely that the previous value is actually the least basis condition number of $\mathcal{P}_2$, which therefore would not equal the projection constant of $\mathcal{P}_2$ in $\mathcal{C}([-1,1])$.

As for trigonometric polynomials of degree at most 1, we present numerical evidence that the Lagrange bases at equidistant points are best conditioned.

Key words: Banach-Mazur distance, condition number, projection constant

Email address: S.Foucart@damtp.cam.ac.uk (Simon Foucart).

1 Preliminaries

Let $\underline{u} := (u_1, \dots, u_n)$ be a basis of a finite-dimensional subspace U of a Banach space $(X, \|\bullet\|)$. The ℓ_∞-condition number of $\underline{u}$ is by definition

$$\kappa_\infty(\underline{u}) := \sup_{a \in \ell_\infty^n \setminus \{0\}} \frac{\|\sum_{i=1}^n a_i u_i\|}{\|a\|_\infty} \times \sup_{a \in \ell_\infty^n \setminus \{0\}} \frac{\|a\|_\infty}{\|\sum_{i=1}^n a_i u_i\|} =: s_1(\underline{u}) \times s_2(\underline{u}) \,. \quad (1)$$

In approximation theory, some efforts were put into evaluating the condition number of certain bases. In particular, Gautschi considered the power basis [8] and orthogonal bases [7] of $\mathcal{P}_n$, the space of all algebraic polynomials of degree at most n, and de Boor initiated the studies of the B-spline basis condition number, which attracted a great deal of attention (see [10,12] and references therein).

The condition number of a basis $\underline{u}$ of U is pertinent not only because it characterizes, to a certain extent, the stability of numerical computations with $\underline{u}$, but also because one can associate to $\underline{u}$ a projection P from X onto U satisfying

$$\|P\| \leq \kappa_\infty(\underline{u}). \quad (2)$$

Indeed, giving the basis $\underline{u}$ is giving an isomorphism $T : U \to \ell_\infty^n$, and by the Hahn-Banach theorem applied componentwise, there exists a norm-preserving extension of T, say $\widetilde{T} : X \to \ell_\infty^n$. Then $P := T^{-1}\widetilde{T}$ is a projection from X onto U which satisfies $\|P\| \leq \|T^{-1}\|\|T\| = \kappa_\infty(\underline{u})$.

Therefore, when searching for a projection with minimal norm, it is reasonable to examine the least of these condition numbers, that is the value

$$\kappa_\infty(U) := \inf \kappa_\infty(\underline{u}) = \inf \{\|T\|\|T^{-1}\|,\ T : U \to \ell_\infty^n \text{ isomorphism}\},$$

which is known to the Banach space geometer as the Banach-Mazur distance between U and ℓ_∞^n. Thus, for the (relative) projection constant of U in X, defined by

$$p(U, X) := \inf \{\|P\|,\ P : X \twoheadrightarrow U \text{ projection}\},$$

equation (2) implies the upper estimate

$$p(U, X) \leq \kappa_\infty(U) \,.$$

We are interested in the possibility of a deeper relation between those two constants, in particular we wonder if equality can occur. With no further assumptions on the spaces, the answer is clearly no. Indeed, even an inequality of the type $\kappa_\infty(U) \leq C\, p(U, X)$ is wrong in general, since for example $p(\ell_2^n, \ell_2) = 1$ while $\kappa_\infty(\ell_2^n) = \sqrt{n}$ [14, chapter 9]. Furthermore, Szarek [13] showed that a much weaker inequality $\kappa_\infty(U) \leq C\, p(U)$ also fails to be true. Here $p(U)$, the (absolute) projection constant of U is, in a sense, the supremum of the relative projection constants of U, more precisely

$$p(U) := \sup \{p(i(U), X),\ X \text{ Banach},\ i : U \to X \text{ isometric embedding}\}.$$

