2 Electrostatics

2.1 Electrostatic potential

Electrostatics is the study of time independent electromagnetic phenomena in the absence of currents and magnetic fields. Then Maxwell's equations are

$$\nabla\times\mathbf{E} = 0 \tag{1}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho. \tag{2}$$

The first equation can be satisfied by defining the (electrostatic) potential ϕ by means of

so that the second equation yields Poisson's equation

$$\nabla^2\phi = -\frac{1}{\epsilon_0}\rho. \tag{3}$$

In this way the study of electrostatics is reduced to the study of a single equation – Poisson's equation. In regions of space where there is no electric charge $\rho = 0$, this reduces to Laplace's equation

$$\nabla^2\phi = 0. \tag{4}$$

Example. Find the electrostatic potential ϕ for the point charge q at O.

Since Poisson's equation is a linear equation for ϕ, the 'superposition principle' applies and tells us that any linear combination (superposition) of solutions is again a solution. An example of this is

The electric dipole.

Find the electrostatic potential ϕ for a system of two point charges: $-q$ at O and $+q$ at $\mathbf{d}$.

If $d = |\mathbf{d}|$ is small we have

The electric dipole arises by taking the limits $q \to \infty, d \to 0$ in such a way that qd remains constant, at a finite value $qd = p$. Then $\mathbf{p} = q\mathbf{d}$ defines the dipole moment of the electrical dipole, and its potential is given by

The electric quadrupole: not lectured

We can easily go further to the linear quadrupole with charges $-q$ at $\pm\mathbf{d}$ and $2q$ at the origin, so that the system has zero total charge and also zero dipole moment. (It looks like a pair of dipoles pointing in opposite directions.)

$$\begin{aligned}
\frac{4\pi\epsilon_0}{q}\phi &= \frac{2}{r} - \frac{1}{|\mathbf{r}+\mathbf{d}|} - \frac{1}{|\mathbf{r}-\mathbf{d}|} \\
&= \frac{2}{r} - [\frac{1}{r} + \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}] - [\frac{1}{r} - \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}] \\
&= -(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}. \qquad (5)
\end{aligned}$$

Note that this approach gets the cancellation of unwanted terms to happen ahead of their evaluation. Hence

$$4\pi\epsilon_0\phi = -q(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r} = -qd^2\frac{\partial^2}{\partial z^2}\frac{1}{r} = -qd^2\frac{\partial}{\partial z}(-\frac{z}{r^3}) = qd^2(\frac{1}{r^3} - \frac{3z^2}{r^5}). \qquad (6)$$

In spherical polars the quadrupole potential is

$$4\pi\epsilon_0\phi = qd^2\frac{1 - 3\cos^2\theta}{r^3}. \qquad (7)$$

We note that the point charge, electric dipole and quadrupole potentials go to zero as r goes to infinity respectively like $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$.

The general charge distribution $\mathcal{D}$

From the superposition principle

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|}. \tag{8}$$

This is consistent with our definition of the electric field $\mathbf{E}$ from Chapter 1, at least for $\mathbf{r} \notin \hat{V}$. We do not have time to provide the proof, by standard methods in vector calculus, that $\phi(\mathbf{r})$ above satisfies Poisson's equation for all $\mathbf{r} \in V$.

Large distance behaviour of ϕ

Using Taylor's theorem we find

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \left(\frac{1}{r} - \mathbf{r}'\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\cdot\boldsymbol{\nabla})^2\frac{1}{r}\ldots\right)\rho(\mathbf{r}')d\tau'. \tag{9}$$

Uniqueness

Suppose we are given a charge distribution $\rho(\mathbf{r})$ throughout a fixed spatial volume V, then Poisson's equation in V has a unique solution provided that, on $S = \partial V$, either

(i) (Dirichlet boundary conditions) $\phi(\mathbf{r})$ is specified for all $\mathbf{r} \in S$,

or

(ii) (Neumann boundary conditions) $\dfrac{\partial\phi}{\partial n} = \mathbf{n}\cdot\boldsymbol{\nabla}\phi(\mathbf{r}) = -\mathbf{n}\cdot\mathbf{E}(\mathbf{r})$ is specified for all $\mathbf{r} \in S$.

