MATHEMATICAL TRIPOS PART IB

ELECTROMAGNETISM: Examples 3

1. It is given, as a consequence of the Maxwell equation $\nabla\wedge\mathbf{E} = -\partial\mathbf{B}/\partial t$, that

$$\oint_{C(t)} (\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) \,.\, d\mathbf{r} = -\frac{d}{dt}\int_{S(t)} \mathbf{B} \,.\, d\mathbf{S},$$

where $C(t)$ is a simple closed curve deforming in time, $\mathbf{v}$ is the velocity of a point on C moving with the curve, and $S(t)$ is a surface spanning C.

Verify this by evaluating the two integrals in the particular case in which $\mathbf{E}$ and $\mathbf{B}$ are given in cylindrical polar coordinates (r, θ, z) by

$$\mathbf{E} = (0, 1, 0)e^{-t} \quad\text{and}\quad \mathbf{B} = (0, 0, 1)r^{-1}e^{-t}$$

and $C(t)$ is the circle in the plane $z = 0$ with centre at 0 and radius $1 + t$.

2. A wire of resistance R per unit length is bent so as to form three sides AB, BC, CD of a rectangle ABCD which is held stationary in a horizontal plane. Another wire EF, of mass m, has length equal to BC, and resistance R per unit length; its ends E and F are constrained to lie on the wires AB and CD respectively, and slide without friction along them. A uniform vertical magnetic field $\mathbf{B}$ is applied to the system; for time $t > 0$, $|\mathbf{B}|$ varies as α/t where α is constant. Denoting by x the perpendicular distance between BC and EF, deduce the differential equation satisfied by x for $t > 0$, $0 < x <$ AB, and write down one solution.

[You may assume that the effect of the magnetic field due to any current flow in the wires is negligible compared to the effect of the applied field $\mathbf{B}$.]

3. Find the magnetic field described by the vector potential $\mathbf{A} = (0, \frac{1}{2}B_0 rz, 0)$ in cylindrical coordinates (r, θ, z) with B_0 constant

Use Stokes's theorem to evaluate the flux of the magnetic field through a conducting loop of radius a and resistance R which lies in the plane $z = h(t)$ with its centre on the axis. Hence find the induced current in the loop as h varies in time, neglecting self inductance.

Suppose now that the loop has negligible mass, so that the external force $\mathbf{F}$ causing the movement of the loop exactly balances the force exerted on the loop by the magnetic field. By calculating this magnetic force show that the rate of working of $\mathbf{F}$ is equal to the rate of dissipation of energy due to the resistance of the loop.

4. Starting from Maxwell's equations and Ohm's Law, show that any charge distribution within a stationary conductor of uniform conductivity σ will decay exponentially in time with a decay constant σ/ϵ_0 independently of any magnetic field that may be present. Can Ohm's law be trusted in this context.

A uniform conducting sphere of radius R is set up with a uniform volume charge density ρ_0 throughout its interior. Obtain the distributions of charge density ρ, current density $\mathbf{j}$ and electric field $\mathbf{E}$ within the sphere as functions of time. What is the electric field outside the sphere?

Show that the rate of ohmic heat generation in the sphere equals the rate of dissipation of electrostatic field energy.

5. A steady current I flows along a cylindrical conductor of constant circular cross-section and uniform conductivity σ. Show, using the relevant equations for $\mathbf{E}$ and $\mathbf{J}$, that the current is distributed uniformly across the cross-section of the cylinder, and calculate the electric and magnetic fields just outside the surface of the cylinder.

Verify that the integral of the Poynting vector over unit length of the surface is equal to the rate per unit length of dissipation of electrical energy as heat.

6. A monochromatic wave with fields

$$\mathbf{E}_{inc} = (E_0, 0, 0)\,\exp i(kz - \omega t), \quad \mathbf{B}_{inc} = (1/c)(0, E_0, 0)\,\exp i(kz - \omega t),$$

is incident from empty space in $z < 0$ on perfectly conducting material in $z > 0$ with surface $z = 0$. Show that, if the reflected fields are

$$\mathbf{E}_{ref} = (-E_0, 0, 0)\,\exp i(-kz - \omega t), \quad \mathbf{B}_{ef} = (1/c)(0, E_0, 0)\,\exp i(-kz - \omega t),$$

then the total fields, $\mathbf{E}_{inc} + \mathbf{E}_{ref}$ and $\mathbf{B}_{inc} + \mathbf{B}_{ref}$, satisfy Maxwell's equations and all the relevant boundary conditions at $z = 0$. What surface current flows in the plane $z = 0$? Given that the physical fields (and the physical surface current) are the real parts of the expressions just considered, calculate the Poynting vector in $z < 0$ and show that its time average (over one period of the wave motion) is zero. Show also that the normal force F per unit area exerted on the surface $z = 0$ has time average given by $\langle F \rangle = \epsilon_0 E_0^2$.

7. Perfectly conducting planes are positioned at $z = 0$ and $z = a$. Show that a monochromatic field, consisting of plane waves independent of x and y, can exist between the planes if and only if the angular frequency takes one of the values $n\pi c/a, \quad n = 1, 2, 3, \ldots$.

8. Write down Maxwell's equations for the electric and magnetic fields in a vacuum, and show that each cartesian component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the wave equation

$$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \ .$$

State the boundary conditions that must be satisfied by $\mathbf{E}$ and $\mathbf{B}$ just outside the surface of a perfect conductor. Verify that a wave may propagate in the direction $0z$ between two perfectly conducting planes $y = 0$ and $y = b$, if the wave has field components

$$E_x = \omega A \sin\left[\frac{n\pi y}{b}\right] \sin(kz - \omega t),$$

$$B_y = kA \sin\left[\frac{n\pi y}{b}\right] \sin(kz - \omega t), \quad B_z = \frac{n\pi A}{b} \cos\left[\frac{n\pi y}{b}\right] \cos(kz - \omega t),$$

where n is an integer and A is constant. If λ is the wavelength of such waves, and λ_∞ is that of waves of the same frequency in the absence of the conducting plates, show that

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_\infty^2} - \frac{n^2}{4b^2}.$$