However, if we impose $X = \mathcal{C}$ ($\mathcal{C}$ representing $\mathcal{C}([-1,1])$ or $\mathcal{C}(\mathbb{T})$), there is the estimate, noticed by de Boor in [1, p. 19],

$$\kappa_\infty(U) \leq p_{\text{int}}(U, \mathcal{C}), \tag{3}$$

where

$$p_{\text{int}}(U, \mathcal{C}) := \inf \{\|P\|,\ P : \mathcal{C} \twoheadrightarrow U \text{ interpolating projection}\}$$

is the interpolating projection constant of U in $\mathcal{C}$. Let us outline the arguments. If $P = \sum_{i=1}^n \bullet(x_i)\ell_i$ is an interpolating projection ($\underline{\ell}$ is the Lagrange basis of U at the points $x_1, \ldots, x_n$, i.e. it satisfies $\ell_i(x_j) = \delta_{i,j}$), then $\|P\| = \|\sum_{i=1}^n |l_i|\|$. On the other hand, with s_1 and s_2 being defined in (1), we easily find $s_1(\underline{l}) = \|\sum_{i=1}^n |l_i|\|$ and $s_2(\underline{l}) \leq 1$. Hence $\kappa_\infty(\underline{\ell}) \leq \|P\|$, with equality if $1 \in U$, and we get (3) by taking the infimum over P.

So, for a finite-dimensional subspace U of $\mathcal{C}$,

$$p(U, \mathcal{C}) \leq \kappa_\infty(U) \leq p_{\text{int}}(U, \mathcal{C}),$$

and an inequality of the type $\kappa_\infty(U) \leq C\, p(U, \mathcal{C})$ becomes true for subspaces U of $\mathcal{C}$ whose projection constants are both of the same order. The first to come to mind are the spaces $\mathcal{P}_n$, since in this case both constants are of order $\mathcal{O}(\ln n)$. More precisely (see Cheney and Light [5, chapter 3]),

$$\frac{2}{\pi^2} \ln n - \frac{1}{2} \leq p(\mathcal{P}_n, \mathcal{C}) \quad \text{and} \quad p_{\text{int}}(\mathcal{P}_n, \mathcal{C}) \leq \frac{2}{\pi} \ln(n+1) + 1,$$

hence the condition number of the Lagrange basis at the Chebyshev extrema, coinciding with the norm of the corresponding interpolating projection, is of optimal order, since it grows like $\frac{2}{\pi} \ln n$ [2].

2 Objectives

While the numerical value of the interpolating projection constant $p_{\text{int}}(\mathcal{P}_n, \mathcal{C})$ is known [2] at least up to $n = 200$, the projection constant $p(\mathcal{P}_n, \mathcal{C})$ has only been calculated [4] in the case $n = 2$. So the only available pair of exact constants is

$$1.2201 \approx p(\mathcal{P}_2, \mathcal{C}) \leq \kappa_\infty(\mathcal{P}_2) \leq p_{\text{int}}(\mathcal{P}_2, \mathcal{C}) = \frac{5}{4}, \tag{4}$$

where $\frac{5}{4}$ is the value of the norm of the interpolating projection at the Chebyshev extrema -1, 0 and 1, and at some of their dilatations and shifts.

We are going to prove that there is a basis $\underline{u}$ of $\mathcal{P}_2$ such that

$$\kappa_\infty(\underline{u}) \approx 1.248394563,$$

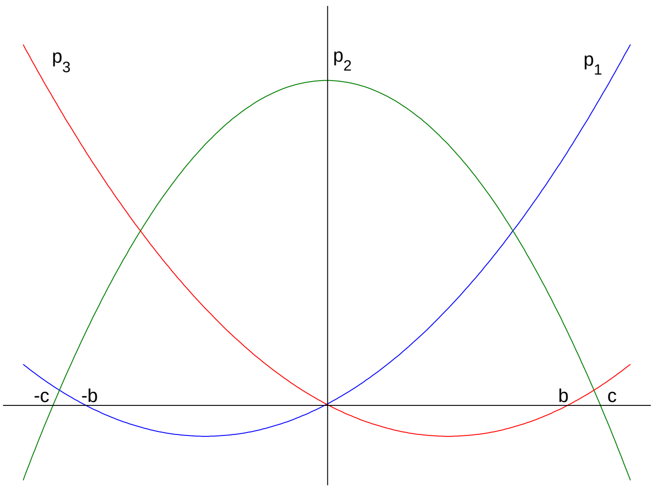
hence the second inequality of (4) is strict, i.e. no Lagrange basis of $\mathcal{P}_2$ is best conditioned. The first inequality of (4) seems to be strict as well, for we believe that this basis $\underline{u}$ is in fact best conditioned.

To this end, let us denote by $\underline{p}(b, c, d)$ the following symmetric basis of $\mathcal{P}_2$:

$$p_1(x) := \frac{x(x+b)}{2d}, \quad p_2(x) := c^2 - x^2, \quad p_3(x) := \frac{x(x-b)}{2d}, \qquad x \in [-1, 1],$$

where

$$b, c, d \in (0, +\infty).$$



For example, $\frac{1}{b^2}\underline{p}(b, b, 1)$ is the interpolating basis at the points $-b$, 0, b.

