Proof of the result $\mathbf{G} = \mathbf{m} \times \mathbf{B}$

Refer to Sec. 3.7, **Force and couples**, and supply the proof that the couple exerted by a uniform magnetic field $\mathbf{B}$ on a plane current loop, of area A, unit normal $\mathbf{n}$, carrying current I, is given by

$$\mathbf{G} = \mathbf{m} \times \mathbf{B}, \quad \mathbf{m} = IA\mathbf{n}. \tag{1}$$

5 Maxwell's equations

5.1 A historical paradox

In magnetostatics, the equation

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}, \tag{2}$$

implies $\nabla\cdot\mathbf{J} = 0$. In magnetostatics, this is compatible with the continuity equation $\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t} = 0$. However naive application of the integral form of (2)

$$\oint_C \mathbf{B}\cdot\mathbf{dr} = \mu_0\int_S \mathbf{J}\cdot\mathbf{dS}, \tag{3}$$

to the following situation produced a contradiction, one that Maxwell resolved by generalising (2). **The 'capacitor' paradox.**

Maxwell proposed that (2) be changed by addition to a term that made it compatible with $\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t} = 0$. This gives rise (in free space or the vacuum) to

$$\nabla\times\mathbf{B} = \mu_0(\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}), \tag{4}$$

as was shown in Sec. 1.4 to be sufficient to achieve consistency.

How does the use of (4) provide resolution of the paradox?

as required for consistency. Here σ is the charge density and A is the plate area. The assumption that $\mathbf{E}$ is uniform is a crude one. It can be avoided by doing a somewhat harder calculation along lines similar to those followed above.

5.2 Energy and energy transport

Recall the field energy formulas

$$W_{el} = \frac{\epsilon_0}{2}\int_V \mathbf{E}^2 d\tau, \quad W_{mag} = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau, \tag{5}$$

and the expression for the rate of Ohmic heat loss *i.e.* the rate of dissipation of electromagnetic energy as heat

$$\int \mathbf{J}\cdot\mathbf{E} d\tau. \tag{6}$$

For the last term the divergence theorem has been applied to a fixed volume V of space bounded by a surface S. The left side here is the rate of decrease of the total field energy $W = W_{el} + W_{mag}$. The first term on the right side represents the rate of loss of energy as Ohmic heat, while the second term there is the rate of energy transport out of V through the surface S.

For the latter, define the Poynting vector **S**

The flux of **S** through a closed surface S, with outward unit normal **n**, is

This is the flux of electromagnetic energy being transported through S out of V.

5.3 Decay of charge density in a medium of high conductivity σ

In Sec. 1.4, we derived the continuity equation

where $\tau = \frac{\epsilon_0}{\sigma}$ is the *relaxation time* of the medium. For copper or silver $\tau \approx 10^{-18}$*sec.*, so that any charge density present – for whatever reason – in the medium at the initial time $t = 0$ quickly goes to zero. It may be expected to flow to the surface of the medium. For a perfect conductor, for which σ is infinite, we have $\rho(t) = 0$ at all times, as has been discussed above.

5.4 Plane wave solutions of Maxwell's equations

We here deal with the vacuum or free-space, *i.e.* $\rho = 0, \quad \mathbf{J} = 0$. We begin as simply as possible by seeking a solution describing a wave propagating in the z-direction with fields that do not depend on x or y.

Here $\nu\lambda = \dfrac{\omega}{2\pi}\lambda = c$ relates the wavelength λ and frequency of the wave in a standard way to other wave variables. Finally, note that the use of complex exponentials is very convenient, but the physical fields must always be identified by taking real parts.

What about the magnetic fields?

So our wave solution of Maxwell's equations is

$$\mathbf{E} = (E_0, 0, 0)\ \exp i(kz - \omega t), \quad \mathbf{B} = \frac{1}{c}(0, E_0, 0)\ \exp i(kz - \omega t). \tag{7}$$

It should be checked that (7) satisfies also (the zero current density version of) the 'fixed' Ampere's Law, although our use of the fact that each component of $\mathbf{E}$ satisfies a wave equation guarantees it. Thus the simplifying assumptions we have made have led us to the valid and simple wave solution (7) of Maxwell's equations. We could similarly have adopted a choice of axes such that that $\mathbf{E} = (0, E, 0)$, and reached, as above, the solution

$$\mathbf{E} = (0, E_0, 0)\ \exp i(kz - \omega t), \quad \mathbf{B} = (-\frac{1}{c}E_0, 0, 0)\ \exp i(kz - \omega t). \tag{8}$$

The solutions (7) and (8) are linearly independent, and the general monochromatic wave of frequency ω is obtained as a linear superposition of them, has fields $\mathbf{E}$ and $\mathbf{B}$ that are transverse to the direction of propagation of the wave.