Field lines and equipotentials

We mention a way of gaining some insight into the nature of the electric field surrounding a system of charges.

One draws the field lines of $\mathbf{E}$ for the system. A field line here is a line at each of whose points $\mathbf{E}$ is tangent to the line.

Also one draws on the same diagram the equipotentials of the system. These are surfaces $\phi =$constant. As $\mathbf{E} = -\boldsymbol{\nabla}\phi$, and $\boldsymbol{\nabla}\phi$ is everywhere normal to such surfaces, it follows that the field lines cut the equipotentials at right angles.

2.2 Gauss's theorem and the calculation of electric fields

In Sec. 1.5 we proved Gauss's theorem

$$\frac{1}{\epsilon_0}Q = \int_S \mathbf{E} \cdot \mathbf{dS}, \tag{10}$$

where

$$Q = \int_V \rho d\tau, \tag{11}$$

is the total charge contained in the spatial volume V, $\partial V = S$. We now apply it to the calculation of the electric fields of simple systems of charge.

a) The point charge q at the origin has been treated in Sec 1.1.

b) Line charge lying along the z-axis with uniform (line) density of charge λ per unit length (compare with a direct calculation in Section 1.1)

c) Plane sheet P occupying the plane $z = 0$, carrying uniform charge density σ per unit area.

For S use the 'Gaussian pillbox': a cylinder of cross-sectional area A, with axis $\mathbf{k} = (0, 0, 1)$, with plane ends at $z = h$ and $z = -h$. By symmetry $\mathbf{E}$ is perpendicular to P. Above P we have $\mathbf{E} = E\mathbf{k}$ and below $\mathbf{E} = -E\mathbf{k}$ for some $E = E(h)$. This time $\mathbf{E} \cdot \mathbf{dS}$ is zero on the curved sides of the pill-box.

d) Parallel plane sheets in the planes $z = 0$ and $z = a$, carrying uniform distributions of charge respectively of charge with surface densities $\pm\sigma$ per unit area.

e) Spherical shell, centre at O, radius r', uniform charge density σ per unit area, and thus total charge $Q = 4\pi r'^2 \sigma$.

Check that $E = \mathbf{E} \cdot \mathbf{e}_r$, the normal component of $\mathbf{E}$, has discontinuity $\frac{1}{\epsilon_0}\sigma$ at $r = r'$.

f) Sphere of radius R carrying uniform charge of density ρ per unit volume, and thus total charge $Q = \frac{4\pi}{3}R^3\rho$.

Note that $E(r)$, the normal (and here only) component of $\mathbf{E}$, is continuous at $r = R$.
We can use $\mathbf{E} = -\boldsymbol{\nabla}\phi = -\mathbf{e}_r\dfrac{\partial\phi}{\partial r}$ to determine the potentials ϕ_1 outside, and ϕ_2 inside, the charge distribution.

g) The discontinuity law at a surface carrying surface charge.

Summary; Electrostatic boundary conditions. Three fundamental quantities: $\rho, \phi, \mathbf{E}$.

BCs: The electric field is discontinuous at the surface charge: $\mathbf{E}_+ - \mathbf{E}_- = \frac{\sigma}{\epsilon_0}\mathbf{n}$. But the potential is continuous across any boundary:

Solutions of Laplace's equations

Very often, we are interested in finding ϕ in a region where $\rho = 0$. Because ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, we can apply all the mathematical machinery of potential theory (solutions of Laplace's equation), developed in 1B Methods:

A: Spherical axisymmetric geometry: (r, θ, ϕ)

Recall the general axisymmetric solution of Laplace's equation obtained by separation-of-variable methods in the usual spherical polar coordinates, by trying $\phi = R(r)\,\Theta(\theta)$, etc.,

$$\phi = \sum_{n=0}^{\infty}\left\{A_n r^n + B_n r^{-(n+1)}\right\}P_n(\cos\theta),$$

where A_n, B_n are arbitrary constants, P_n is the Legendre polynomial of degree n, r is the spherical radius ($r^2 = x^2 + y^2 + z^2$), so that $z = r\cos\theta$, and θ is the co-latitude. (Recall that $P_0(\mu) = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2}$, etc.)