The main part of this paper is devoted to the proof of the following statement:

Theorem 1 $\min\limits_{b,c,d>0} \kappa_\infty(\underline{p}(b, c, d)) \approx 1.248394563 < \frac{5}{4}$.

The referee pointed out to us that the latter value, provided here by theoretical means, coincides with the value of the symmetric generalized interpolating projection constant of $\mathcal{P}_2$ in $\mathcal{C}$ which was obtained by Chalmers and Metcalf in [3] from computational procedures. Once we have underlined the best conditioned normalization of a basis, the equality between least basis condition number and generalized interpolating projection constant is established, and we finally proceed with the precise minimization of theorem 1.

3 Optimal normalization

We consider a basis $\underline{u} = (u_1, \ldots, u_n)$ of a subspace U of a Banach space $(X, \| \bullet \|)$, and we denote by $(\mu_1, \ldots, \mu_n)$ the dual basis of U^*. One easily gets

$$s_1(\underline{u}) = \max_{\varepsilon_1,\ldots,\varepsilon_n=\pm1} \left\| \sum_{i=1}^n \varepsilon_i u_i \right\| \left(= \left\| \sum_{i=1}^n |u_i| \right\| \text{ if } X = \mathcal{C} \right) \text{ and } s_2(\underline{u}) = \max_{i\in\{1,\ldots,n\}} \|\mu_i\| \,.$$

Hence, normalizing the dual functionals to 1, i.e. introducing the basis $\underline{u^N}$ defined by $u_i^N := \|\mu_i\| u_i$, $i \in \{1, \ldots, n\}$, we have $s_2(\underline{u^N}) = 1$ and

$$s_1(\underline{u^N}) = \max_{\varepsilon_1,\ldots,\varepsilon_n=\pm1} \left\| \sum_{i=1}^n \varepsilon_i \|\mu_i\| u_i \right\| \leq \max_{i\in\{1,\ldots,n\}} \|\mu_i\| \times \max_{\varepsilon_1,\ldots,\varepsilon_n=\pm1} \left\| \sum_{i=1}^n \varepsilon_i u_i \right\| .$$

Therefore we derive the estimate

$$\kappa_\infty(\underline{u^N}) \leq \kappa_\infty(\underline{u}). \tag{5}$$

4 Generalized interpolating projections

This notion was introduced by Cheney and Price in [6]. For a finite dimensional subspace U of $\mathcal{C}$, a projection $P : \mathcal{C} \twoheadrightarrow U$ is said to be a generalized interpolating projection if we can find a basis $\underline{u} = (u_1, \ldots, u_n)$ of U such that the functionals $\widetilde{\mu}_i \in \mathcal{C}^*$ in the representation $P = \sum_{i=1}^n \widetilde{\mu}_i(\bullet) u_i$ have disjoint carriers. We define the generalized interpolating projection constant of U in $\mathcal{C}$ by

$$p_{\text{g.int}}(U, \mathcal{C}) := \inf\{\|P\|, P : \mathcal{C} \twoheadrightarrow U \text{ generalized interpolating projection}\}.$$

The following theorem holds (see also [3, theorem 2]):

Theorem 2 *For a finite dimensional subspace U of $\mathcal{C}$, we have*

$$\kappa_\infty(U) = p_{\text{g.int}}(U, \mathcal{C}).$$

Proof. Let $P = \sum_{i=1}^n \widetilde{\mu}_i(\bullet)u_i$ be a generalized interpolating projection from $\mathcal{C}$ onto U, where the carriers of the $\widetilde{\mu}_i$'s are disjoint. It was established in [6, lemma 9] that

$$\|P\| = \left\| \sum_{i=1}^n \|\widetilde{\mu}_i\| |u_i| \right\|.$$

Hence, $(\mu_1 := \widetilde{\mu}_{1|U}, \ldots, \mu_n := \widetilde{\mu}_{n|U})$ being the dual basis of $\underline{u}$, we have $\|\sum_{i=1}^n \|\mu_i\||u_i|\| \leq \|P\|$, i.e. $\kappa_\infty(\underline{u^N}) \leq \|P\|$. By taking the infimum over P, we obtain

$$\kappa_\infty(U) \leq p_{\text{g.int}}(U, \mathcal{C}).$$

Let now $\underline{u} = (u_1, \ldots, u_n)$ be a basis of U, and let $(\mu_1, \ldots, \mu_n)$ be the dual basis of U^*. Each μ_i has a norm-preserving extension to the whole $\mathcal{C}$ which can be written (see e.g. [11, theorem 2.13])

$$\widetilde{\mu}_i = \sum_{j=1}^{m_i} \alpha_{i,j} \bullet (t_{i,j}), \qquad m_i \leq n,\ t_{i,j} \in [-1, 1],\ \alpha_{i,j} \neq 0.$$

