3 Steady electric currents and magnetism

3.1 Steady current flow

Here we study steady current flow in conducting material. This is governed by Maxwell's equations without $\frac{\partial}{\partial t}$ terms, so that we have

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \boldsymbol{\nabla} \times \mathbf{E} = 0, \tag{1}$$

together with the experimental law, valid for simple conductors, but not, for example, for nonisotropic materials such as crystalline material,

$$\mathbf{J} = \sigma \mathbf{E},\tag{2}$$

where σ is the conductivity of the material. (Both conductivity and surface charge are normally denoted by the same symbol σ . We seldom have contexts in which both arise.)

Consider steady current flow in regions of conducting material, outside of batteries.

We might ask: can we obtain an understanding of the elementary form

$$V = IR \tag{3}$$

of Ohm's law, relating the potential difference across the ends of a conductor to the current that flows within it?

We do this here for a simple example; there are two others in Problem Set 2.

Example: Uniform current enters the plate of uniform thickness δ shown in the diagram. Relate V and I via Ohm's law.

Generation of heat by steady current flow

Consider steady current in a tube. How much energy is generated by the flow?

Consider the tube of flow shown, *i.e.* the cylinder whose sides are lines of **E** and **J** and whose ends are equipotentials. Current of density **J** enters at the end A where the potential is ϕ_A and leaves at B where the potential is $\phi_B < \phi_A$. The potential difference is

In unit time charge $J\delta S$ enters the tube at A in unit time and leaves at B. The work done on this charge moving it through the potential difference V in unit time is

$$(J\delta S)V = (J\delta S)(E\delta r) = JE(\delta S \ \delta r) = (\mathbf{J} \cdot \mathbf{E})\delta\tau.$$
(4)

This work done corresponds to the conversion of electrical energy into heat, *i.e.* to the loss of electrical energy. The energy loss per unit time in volume τ , with surface S is

Consider a conductor with current entering it and leaving it at ends S_1 and S_2 , which are equipotentials of potentials ϕ_1 and ϕ_2 .

Using the elementary form (3) of Ohm's law, we have shown that the energy generation per unit time in a conductor of resistance R through which flows a current I is

$$W = RI^2.$$
 (5)

This is a formula familiar from elementary studies for the energy dissipated in unit time as heat.

3.2 Magnetostatics

This deals with steady currents and the associated (time independent) magnetic fields. It is governed by the equations

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \, \mathbf{J}, \quad (\Rightarrow \boldsymbol{\nabla} \cdot \mathbf{J} = 0) \tag{6}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{7}$$

Eq. (7) is automatically satisfied when the vector potential \mathbf{A} is introduced via

Recall Section 1.6: A is not unique, because we can transform the vector potential according to

In what follows, we therefore assume that we can deal with vector potentials A which obey

$$\nabla \cdot \mathbf{A} = 0.$$
 (Coulomb gauge) (8)

Biot-Savart law.

So we can write down the solution of these Poisson's equations as

$$A_{k}(\mathbf{r}) = \frac{\mu_{0}}{4\pi} \int_{V} \frac{J_{k}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_{0}}{4\pi} \int_{V} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'.$$
 (9)

Since it is not obvious that the expression (9) for **A** satisfies (8), we ought to prove that it does. When this is done, it follows that

satisfies (6). In calculating **B**, note that $\nabla \times$ acts only on the **r** variable, found only in the denominator factor of expression (9) for **A**.

Consider a current of density **J** flowing in an element $\delta \mathbf{r}$ of a very thin wire of cross-sectional area A. Neglecting the thickness of the wire, we can write, for the vector-potential and the magnetic field due to a wire which carries a current I and takes the form of a simple curve C, the expressions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathbf{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{10}$$

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \int_C \frac{(\mathbf{r} - \mathbf{r}') \times \mathbf{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}.$$
 (11)

These results for \mathbf{B} are each often referred to as the Biot-Savart law.

Proof that (9) satisfies (8).

