Fluid Dynamics IB
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Governing Equations: The Rough Guide

Note: The following thumbnail sketch is *not* an exhaustive list of all you need to know. The chief omissions are probably path, streak and streamlines; the streamfunction; the various simple solutions to Laplace's equation; and, of course, all the derivations and applications!

Material Derivative:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla \tag{1}$$

The rate of change observed moving with a parcel of fluid is equal to the rate of change at a fixed point plus the rate of change due to moving (being advected) to a different position.

Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \qquad or \qquad \frac{D\rho}{Dt} + \rho\nabla.\mathbf{u} = 0 \tag{2}$$

The rate of change of density ρ in a small fixed volume is due to the divergence of the mass flux $\rho\mathbf{u}$ out of the volume; *or* the rate of change of density moving with a small parcel of fluid is due to the rate of change of volume of the parcel. If every parcel of fluid retains the same density then the flow is called 'incompressible' and

$$\nabla.\mathbf{u} = 0 \tag{3}$$

Conservation of Momentum:

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F} \tag{4}$$

The acceleration of a small fluid parcel is due to the body force, $\mathbf{F}$ (per unit volume), acting on the parcel and to the difference between the pressure acting on opposite sides of the parcel.

Boundary Conditions: Boundary conditions appropriate for (3) and (4) are that the normal velocity of the fluid equals the normal velocity of the boundary (the kinematic b.c.)

$$\mathbf{u}.\mathbf{n} = \mathbf{U}.\mathbf{n} \tag{5}$$

and that the pressure in the fluid equals the force per unit area on the boundary (the dynamic b.c.). For a free surface at $z = \zeta(x, y, t)$ the kinematic b.c. becomes

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = w \qquad or \qquad \frac{D}{Dt}(z - \zeta) = 0 \tag{6}$$

and the dynamic b.c. is that p is continuous across the free surface.

Material line elements:

$$\frac{d}{dt}\delta\mathbf{l} = (\delta\mathbf{l}.\nabla)\mathbf{u} \tag{7}$$

The length and orientation of a material line element $\delta\mathbf{l}$ changes due to variation of $\mathbf{u}$ in the direction of $\delta\mathbf{l}$ i.e. due to the difference in velocity between the ends of the element.

Direct solution of the nonlinear equation of motion (4) is difficult and most of the solutions in the course use the momentum integral equation or Bernoulli's theorem or the potential flow equations (each of which is derived from (4) under certain conditions.)

Momentum Integral Equation:

$$\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = -\int_S \rho\mathbf{u}(\mathbf{u}.\mathbf{n})dS - \int_S p\mathbf{n}\,dS + \int_V \mathbf{F}\,dV \tag{8}$$

The rate of change of momentum $\rho\mathbf{u}$ within a fixed volume is due to the flux of momentum $\rho\mathbf{uu}$ out of the volume, the pressure forces on the bounding surface and the volume forces on the interior.

Bernoulli's Theorem: If the flow is steady and $\mathbf{F} = -\nabla\chi$ then

$$\mathbf{u}.\nabla(\frac{1}{2}\rho u^2 + p + \chi) = 0 \tag{9}$$

Moving along a streamline a fluid parcel will speed up if it moves from a region of high pressure and potential energy to a region of lower pressure and potential energy. This can be interpreted loosely as conservation of energy. (The result is not valid if there is energy loss due to friction in unsteady, turbulent motion.)

Potential Flow: If the flow is irrotational (no vorticity) and incompressible then

$$\mathbf{u} = \nabla\phi \quad \text{with} \quad \nabla^2\phi = 0 \tag{10}$$

The flow is found by solving Laplace's equation with boundary conditions on the normal velocity $\mathbf{n}.\nabla\phi$. The pressure (and forces) are then found from

$$\nabla(\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi) = 0\ . \tag{11}$$

Forces on bodies on potential flow: In potential flow there is no *drag force* on a steadily translating sphere or cylinder (or any other body) in fluid at rest at infinity. This is because friction has been neglected and the kinetic energy of the fluid is constant. There is a drag force on an accelerating body of the form $m^*\dot{\mathbf{U}}$, where m^* corresponds to the mass of fluid around the body that needs to be accelerated with it. There is a *lift force* of magnitude $\rho\kappa U$ on a two-dimensional body with circulation κ in steady translation.

Vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ and circulation: The vorticity equation (when $\mathbf{F} = -\nabla\chi$)

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}.\nabla)\mathbf{u} \tag{12}$$

is derived by taking the curl of the momentum equation. The vorticity of a small fluid parcel changes due to the variation of $\mathbf{u}$ in the direction of $\boldsymbol{\omega}$; i.e. if the parcel is being stretched in the direction of $\boldsymbol{\omega}$ it becomes thinner in the other directions (by mass conservation) and so must rotate faster in order to conserve angular momentum. The circulation around a closed curve Γ is given by $\oint_\Gamma \mathbf{u}.\mathbf{dl}$ and if Γ is a material curve then the circulation is constant.

Linear water waves: Use a linear combination of separable solutions of the form $\phi = Ae^{\pm ik_x x}e^{\pm ik_y y}e^{\pm kz}e^{\pm i\omega t}$, where $k_x^2 + k_y^2 = k^2$ to satisfy (11), the linear combination is chosen to satisfy the kinematic b.c. (5) on side and bottom boundaries (if any), and $\omega(k)$ obeys $\omega^2 = gk\tanh(kh)$ in order to satisfy the linearised form of (6) on the free surface.