Mathematical Tripos Part IB N. G. Berloff
FLUID DYNAMICS (P2)

Example Sheet 3

1. Two line vortices are initially at $(0, d)$ and $(0, -d)$ in Cartesian coordinates. Describe the motion of the vortices if their strengths are: (a) $+\kappa$ and $-\kappa$; (b) $+\kappa$ and $+\kappa$; and (c) $+2\kappa$ and $-\kappa$.

A two-dimensional fluid flow contains a line source of strength m fixed at the origin and two freely moving line vortices of strength $\pm\kappa$. The motion of each vortex is due to the combined velocity field of the source and the other vortex. Suppose that the two vortices are symmetrically placed at $(r(t), \pm\theta(t))$ in polar coordinates. Find $\dot{r}$ and $\dot{\theta}$ and hence show that

$$r \sin\theta = A\mathrm{e}^{-2m\theta/\kappa}$$

where A is a constant. Sketch the vortex paths.

2. Write down the equation and linearized boundary conditions for the velocity potential and the motion of the free surface, for small amplitude oscillations of the water surface in a square container $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$ of depth h, i.e. water in $-h \leqslant z \leqslant \zeta(x, y, t)$ with $\zeta \ll 1$. Find the frequencies of the oscillations. Show the sign of the surface displacement in plan view for the five lowest frequency modes.

3. Fluid of density ρ_1 occupies the region $z > 0$ and overlies another fluid of density ρ_2 (with $\rho_2 > \rho_1$), which occupies the region $z < 0$. Show that the frequencies of small amplitude oscillations of the interface between the regions are given by

$$\omega^2 = gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) .$$

[Hint: You will need different potentials ϕ_1 and ϕ_2 for the two regions, and you should apply the kinematic boundary condition to the flow in each region.]

4. Find the frequencies of small-amplitude oscillations for the water surface in a vertical cylinder of radius a and of large depth. [You may assume that the velocity potential is given by separable solutions to Laplace's equation of the form $\phi(r, z, t) = J_0(kr)e^{kz - i\omega t}$ for axisymmetric oscillations and $\phi(r, \theta, z, t) = J_n(kr)\cos(n\theta)e^{kz - i\omega t}$ for non-axisymmetric oscillations, where the functions J_0, J_1 etc. are called Bessel functions; see over for graphs of J_0, J_1 and J_2.] What restriction does the kinematic boundary condition at $r = a$ place on the value of k?

What is the lowest frequency for a cup of tea? Show the sign of the surface displacement in plan view for the three lowest frequency modes. [The first root of $J_0'(z) = 0$ is $z = 3.83$ while the first root of $J_1'(z) = 0$ is $z = 1.84$.]

5. Water from a large deep reservoir flows over a weir. The water is of depth d where the free surface has fallen to a level η below that far upstream in the reservoir. Assume that the depth of the water varies sufficiently slowly so that the velocity can be taken to be horizontal and uniform in depth. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2g\eta}$. From the condition that Q does not vary along the flow, and the condition that $d + \eta$ is a minimum at the crest of the weir [differentiate], show that $\eta = \frac{1}{2}d$ at the crest. Deduce that $Q^2 = 8gh^3/27$ where h is the minimum value of $d + \eta$.

6. An idealized river flows in a channel of rectangular cross-section, with a flat, horizontal bottom, and with width $w(x)$ varying slowly along the channel. Far upstream, the fluid velocity u has the constant value U, the depth of the water has the constant value H, and the width of the channel has the constant value W. Taking $u(x)$ to be constant over each cross-section of the channel, show [from mass conservation and Bernoulli] that

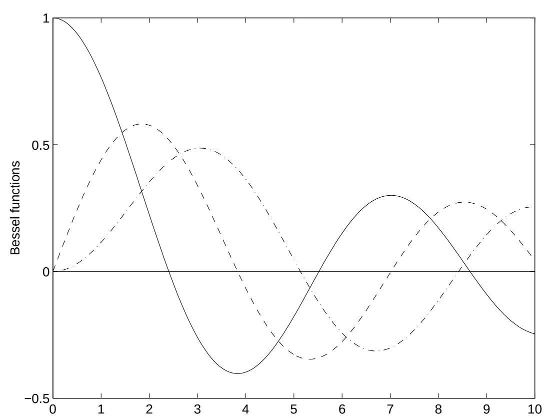
$$\frac{W}{w} = \frac{u}{U}\left(1 + \tfrac{1}{2}F^2 - \tfrac{1}{2}F^2\frac{u^2}{U^2}\right) , \quad \text{where} \quad F^2 = \frac{U^2}{gH} .$$

Sketch this relationship. Observation of the river shows that $u(x)$ is steady and slowly varying and that far downstream $w \to W$ but $u \to V \neq U$. What can be deduced about W/w in the region of varying width? Find V and the depth far downstream.

7. Rework the analysis of a hydraulic jump in your lecture notes using the reference frame in which the jump is stationary. In that reference frame, write down the flux of energy (kinetic plus potential) and the rate of working of pressure forces at control surfaces on either side on the jump. The difference between the fluxes at the two sides is the rate of energy dissipation D due to friction (viscosity) in the jump. By eliminating the speed of the jump relative to the fluid behind it, $(V - U_2)$, show that

$$D = \frac{\rho g}{4}(V + U_1)\frac{(h_2 - h_1)^3}{h_2} ,$$

where h_1 and h_2 are, respectively, the fluid depths ahead of and behind the jump and $V + U_1$ is the speed of the jump relative to the fluid ahead of it.



Bessel functions for Q4, $J_0(x)$ (solid curve), $J_1(x)$ (dashed), $J_2(x)$ (dash-dotted).

Please email corrections/comments to N.G.Berloff@damtp.cam.ac.uk