[Care with $\nabla^2 \mathbf{F}$ for a vector field \mathbf{F} may be needed. There is no problem in Cartesians, and hence probably not in the material of this course:

$$(\boldsymbol{\nabla}^2 \mathbf{F})_k = (\partial_j \partial_j) F_k \tag{12}$$

where $\nabla^2 = \partial_j \partial_j$ is the usual expression used in Laplace's equation. In other coordinate systems, where the unit basis vectors are themselves coordinate dependent, $(\nabla^2 \mathbf{F})_{\alpha}$, the component of the vector $\nabla^2 \mathbf{F}$ along the unit vector \mathbf{e}_{α} , is no longer given by $(\nabla^2)\mathbf{F}_{\alpha}$. The correct result however follows from use of $\nabla^2 \mathbf{F} = -\nabla \times (\nabla \times \mathbf{F}) + \nabla (\nabla \cdot \mathbf{F})$ where each of the two terms on the right is calculable by two well-defined steps in any system of orthogonal curvilinear co-ordinates.

3.3 Magnetic fields of simple current distributions

To calculate these one may use Ampère's law, the Biot-Savart law or perhaps first calculate A.

a) Infinite straight wire carrying current I. Calculate **B** at point P, distance s away from the wire using the Biot-Savart law. (Previously calculated in Section 1.5 using Ampère's law.)

b) Long solenoid

This is a continuous wire carrying current I wound round a very long right circular cylinder, so long that end effects can be ignored. Assume there are N turns of wire per unit length, with N large, wound in a spiral of very small pitch, so that we can regard the cylindrical surface as carrying a surface current. Use cylindrical polars (r, ϕ, z) , with z-axis at the axis of the cylinder. Then $\mathbf{s} = NI\mathbf{e}_{\phi}$ gives the current density, *i.e.* the current per unit length, measuring the charge crossing unit length in unit time. The answer obtained here illustrates the general discontinuity law given in Sec. 1.8, and proved in Sec. 3.8

$$\mathbf{n} \times \mathbf{B}|_{-}^{+} = \mu_0 \,\mathbf{s},\tag{13}$$

at a surface of discontinuity carrying a surface current density s per unit length. We have $n \times B|^+ = 0$, and

$$\mathbf{n} \times \mathbf{B}|_{-} = (-)\mathbf{e}_r \times (\mu_0 \, N I \mathbf{k}) = \mu_0 \, N I \mathbf{e}_\phi = \mu_0 \mathbf{s}. \tag{14}$$

c) Long cylindrical conductor

Consider current, flowing in a long right circular cylinder and distributed uniformly over its circular cross-section, of area $A = \pi a^2$. Assume that magnetic fields can be calculated within the conducting material by the same formulas as apply in the vacuum or free-space. This is a good approximation for good conductors, which do have similar magnetic properties to free-space.

3.4 The large distance expansion of the vector potential

Consider the formula (9) for the vector potential in the situation in which the distribution of current density is confined to a subset $\hat{V} \subset V$ =allspace. Chose the origin near to it. Then a long calculation based on Taylor's theorem yields the following leading large r approximation to $\mathbf{A}(\mathbf{r})$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \mathbf{m} \times \mathbf{r},\tag{15}$$

where the magnetic dipole moment of the current distribution is defined by

$$\mathbf{m} = \frac{1}{2} \int_{V} \mathbf{r} \times \mathbf{J}(\mathbf{r}) d\tau.$$
(16)

We consider in detail only the following case.

3.5 The current loop

Here we look at the vector potential \mathbf{A} of a current loop, *i.e.* a wire of negligible cross-section shaped in the form of a closed contour C, carrying a current I. For simplicity, let C define a plane loop of area $\mathbf{S} = S\mathbf{n}$ of unit normal \mathbf{n} .

Chose an origin near the loop and seek the vector potential of its magnetic field, at distances large on a scale set by the physical dimensions of the loop. (Or, consider $\mathbf{A}(\mathbf{r})$ due to a small loop.)

Let S be a surface such that $\partial S = C$. Let **c** be an arbitrary constant vector, and work on $\mathbf{c} \cdot \mathbf{A}$

We have reached a result which agrees exactly with (15). To see this recall the usual conversion $\int_V (\ldots) \mathbf{J}(\mathbf{r}) d\tau \to \int_C (\ldots) I \mathbf{dr}$. Then (16) gives

$$\mathbf{m} = \frac{1}{2}I \int_C \mathbf{r} \times \mathbf{dr} = IS\mathbf{n}.$$
 (17)

We have obtained a result crucial to the understanding of magnetism at all levels: a small current loop gives, via (17), a physical realisation of a magnetic moment.