We consider sequences of points $(t_{i,j}^k)_{k\in\mathbb{N}}$ converging to $t_{i,j}$ and such that, for a fixed k, the $t_{i,j}^k$'s are all distinct. We set $\widetilde{\mu}_i^k := \sum_{j=1}^{m_i} \alpha_{i,j} \bullet (t_{i,j}^k)$ and $\mu_i^k := \widetilde{\mu}_{i|U}^k$. Since $\|\mu_i^k\| \leq \|\widetilde{\mu}_i^k\| = \sum_{j=1}^{m_i} |\alpha_{i,j}| = \|\mu_i\|$, extracting a convergent subsequence if necessary, we can assume that μ_i^k converges in norm. The limit must be μ_i, in view of $\widetilde{\mu}_i^k \underset{k\to\infty}{\overset{\sigma(\mathcal{C}^*,\mathcal{C})}{\longrightarrow}} \widetilde{\mu}_i$. Writing $\begin{pmatrix} \mu_1^k & \cdots & \mu_n^k \end{pmatrix} =: \begin{pmatrix} \mu_1 & \cdots & \mu_n \end{pmatrix} A(k)$, we have $A(k) \underset{k\to\infty}{\longrightarrow} I$, hence $A(k)$ is invertible, at least for k large enough, and $A(k)^{-1} \underset{k\to\infty}{\longrightarrow} I$. Thus, with $\begin{pmatrix} u_1^k & \cdots & u_n^k \end{pmatrix} := \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} A(k)^{-1}$, we get $\mu_j^k(u_i^k) = \delta_{i,j}$ and $u_i^k \underset{k\to\infty}{\longrightarrow} u_i$. Therefore $P^k := \sum_{i=1}^n \widetilde{\mu}_i^k(\bullet)u_i^k$ is a generalized interpolating projection from $\mathcal{C}$ onto U and then

$$\begin{aligned} p_{\text{g.int}}(U, \mathcal{C}) &\leq \lim_{k\to\infty} \|P^k\| = \lim_{k\to\infty} \left\| \sum_{i=1}^n \|\widetilde{\mu}_i^k\| |u_i^k| \right\| = \lim_{k\to\infty} \left\| \sum_{i=1}^n \|\mu_i\| |u_i^k| \right\| \\ &= \left\| \sum_{i=1}^n \|\mu_i\| |u_i| \right\| = \kappa_\infty(\underline{u^N}) \leq \kappa_\infty(\underline{u}). \end{aligned}$$

By taking the infimum over $\underline{u}$, we derive

$$p_{\text{g.int}}(U, \mathcal{C}) \leq \kappa_\infty(U).$$

5 Determining the first supremum

For the basis $\underline{p}(b,c,d)$, it is readily seen that the first supremum in (1) is

$$s_1 := s_1(\underline{p}(b,c,d)) = \max_{\pm,\mp}(\|p_{\pm,\mp}\|_{[0,1]}), \qquad \text{where} \qquad p_{\pm,\mp} := p_1 \pm p_2 \mp p_3.$$

More precisely, we have, for $x \in [0,1]$ and with $C := c^2$,

$$\begin{aligned}
p_{+,+}(x) &= \left(\frac{1}{d} - 1\right) x^2 + C \text{ either increases or decreases with } x,\\
p_{-,+}(x) &= \left(\frac{1}{d} + 1\right) x^2 - C \text{ increases with } x,\\
p_{-,-}(x) &= \ \ x^2 + \frac{b}{d}x - C \quad \text{increases with } x,\\
p_{+,-}(x) &= -x^2 + \frac{b}{d}x + C\,.
\end{aligned}$$