B: Cylindrical (circular 2D) geometry: (s, ϕ) Much the same pattern as before, except that, as always in 2D potential theory, the solutions involve logarithms as well as powers of the radial coordinate s; and potentials can be multi-valued. Also, there is no particular direction corresponding to the axis of symmetry in case **A**.

Recall the general solution obtained by separation of variables in 2D cylindrical polars s, θ:

$$\phi = A_0 \log s + B_0\theta + \sum_{n=1}^{\infty} \{A_n s^n \cos(n\theta + \alpha_n) + B_n s^{-n} \cos(n\theta + \beta_n)\}$$

2.3 Perfect conductors

In an **insulator**, such as glass or rubber, each electron is attached to a particular atom. In a metallic **conductor**, in contrast, one or more electrons per atom are free to move through the material. A **perfect conductor** is a material containing an unlimited supply of free charges.

(a) $\mathbf{E} = 0$ inside a conductor.

(b) $\rho = 0$ inside a conductor.

(c) Any net charge resides on the surface.

(d) A conductor is an equipotential.

(e) $\mathbf{E}$ is perpendicular to the surface, just outside a conductor.

(f) From g) of Section 2.2

$$\frac{1}{\epsilon_0}\sigma = \mathbf{n}\cdot\mathbf{E}|_-^+ = \mathbf{n}\cdot\mathbf{E} = E \tag{12}$$

This uses the fact that $\mathbf{E} = 0$ inside $\mathcal{C}$, (*i.e.* on the minus side of the surface S of $\mathcal{C}$).

The Force on a charged conductor

2.4 Electrostatic energy

The potential energy (PE) of a point charge Q at $\mathbf{r}$ in an electric field of potential $\phi(\mathbf{r})$ is the work that must be done on Q to bring it from infinity (where $\phi = 0$) to $\mathbf{r}$.

$$PE = W = -\int_{\infty}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} = \tag{13}$$

Consider a system of point charges q_i, $i = 1, 2, \ldots, n$, bringing them from infinity to their final positions in order, doing work (denoting $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, $r_{ij} = |\mathbf{r}_{ij}|$, and $\sum_{i=1}^{n}\sum_{j<i} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}$.)

Thus W by construction gives the electrostatic energy of the system.
But the potential at q_i due to all the other charges is

so that

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\phi_i. \tag{14}$$

The corresponding result for a continuous distribution of charge density $\rho(\mathbf{r})$ in volume V then is

If there are conductors $\mathcal{C}_i$ with charges Q_i at potentials ϕ_i, then the contribution which they make to W is given by

(Recall that the potential is constant on a conductor).

Field energy in electrostatics

Given a charge distribution $\rho(\mathbf{r}')$ distributed over a finite volume $\hat{V}$ and a set of conductors all in some finite region of space in which an origin is taken. Let V be all space bounded by a sphere S at infinity, but excluding the interiors of the conductors. Find the energy of the electrostatic field.

2.5 Capacitors and capacitance

A pair of conductors carrying charges $\pm Q$ constitute a capacitor (or a condenser). Since their potentials are proportional to Q, the same applies to their potential difference $V = \phi_1 - \phi_2$.

Therefore we define the **capacitance** C of the capacitor by

$$C \equiv \frac{Q}{V}. \tag{15}$$

It turns out always to be a constant that depends on the configuration of the two conductors.

a) Parallel-plate capacitor (two metal surfaces of areas A held a distance a apart.) Find the capacitance.

The field lines are mainly straight lines perepndicular to the plates. We assume the distance a between the plates is small on a scale set by the area A of the plates. Thus we may neglect ' edge effects', so called because the electric field lines near to the edges of the plates bulge out from between the plates.

b) Concentric spheres S_1 and S_2 of radii a and $b > a$, carrying charges Q and $-Q$. Find the capacitance.
Take $\phi = 0$ at $r = b$ and $\phi = V$ at $r = a$.