3.6 Dipole view of m

We now show why, in the previous section, we referred to \mathbf{m} as a magnetic dipole moment. At points where there is no charge density $\mathbf{J} = 0$, the magnetic field \mathbf{B} obeys

We have found a certain analogy between the magnetic dipole moment \mathbf{m} that determines the leading large r behaviour of the vector potential $\mathbf{A}(\mathbf{r})$ of a current distribution localised near the origin of space, and the electric dipole. Since the 'dipole term' gives the leading contribution to $\mathbf{A}(\mathbf{r})$, this underlines the fact that magnetism has no analogue of the point charge: as far as is known at present magnetic monopoles do not exist. But the small current loop provides a physical realisation of a magnetic dipole.

[* A brief informal aside

If one considers atoms which possess spin about some axis, one can see roughly that the motion of their electrons approximate to current loops with moments parallel to this axis. If the spin axes of all the atoms, in some material made up of such atoms, can be made to line up parallel, then the material acquires a macroscopic magnetic moment. This offers a little insight into the origin of permanent or (ferro-)magnetism. *]

3.7 Forces and couples

In Sec. 1.2, we find that the force, felt by an element of volume δV of medium in which the current density is $\mathbf{J}(\mathbf{r})$, because of a given magnetic field $\mathbf{B}(\mathbf{r})$ is (Lorentz force)

for an element $\delta \mathbf{r}$ of thin conducting wire carrying current I. For a loop C_1 , carrying current I_1 , in a given field **B**, the total force and couple felt are It can be shown (see problem set 2) that, for C_1 a current loop of moment $\mathbf{m} = IS\mathbf{n}$ in a uniform magnetic field,

$$\mathbf{F} = 0, \quad \mathbf{G} = \mathbf{m} \times \mathbf{B}. \tag{18}$$

If $\mathbf{B}_2(\mathbf{r})$ is the field due to a current loop C_2 carrying current I_2

$$\mathbf{B}_{2}(\mathbf{r}) = \frac{\mu_{0}}{4\pi} \oint_{C_{2}} \frac{I_{2} \mathbf{d} \mathbf{r}_{2} \times (\mathbf{r} - \mathbf{r}_{2})}{|\mathbf{r} - \mathbf{r}_{2}|^{3}},\tag{19}$$

then the force \mathbf{F}_{12} , exerted on loop C_1 by (the magnetic field due to the current in) the loop C_2 , is

It is not obvious, but true, that this expression is compatible with Newton's third law. Proof, which requires the application of Stokes's theorem, is asked for in Problem set 2.

Example: parallel wires

Suppose $C_{1,2}$ are infinite wires carrying currents $I_{1,2}$, the former along the x-axis, the latter parallel to it and through (0, 0, a). Find force.

3.8 Proof of (13)

Let S be a surface with unit normal **n** which separates regions V_{\pm} of space, with **n** pointing from S into V_{\pm} . Let \mathbf{B}_{\pm} be the magnetic fields in V_{\pm} , and let current **s** per unit length flow in S.

Consider an area A of S sufficiently small to be considered plane. Apply Ampére's law

$$\int_C \mathbf{B}.\mathbf{dr} = \mu_0 I \tag{20}$$

to the plane needle shaped contour C shown, in the limit $\delta \to 0$ for finite (small) l. l should be small enough for variation of \mathbf{B}_{\pm} and \mathbf{s} across it, but non-zero. Also the plane of the needle C contains \mathbf{n} , but its orientation, *i.e.* the direction of the normal \mathbf{b} , $|\mathbf{b}| = 1$ to the plane, is arbitrary. The direction \mathbf{t} , $|\mathbf{t}| = 1$ of the needle is then chosen so that $\mathbf{b}, \mathbf{t}, \mathbf{n}$ form an orthonormal right-handed triad. We get

$$(-\mathbf{B}_{+} + \mathbf{B}_{-}).\mathbf{t}l = \mu_{0}\mathbf{s}.\mathbf{b}l$$

$$(-\mathbf{B}_{+} + \mathbf{B}_{-}).\mathbf{n} \times \mathbf{b} = \mu_{0}\mathbf{s}.\mathbf{b}$$

$$[(-\mathbf{B}_{+} + \mathbf{B}_{-}) \times \mathbf{n} - \mu_{0}\mathbf{s}].\mathbf{b} = 0.$$
(21)

Since **b** was taken to be in an arbitrary direction in S it follows that

$$\mathbf{n} \times \mathbf{B}|_{-}^{+} = \mu_0 \mathbf{s}. \tag{22}$$