We get $\|p_{+,+}\|_{[0,1]} = \max(|p_{+,+}(0)|, |p_{+,+}(1)|) = \max\left(C, \left|\frac{1}{d} - 1 + C\right|\right)$ and $\|p_{-,+}\|_{[0,1]} = \max(|p_{-,+}(0)|, |p_{-,+}(1)|) = \max\left(C, \left|\frac{1}{d} + 1 - C\right|\right)$, hence

$$\max(\|p_{+,+}\|_{[0,1]}, \|p_{-,+}\|_{[0,1]}) = \max\left(C, \frac{1}{d} + |C-1|\right). \tag{6}$$

We also have $\|p_{-,-}\|_{[0,1]} = \max(|p_{-,-}(0)|, |p_{-,-}(1)|) = \max\left(C, \left|\frac{b}{d} + 1 - C\right|\right)$. Now, if $x^* := \frac{b}{2d}$, the critical point of $p_{+,-}$, is in the interval $[0,1]$, we get $\|p_{+,-}\|_{[0,1]} = \max(|p_{+,-}(x^*)|, |p_{+,-}(1)|) = \max\left(\frac{b^2}{4d^2} + C, \left|\frac{b}{d} - 1 + C\right|\right)$, and then

$$\max(\|p_{-,-}\|_{[0,1]}, \|p_{+,-}\|_{[0,1]}) = \max\left(\frac{b}{d} + |C-1|, \frac{b^2}{4d^2} + C\right). \tag{7}$$

If otherwise $x^* = \frac{b}{2d} \geq 1$, we have

$$\max(\|p_{-,-}\|_{[0,1]}, \|p_{+,-}\|_{[0,1]}) = \max\left(C, \frac{b}{d} + |C-1|\right). \tag{8}$$

In view of (6), (7) and (8), the following proposition holds:

Proposition 3 *If $b \leq 2d$, we have*

$$s_1 = \max\left(\frac{\max(1,b)}{d} + |C-1|, \frac{b^2}{4d^2} + C\right).$$

If otherwise $b \geq 2d$, we have

$$s_1 = \max\left(\frac{\max(1,b)}{d} + |C-1|, C\right).$$

6 Reducing the minimization domain

We show in this section that only the case $b \leq c \leq 1$ can lead to a condition number $\kappa := \kappa_\infty(\underline{p}(b,c,d))$ smaller than $\frac{5}{4}$. First, we note that the dual basis of $\underline{p}(b,c,d)$ has the expression, for $f \in \mathcal{P}_2$ and with $t^* := \frac{C(1+b)}{b+C}$,

$$\begin{aligned}
\mu_1(f) &= \frac{d}{bC(b+2C+bC)}\left(-C(C-b^2)f(-1) + (b+C)^2 f(t^*)\right),\\
\mu_2(f) &= \frac{1}{C}f(0),\\
\mu_3(f) &= \frac{d}{bC(b+2C+bC)}\left(-C(C-b^2)f(1) + (b+C)^2 f(-t^*)\right).
\end{aligned}$$

We will use the upper bound $s_2 := s_2(\underline{p}(b,c,d)) \geq \max(\frac{1}{C}, \frac{d}{b})$, as we remark that $\|\mu_2\| = \frac{1}{C}$ and that $\|\mu_1\| \geq \frac{d}{b}$, considering $\mu_1(f)$ for $f(x) = x$.

1) **The case $c \geq 1$.**

According to proposition 3, we separate two subcases.

1a) *The case $b \geq 2d$.*

Since $s_1 \geq \frac{b}{d} + C - 1 \geq C + 1$, we have $\kappa \geq 1 + \frac{1}{C}$, so that we can assume $C \geq 4$ in order to get $\kappa \leq \frac{5}{4}$. Consequently, in view of $s_2 \geq \max(\frac{1}{C}, \frac{d}{b})$,

$$\begin{aligned}
&\text{i) if } \frac{b}{d} \geq C, && \text{then } s_1 \geq 2C - 1, && \text{and } \kappa \geq 2 - \frac{1}{C} \geq \frac{7}{4},\\
&\text{ii) if } C \geq \frac{b}{d}, && \text{then } s_1 \geq 2\frac{b}{d} - 1, && \text{and } \kappa \geq 2 - \frac{d}{b} \geq \frac{3}{2}.
\end{aligned}$$

