Mathematical Tripos, Part IB : Electromagnetism

4 Electromagnetic induction

Recall the paragraph from Sec. 1.5, repeated here: The Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1}$$

implies

by applying Stokes's theorem to a fixed curve $C = \partial S$ bounding a fixed open surface S. If we define the electromotive force (or electromotance) acting in C by

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr},\tag{2}$$

and the flux of \mathbf{B} through (the open) surface S by

$$\Phi = \int_{S} \mathbf{B} \cdot \mathbf{dS},\tag{3}$$

then we get Faraday's Law of induction

$$\mathcal{E} = -\frac{d\Phi}{dt}.\tag{4}$$

This will be studied now.

In chapter two we studied electric fields \mathbf{E} such that

called conservative, since, in virtue of $\nabla \times \mathbf{E} = 0$, there exists the electrostatic potential ϕ such that $\mathbf{E} = -\nabla \phi$. In chapter two it was assumed implicitly that there were no magnetic fields in the discussion, but it could equally have been assumed that we were dealing with non-conducting material (*e.g.* the vacuum or free space) and time-independent magnetic fields, since the latter would then be entirely uncoupled from the electrostatics.

Here we study time-dependent magnetic fields and the the non-conservative electric fields that accompany them. The latter may give rise to non-zero electromotive forces (or electromotances, or EMFs for short), and hence cause current flow.

We first make this study in the (pre-Maxwellian) approximation to the full Maxwell theory, in which

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}.$$
 (5)

In other words, we neglect the displacement current, even though it was seen in Sec. 1.4 to be an essential ingredient of a consistent theory. It can be shown however that this is justified in the practically significant context in which there are alternating currents of low enough frequency flowing in media of high enough conductivity.

We look first at simple situations wherein it can be seen how time-dependent magnetic fields can produce non-zero EMFs and cause current flow.

4.1 Simple examples

If you think about a typical electric circuit – a battery hooked up to a light bulb. Why is the current the same all the way around the loop?

There are really two forces involved in driving current around a circuit: the *source force* and the *electrostatic force*, which serves to smooth out the flow and communicate the influence of source to distant parts of the circuit.

In 1831 Faraday reported on a series of experiments, including these three:

4.2 Faraday's law of induction

Let ${\cal C}$ be either

- (a) a fixed closed geometrical curve, or
- (b) a physical, possibly moving circuit.

Let S be a surface bounded by $C = \partial S$. Faraday's experimental law, valid in both the contexts (a) and (b), with an appropriate definition in each case of the EMF \mathcal{E} in C, is

Note: The significance of the minus sign in the definition of the EMF reflects Lenz's law, which states that any EMF induced in a circuit by a change of flux through it tends to oppose any EMF (*e.g.* due to a battery) that already exists in the circuit.

4.3 The Faraday experiment

In the set-up shown the crossbar LM can slide with negligible friction parallel to ON. The uniform time independent magnetic field $\mathbf{B} = (0, 0, B)$ points upwards from the plane of the page.

We shall neglect the resistance of the wire $QMN(\mathcal{E}_0)OLP$.

Lenz's law is a special case of more general belief: le Châtelier's principle. This can be stated as follows: a physical system in a steady state reacts by opposing any change imposed on it from outside.

We neglected the magnetic field due to the current induced in C, which opposes the battery produced I_0 . But (using the result (38) from Sec. 3.3 a), we see that the field due to the induced current in LM e.g. points downwards on the plane of the diagram, and opposes **B**. This too exemplifies a Lenz view: flux change of one sign produces currents which create flux of the the opposite sign.

4.4 Coil rotating in a fixed magnetic field

Let C be a closed rectangular curve PQRS of area A. Very thin conducting wire is wrapped N times around the curve C with free ends connected to some external circuit.

Suppose C can rotate rigidly about a fixed axis $\mathbf{j} = (0, 1, 0)$ with angular velocity ω in the presence of a uniform time-independent magnetic field $\mathbf{B} = (0, 0, B)$.

When the normal to the coil makes an angle $\theta = \omega t$ to **B** as shown, so that $\mathbf{n} = \cos \theta \mathbf{k} + \sin \theta \mathbf{i}$, then the flux of **B** through the coil is

$$\int \mathbf{B} \cdot \mathbf{dS} = N\mathbf{B} \cdot \mathbf{n}A = NB\cos\theta A.$$
 (6)

4.5 Inductance and magnetic energy

We will illustrate these concepts by reference to the long solenoid of Sec. 3.3. First we recall the context and some of the results obtained there.

The solenoid has N turns of wire per unit length and length l very large so that end effects can be neglected. It carries current I. It is cylindrical with axis $\mathbf{k} = (0, 0, 1)$, and cross-sectional area A. The magnetic field due to the current flow is

$$\mathbf{B} = \mu_0 N I \mathbf{k} \tag{7}$$

inside the solenoid and zero outside. The flux of ${f B}$ through one turn of the solenoid is

and through all Nl turns is

This is proportional to I, and we define the (self)-inductance L of the coil

Suppose now that the long solenoid, which remains stationary in this discussion, is attached to a battery of EMF \mathcal{E}_0 , so that the total EMF in the circuit is

The first term corresponds to Ohmic heat generation. As it is not of immediate interest, we suppose the circuit is of negligible resistance and neglect it. Hence

This leads to an important alternative expression for the magnetic energy.

$$W = \frac{1}{2\mu_0} \int_V \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B} d\tau = \frac{1}{2\mu_0} \int_V \left[-\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) + \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) \right] d\tau.$$
(8)

We may use the divergence theorem on the first term and see that it vanishes provided that (as can be checked) its integrand goes to zero fast enough as r goes to infinity. Now from (8) we get the important result

$$W = \frac{1}{2\mu_0} \int_V \mathbf{B}^2 d\tau.$$
(9)

It can easily be seen, using the expression for L above and (7) for $B = |\mathbf{B}|$, that the two expressions for the magnetic energy W give the same result in the case of the long solenoid

$$\frac{1}{2}LI^2 = \frac{1}{2}(\mu_0 N^2 V)I^2 = \frac{1}{2\mu_0}B^2 V = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau.$$
 (10)