Summary of Vector Calculus

The following results apply to any (suitably differentiable) scalar field $\phi(\mathbf{x})$ and vector fields $\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$. They all have important applications in electromagnetism.

Derivatives of a vector field:

$$div\,\mathbf{E} \equiv \nabla\cdot\mathbf{E} = \frac{\partial E_i}{\partial x_i}$$ has one component and is a scalar

$$curl\,\mathbf{E} \equiv (\nabla \times \mathbf{E})_i = \epsilon_{ijk}\frac{\partial E_k}{\partial x_j}$$ has three components and is a vector

$$grad\,\mathbf{E} \equiv (\nabla\mathbf{E})_{ij} = \frac{\partial E_i}{\partial x_j}$$ has nine components and is a 2nd-rank tensor

Two identities:

$$curl\,grad\,\phi = \nabla \times \nabla\phi \equiv 0\ . \qquad div\,curl\,\mathbf{E} = \nabla\cdot\nabla \times \mathbf{E} \equiv 0\ .$$

Three vector triple products:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$$

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \tfrac{1}{2}\nabla E^2 - (\mathbf{E}\cdot\nabla)\mathbf{E}$$

$$\nabla \times (\mathbf{E} \times \mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{E} - (\mathbf{E}\cdot\nabla)\mathbf{B} + \mathbf{E}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{E})$$

Position vector: let $\mathbf{r}$ or r_i denote the position vector, and $r = |\mathbf{r}|$. Then $\hat{\mathbf{r}} = \mathbf{r}/r$ defines a unit vector in the direction or $\mathbf{r}$.

$$\partial_i r_j = \delta_{ij}, \qquad \partial_i r = \frac{r_i}{r}, \quad \text{or} \quad \nabla r = \hat{\mathbf{r}}$$

$$\frac{\partial}{\partial r_i}\left(\frac{1}{r}\right) = -\frac{r_i}{r^3}, \quad \frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right) = \frac{3r_i r_j - r^2\delta_{ij}}{r^5}, \quad \nabla^2\left(\frac{1}{r}\right) = 0.$$

Divergence Theorem (Gauss): If V is a simply connected domain with surface S and outward normal $\mathbf{n}$ then

$$\int_V (\nabla \cdot \mathbf{E})\, dV = \int_S \mathbf{E} \cdot \mathbf{n} dS.$$

Stokes' Theorem: If C is a closed curve spanned by the surface S then

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E})\cdot\mathbf{dS}$$

Gradient Theorem:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla\phi) \cdot d\mathbf{l} = \phi(\mathbf{b}) - \phi(\mathbf{a}).$$

Curl in curvilinear coordinates (ξ_1, ξ_2, ξ_3)**:**

$$\nabla \times \mathbf{E} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \dfrac{\partial}{\partial \xi_1} & \dfrac{\partial}{\partial \xi_2} & \dfrac{\partial}{\partial \xi_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix} ,$$

where: $h_1 = 1$, $h_2 = r$, $h_3 = 1$ in cylindrical polars (r, θ, z); and
$h_1 = 1$,$h_2 = r$, $h_3 = r \sin\theta$ in spherical polars (r, θ, φ).

Cylindrical coordinates (r, θ, z)

Grad, div and Laplacian:

$$\nabla \phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right)$$

$$\nabla \cdot \mathbf{E} = \frac{1}{r} \frac{\partial (r E_r)}{\partial r} + \frac{1}{r} \frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Volume and line elements, and normal to cylinder:

$$dV = r\, dr\, d\theta\, dz \qquad \mathbf{dl} = (dr, r\, d\theta, dz)$$

$$\mathbf{n} = (\cos\theta, \sin\theta, 0)$$

Spherical coordinates (r, θ, φ)

Note that r and θ denote different quantities in cylindrical and spherical polars.

Grad, div and Laplacian:

$$\nabla \phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin\theta} \frac{\partial \phi}{\partial \varphi} \right)$$

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial (r^2 E_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta\, E_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial E_\varphi}{\partial \varphi}$$

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi}{\partial \varphi^2}$$

Volume and line elements, and normal to sphere:

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi \qquad \mathbf{dl} = (dr, r\, d\theta, r \sin\theta\, d\varphi)$$

$$\mathbf{n} = (\cos\theta, \sin\theta \cos\varphi, \sin\theta \sin\varphi)$$

Please email corrections/comments to N.G.Berloff@damtp.cam.ac.uk