1b) *The case $b \leq 2d$.*

If $C \geq 4$, we obtain $s_1 \geq \frac{b}{d} + 3$, so that $\kappa \geq 1 + \frac{3d}{b} \geq \frac{5}{2}$. Hence we can assume $C \leq 4$. Since $s_1 \geq \frac{b^2}{4d^2} + C$ and $s_2 \geq \max(\frac{1}{C}, \frac{d}{b})$,

$$\text{i) if } d \le \frac{b}{C}, \qquad \text{we get } \kappa \ge \frac{1}{C}\frac{b^2}{4d^2} + 1 \ge \frac{C}{4} + 1 \ge \frac{5}{4},$$
$$\text{ii) if } d \ge \frac{b}{C}, \qquad \text{we get } \kappa \ge \frac{b}{4d} + C\frac{d}{b} \ge \frac{C}{4} + 1 \ge \frac{5}{4},$$

the latter holding because $\frac{b}{4}\frac{1}{d} + \frac{C}{b}d$ is an increasing function of d on $\left[\frac{b}{2c}, +\infty\right)$, and $\frac{b}{2c} \le \frac{b}{C}$ as $C \le 4$.

2) **The case $c \le 1$ and $b \ge c$.**

On account of $C \le 1$, the point t^* lies in $[0,1]$. This implies that

$$\|\mu_1\| = \frac{d}{bC(b+2C+bC)}\left(|C(C-b^2)| + (b+C)^2\right).$$

With $b \ge c$, we get $\|\mu_1\| = \frac{d}{C} = \|\mu_3\|$. In view of (5) and of $\|\mu_2\| = \frac{1}{C}$, for fixed b and c, the choice $d = 1$ minimizes κ. Two cases have to be considered, according to proposition 3.

$$\text{2a) If } b \le 2, \qquad \text{then } \kappa \ge \left(\frac{b^2}{4} + C\right)\frac{1}{C} \ge \left(\frac{C}{4} + C\right)\frac{1}{C} = \frac{5}{4}.$$
$$\text{2b) If } b \ge 2, \qquad \text{then } \kappa \ge (b+1-C)\frac{1}{C} \ge \frac{3}{C} - 1 \ge 2.$$

7 Minimizing the condition number

We suppose now that $b \le c \le 1$, so that $\|\mu_1\| = \frac{d\lambda}{C} = \|\mu_3\|$, where

$$\lambda := \lambda(b,C) := \frac{b^2 + 2bC + 2C^2 - b^2C}{b(b+2C+bC)} = \frac{(b+C)^2 + C(C-b^2)}{(b+C)^2 - C(C-b^2)}.$$

Thus, for fixed b and c, the optimal choice is $d = \frac{1}{\lambda}$. We note that $b \le \frac{2}{\lambda}$, so that, according to proposition 3, it remains to minimize, under the conditions $0 < b \le c \le 1$, $C = c^2$,

$$\kappa_\infty\left(\underline{p}\left(b, c, \frac{1}{\lambda}\right)\right) = \max(F(b,C), G(b,C)),$$

where

$$F(b,C) := \frac{\lambda+1}{C} - 1 \quad \text{and} \quad G(b,C) := \frac{b^2\lambda^2}{4C} + 1.$$

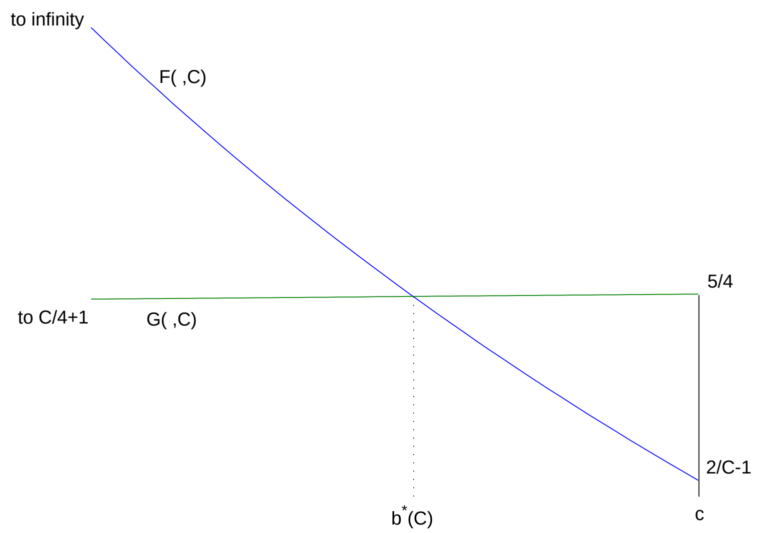
One calculates

$$(b^2 + 2bC + b^2C)^2\frac{\partial\lambda}{\partial b} = -4C^2(b+C)(b+1) \le 0,$$

hence λ, and therefore F, decreases with b. One also gets

$$(b+2C+bC)^2\frac{\partial(b\lambda)}{\partial b} = (1-C)(b(b+bC+4C)+2C^2) \geq 0,$$

hence $b\lambda$, and therefore G, increases with b.