Also

To discuss the transport of energy by the wave (7) obtained above, we require the real parts

$$\mathbf{E} = (E_0, 0, 0)\ \cos(kz - \omega t), \quad \mathbf{B} = (0, \frac{1}{c}E_0, 0)\ \cos(kz - \omega t), \tag{9}$$

For the time average of this we have

For the simple plane wave (9), it follows that the energy density travels at the speed of light across unit area normal to the wave.

Of course, similar results holds for the wave (8).

If we consider a linearly polarised wave with fields

We have merely reproduced our wave in an arbitrary Cartesian basis.

[**Circularly polarised waves**

Take a solution that is (7) minus i-times-(8), with E_0 real. This has physical fields

where $\mathbf{e}_s(\phi)$ and $\mathbf{e}_\phi(\phi)$ are the unit vectors of cylindrical polar coordinates (s, ϕ, z) with the z-axis in the direction of propagation of the wave. This wave is said to be (positively) circularly polarised. A wave of negative circular polarisation linearly independent of this can be constructed, using (7) plus i-times-(8) with E_0 real, but we do not need the details contained in this parenthesis].

5.5 Boundary conditions

Sec. 1.8 should perhaps be reviewed at this point.

Suppose a surface S carries either a charge density σ per unit area, or a surface current s per unit length. Let the unit normal $\mathbf{n}$ to S point from the negative $(-)$ to the positive $(+)$ side of S.

We proved

It should be clear that the proofs can be applied to deriving

$$\mathbf{n}.\mathbf{B}|_{-}^{+} = 0, \tag{10}$$

and

$$\mathbf{n}\times\mathbf{E}|_{-}^{+} = 0. \tag{11}$$

As an aid to remembering these results, we noted in Sec. 1.8, their exact correspondence with Maxwell's equations themselves.

5.6 Reflection at the surface of a perfect conductor

We consider a monochromatic wave (7) propagating in the z-direction from the half-space $z < 0$, towards perfectly conducting material in $z > 0$, whose surface is the plane $z = 0$. In fact the solution of Maxwell's equations plus the boundary conditions (BC) on $z = 0$ will comprise not only an incident wave but also (at least) a suitably matched reflected wave. The fields of the former will have argument $(kz - \omega t)$, where $kc = \omega$, while those of the latter (moving in the negative z-direction) are $(-kz - \omega t)$. All fields in the problem have the same t-dependence $\propto e^{-i\omega t}$.

We also see that the physical fields are the real parts of $\mathbf{E}$ and $\mathbf{B}$, *i.e.* for E_0 real,

$$(\mathbf{E}_{phys})_x = 2E_0 \sin kz \, \sin \omega t, \quad \text{and} \quad (\mathbf{B}_{phys})_y = (2/c)E_0 \cos kz \, \cos \omega t. \tag{12}$$

Hence the mean force per unit area is

$$\langle f \rangle = \epsilon_0 {E_0}^2, \tag{13}$$

using the result $\langle \cos^2 \omega t \rangle = \frac{1}{2}$. Since the force is normal to the surface, (13) gives the mean pressure (**radiation pressure**) at the surface.

Also, from (12), $(\mathbf{E}_{phys})_x = 0$ for $z = -a$, when k is given by $k = n\pi/a, \quad n = 1, 2, 3, \dots$. For such k, (12) gives a solution of Maxwell's equations in empty space $-a < z < 0$ between the surfaces $z = -a,\ 0$ of perfect conductors. The surface current at $z = -a$ etc., can be calculated.

5.7 The historical paradox revisited

Here we provide a treatment which does not make the (crude) assumption that the the electric field $\mathbf{E}$ between the plates is uniform. Assume the plates are circular of radius a, and neglect edge effects. Use cylindrical polars (s, ϕ, z).

The surface charge density on the the lower plate is

$$\sigma = \epsilon_0 \mathbf{k} \cdot \mathbf{E}|_-^+ = \epsilon_0 \alpha J_0(ks) \exp(-i\omega t), \quad 0 \le s \le a. \tag{14}$$

vv

We now show that the integral form of (*) can be applied consistently to $\oint_C \mathbf{B}\cdot\mathbf{dr}$ whether or not the surface $S, \partial C = S$, chosen passes between the plates or not. Let C be the circumference of the lower plate, S_2 the lower plate itself, and S_1 a surface bounded by C but lying entirely outside the region between the plates and so pierced by the current I. As before, for S_1 $\oint_C \mathbf{B}\cdot\mathbf{dr} = \mu_0 I$. For S_2, on the other hand, we have