MATHEMATICAL TRIPOS PART IB

ELECTROMAGNETISM

Examples 2

1. A potential difference V is maintained between two coaxial cylinders of radii $r = a, b, \quad b > a$ and the gap $a < r < b$ is filled with material of uniform conductivity σ. Find the potential in $a < r < b$ as a function of r. Hence find the total current I per unit length that flows from $r = a$ to $r = b$, and deduce the effective resistance R per unit length as given by $V = IR$.

Repeat the calculation in spherical geometry, *i.e.* for concentric spheres of radii $r = a, b, \quad b > a$.

2. If $\mathbf{B} = (0, 0, B)$ in cartesians, with B constant, verify that the following vector potentials satisfy $\mathbf{B} = \nabla\times\mathbf{A}$:

(i) in cartesians, $\mathbf{A} = (0, xB, 0)$

(ii) in cylindrical polars, $\mathbf{A} = (0, \frac{1}{2}B\,s, 0)$

(iii) in spherical polars, $\mathbf{A} = (0, 0, \frac{1}{2}B\,r\mathrm{sin}\theta)$

[For (iii), from a decent diagram of spherical polars, find $\mathbf{k} = \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta$.]

3. Steady current I in the z-direction flows unifromly in the the region between the cylinders $x^2 + y^2 = a^2$ and $(x + d)^2 + y^2 = b^2$, where $0 < d < (b - a)$. Show that the associated magnetic field $\mathbf{B}$ is uniform throughout the region $x^2 + y^2 < a^2$, being in the y-direction and of magnitude $\mu_0\, I\, d\, \left[2\pi\left(b^2 - a^2\right)\right]^{-1}$.

[We assume that magnetic fields can be calculated within conducting media by same formulas that apply to their calculation in free space. Thus the field in the cavity may be calculated by the *superposition* of the fields due to a current density $(0, 0, J)$ flowing in the complete large cylinder, and (2) $(0, 0, -J)$ in the smaller cylinder.]

4. State the Biot-Savart Law for the magnetic field due to a current loop, and use it to obtain the magnetic field at the centre of a square loop of wire, with sides of length a, which carries a current I.

5. State the expression for the force on a current element in a magnetic field. A rigid wire, in the form of a plane closed loop of area A, carries a current I. Use the stated expression to find the couple required to keep the wire stationary in a uniform magnetic field $\mathbf{B}$, which makes an angle θ with the normal to the plane of the loop.

6. A circular loop of wire of radius a carries current I. Take cartesian coordinates with O at the centre of the loop and the z-axis normal to the loop at O. Use the

Biot-Savart law to show that the magnetic field at $(0, 0, z)$ is $(0, 0, B)$ where

$$B = \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}}.$$

Show that, for $z \gg a$, this agrees with the result of calculating $B = -\dfrac{\partial \Omega}{\partial z}$ for the magnetic scalar potential Ω of the current loop viewed as magnetic dipole of moment $\mathbf{m}$.

7. A cylindrical conductor of radius a, with axis along the z-axis, carrying a current density I of uniform density J, $\quad I = \pi a^2 J$, is situated in empty space. Use Ampére's law to show that the magnetic field within the conductor is given by

$$\mathbf{B} = B\mathbf{e}_\phi, \quad B = \frac{1}{2}\mu_0 J s,$$

in terms of cylindical polars. Calculate the force $\mathbf{F}$ per unit volume acting on the conducting material.

Suppose the conducting medium is a plasma (a gaseous conductor) held in hydrostatic equilibrium by the magnetic forces. Using the equation

$$-\nabla p + \mathbf{F} = 0$$

of hydrostatic equilibrium, and the boundary condition $p(a) = 0$, show that the pressure $p(s)$ is given by

$$p(s) = \frac{1}{4}\mu_0 J^2 (a^2 - s^2).$$

8. State the boundary conditions that apply at a surface current, and the expression for the force per unit area acting on such a surface.

A uniform steady surface current J flows azimuthally around the surface of a long conducting hollow cylinder of radius a. Find the magnetic field $\mathbf{B}(\mathbf{r})$ at all $\mathbf{r}$. Show that there is an outward force $\frac{1}{2}\mu_0 J^2$ per unit area on the cylinder.

9. A steady current I_1 flows around a closed loop C_1. Write down the Biot-Savart expression for the magnetic field at a point $\mathbf{r}$ due to this current. Write down an expression for the force exerted on a second loop I_2, C_2. Transform this (with the aid of Stokes's theorem) to a form which exhibits its antisymmetry with respect to $(1, 2)$, in agreement with Newton's Third Law.