Let us remark that $\lambda(c,C)=1$, thus $F(b,C) \geq F(c,C) = \frac{2}{C}-1$, so that, in order to get $\kappa \leq \frac{5}{4}$, we may assume $C \geq \frac{8}{9}$. Then we get $F(c,C) \leq \frac{5}{4} = G(c,C)$, and on the other hand, since $b\lambda(b,C) \underset{b\downarrow 0}{\longrightarrow} C$, we have $G(0,C) = \frac{C}{4}+1 < \lim_{b\downarrow 0} F(b,C) = +\infty$. Therefore, for each $C \in \left[\frac{8}{9},1\right]$, there exists a unique $b^*(C) \in [0,c]$ such that $F(b^*(C),C) = G(b^*(C),C) =: H(C)$, and we have

$$\kappa \geq \kappa_\infty\left(\underline{p}\left(b^*(C),c,\frac{1}{\lambda(b^*(C),C)}\right)\right) = H(C).$$

Finally, for any $C \in \left(\frac{8}{9},1\right)$, one has $H(C) = G(b^*(C),C) < G(c,C) = \frac{5}{4}$, which already proves that

$$\min_{b,c,d>0} \kappa_\infty(\underline{p}(b,c,d)) < \frac{5}{4}, \qquad \text{so that} \qquad \kappa_\infty(\mathcal{P}_2) < p_{\mathrm{int}}(\mathcal{P}_2,\mathcal{C}).$$

Let us now evaluate the minimal value of $\kappa_\infty(\underline{p}(b,c,d))$, i.e. the minimal value of H on $\left[\frac{8}{9},1\right]$. Let $C^* \in \left[\frac{8}{9},1\right]$ be such that $H(C^*) = \min_{\frac{8}{9}\leq C\leq 1} H(C)$, it must satisfies $H'(C^*)=0$, because C^* is neither $\frac{8}{9}$ nor 1. Now, differentiating the relation $F(b^*(C),C) = G(b^*(C),C)$ with respect to C, we get, for all

$C \in \left[\frac{8}{9}, 1\right]$,

$$b^{*\prime}(C) \times \left[\frac{\partial F}{\partial b} - \frac{\partial G}{\partial b}\right]\Big(b^*(C), C\Big) + \left[\frac{\partial F}{\partial C} - \frac{\partial G}{\partial C}\right]\Big(b^*(C), C\Big) = 0.$$

On the other hand, we have, for all $C \in \left[\frac{8}{9}, 1\right]$,

$$H'(C) = b^{*\prime}(C) \times \frac{\partial F}{\partial b}\Big(b^*(C), C\Big) + \frac{\partial F}{\partial C}\Big(b^*(C), C\Big).$$

Hence, annihilating the determinant of the previous system, we conclude that $H'(C) = 0$ if and only if $\left[\frac{\partial F}{\partial b}\frac{\partial G}{\partial C} - \frac{\partial F}{\partial C}\frac{\partial G}{\partial b}\right](b^*(C), C) = 0$. As a result, $C \in \left[\frac{8}{9}, 1\right]$ satisfies $H'(C) = 0$ if and only if $(b^*(C), C)$ is solution of the following (polynomial, after simplification) system:

$$\begin{cases} F(b, C) - G(b, C) &= 0, \\ \left[\frac{\partial F}{\partial b}\frac{\partial G}{\partial C} - \frac{\partial F}{\partial C}\frac{\partial G}{\partial b}\right](b, C) = 0. \end{cases}$$

Using the Groebner package from Maple, one finds that this system is equivalent to

$$144\,C^8 + 6498\,C^7 + 25839\,C^6 - 25108\,C^5 + 9827\,C^4 \\ - 17192\,C^3 + 2336\,C^2 + 1088\,C - 192 = 0,$$

and

$$60\,b^8 - 906\,b^7 - 1452\,b^6 + 2261\,b^5 + 6451\,b^4 \\ + 568\,b^3 - 3704\,b^2 - 1408\,b - 192 = 0,$$

which, in the prescribed domain, have the unique solution $C \approx 0.9402938300$, $b \approx 0.8675381234$. Computing F for these b and C gives us the value

$$\min_{b,c,d>0} \kappa_\infty(\underline{p}(b, c, d)) \approx 1.248394563.$$

8 Concluding remarks

8.1 About the assumption of symmetry

We tried to cover the case of symmetric bases completely by introducing an additional parameter a, $|a| \le b$, and minimizing the condition number over the bases:

$$p_1(x) := \frac{(x+a)(x+b)}{2d},\ p_2(x) := c^2 - x^2,\ p_3(x) := \frac{(x-a)(x-b)}{2d},\ x \in [-1,1].$$

