MATHEMATICAL TRIPOS PART IB

ELECTROMAGNETISM Lent 2008

Examples 1

1. A current density, as a function of position $\mathbf{r}$ and time t, has the form

$$\mathbf{J} = C\,\mathbf{r}\,e^{-atr^2}$$

where C and a are constants. Show that the equation of conservation of charge can be satisfied by writing the charge density in the form

$$\rho = (f + t\,g)\,e^{-atr^2}$$

where f and g are functions of position, to be determined.

2. Sketch roughly the field lines (including arrows to denote sense) and equipotentials for the following systems of point charges:

(a) A single charge $+q$;

(b) Two charges $+q$ separated by a distance $2a$;

(c) Two charges $\pm q$ separated by a distance $2a$.

3. Use Gauss's theorem to evaluate the electric field due to a charge distribution of density $\rho = \rho_0\, e^{-k|z|}$, where ρ_0 and k are positive constants. Show that it is of the form $\mathbf{E} = (0, 0, E(z))$, where $E(-z) = -E(z)$, given for $z > 0$ by

$$E(z) = \frac{\rho_0}{\epsilon_0 k}\left(1 - e^{-kz}\right) .$$

4. Use Gauss's theorem to obtain the field everywhere of a charge of uniform density ρ occupying the region $a < r < b$, r being the distance from the origin.

Show that in the limit $b \to a$, $\rho \to \infty$ with $(b-a)\,\rho = \sigma$ remaining finite, the electric field suffers a discontinuity of amount σ/ϵ_0 in crossing the layer of charge.

5. Compute the electric field due to an infinite line charge by integrating the expression obtained from the inverse square law.

6. A circular disk of radius a has uniform surface charge density σ.

Compute the potential at a point of the axis of symmetry at distance z from the centre, and hence the electric field there. Find the discontinuity in the normal electric field at the centre of the disk.

7. Calculate the potential at a point $\mathbf{r}$ due to an electric dipole of moment $\mathbf{p}$ at the origin.

Calculate (see Sec. 2.1 of the lecture notes) the potential at a point P with a spherical polar coordinates $(r,\ \theta,\ \phi)$ due to charges $-e$, $2e$ and $-e$ at points with cartesian coordinates $(0,\ 0,\ -a)$,
$(0,\ 0,\ 0)$ and $(0,\ 0,\ a)$ respectively, where $a \ll r$.

8. An electrostatic charge density $\rho(\mathbf{r})$, which does not extend to infinity in space, has an associated potential $\phi(\mathbf{r})$; there are no point- line- or surface-charges. The energy may be assumed to be given by

$$W = \tfrac{1}{2} \int_V \rho\, \phi\, d\tau \ .$$

Derive the alternative formula

$$W = \tfrac{1}{2}\, \epsilon_0 \int_V |\mathbf{E}|^2 d\tau \ .$$

As a model of the atomic nucleus, a total charge Q is assumed to be uniformly distributed inside a sphere of radius R. Find both ϕ and $\mathbf{E}$ both inside and outside the charge distribution, and show from the results obtained that the two expressions for W agree.

9. The potentials of three concentric spherical conductors of radii $r = a, b, c,\quad a < b < c$, are held at the values $\phi = 0, V, 0$. Solve the potential problem by using solutions of the type $\phi = A + B/r$ and $\phi = C + D/r$ for $a < r < b$ and $b < r < c$, and continuity at $r = b$. What is the total charge on the conductor of radius b, and the capacitance of the system?

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