The technique is the same, though the calculations become rather more intricated: whereas, when $b \le 1$ and $c \le 1$, we could show theoretically that the case $a \le 0$ does not lead to any improvement, the same conclusion for $a \ge 0$ was obtained numerically, using the Matlab function fminimax. Hence if, as we believe, best conditioned bases are symmetric, it is very likely that our optimal basis is actually the best conditioned basis of $\mathcal{P}_2$, implying the strict inequality $p(\mathcal{P}_2, \mathcal{C}) < \kappa_\infty(\mathcal{P}_2)$.

8.2 About the trigonometric case

A slightly different technique, based on the determination of the norms of the dual functionals via the extreme points of the unit ball of $\mathcal{P}_2$, can also be used. Let us note that the description of the extreme points of the unit ball of $\mathcal{P}_n$, for any $n \in \mathbb{N}$, was given by Konheim and Rivlin in [9]. For $\mathcal{T}_1$, the space of trigonometric polynomials of degree at most 1, this technique can also be applied, as we find easily that the extreme points of the unit ball of $\mathcal{T}_1$ are the family $\{\pm 1\} \cup \{\sin(\bullet - t), t \in \mathbb{T}\}$. If we trust once again the Matlab function fminimax, we conclude, quite surpisingly, that the Lagrange bases at equidistant points are best conditioned in $\mathcal{T}_1$, with

$$\kappa_\infty(\mathcal{T}_1) = p_{\mathrm{int}}(\mathcal{T}_1, \mathcal{C}) = \frac{5}{3}.$$

Let us note that this is not the projection constant of $\mathcal{T}_1$ in $\mathcal{C}$. Indeed, it is known that the Fourier projection from $\mathcal{C}$ onto $\mathcal{T}_n$ is minimal, and we easily derive

$$p(\mathcal{T}_1, \mathcal{C}) = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} \approx 1.435991124.$$

Acknowledgements

I am indebted to Dr A. Shadrin for his guidance and encouragements.

References

[1] C. de Boor. “A practical guide to splines”. Springer–Verlag, 1978.

[2] L. Brutman. *Lebesgue functions for polynomial interpolation – a survey*. Annals of Numerical Mathematics, 4:111-127, 1997.

[3] B.L. Chalmers and F.T. Metcalf. *Minimal generalized interpolation projections*. Journal of Approximation Theory, 20:302-313, 1977.

[4] B.L. Chalmers and F.T. Metcalf. *Determination of a minimal projection from $C([-1,1])$ onto the quadratics*. Numerical Functional Analysis and Optimization, 11:1-10, 1990.

[5] E.W. Cheney and W.L. Light. "A course in approximation theory". Brooks/Cole, 2000.

[6] E. W. Cheney and K. H. Price. *Minimal projections* in "Approximation Theory" (A. Talbot, Ed.), Academic Press, New York, 261-289, 1970.

[7] W. Gautschi. *The condition of orthogonal polynomials*. Mathematics of Computation, 26:923-924, 1972.

[8] W. Gautschi. *The condition of polynomials in power form*. Mathematics of Computation, 33:343-352, 1979.

[9] A. G. Konheim and T. J. Rivlin. *Extreme points of the unit ball in a space of real polynomials*. American Mathematical Monthly, 73:505-507, 1966.

[10] T. Lyche. *A note on the condition numbers of the B-spline bases*. Journal of Approximation Theory, 22:202-205, 1978.

[11] T.J. Rivlin "Chebyshev polynomials: from approximation theory to algebra and number theory". Wiley–Interscience, 1990.

[12] K. Scherer and A. Shadrin. *New upper bound for the B-spline basis condition number. II. A proof of de Boor's 2^k-conjecture*. Journal of Approximation Theory, 99:217-229, 1999.

[13] S.S. Szarek. *Spaces with large distance to ℓ_∞^n and random matrices*. American Journal of Mathematics, 112:899-942, 1990.

[14] N. Tomczak–Jaegermann. "Banach-Mazur distances and finite-dimensional operator ideals". Longman Scientific & Technical, 1989.