Fluid Dynamics IB
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§4 FLOWS WITH A FREE SURFACE

Water waves, river flow including flow over a weir, and ocean tides, storm surges and tsunamis. We shall see how the height of a weir precisely controls the flow rate over it — a matter of some importance in hydraulic engineering. If an earthquake generates a tsunami or 'tidal wave' near Japan, at one side of the Pacific, it is fairly simple to estimate how long the wave takes to reach California, or southern Chile.
(We shall find that the answer is roughly half a day to California and a day to southern Chile. Such waves travel at speeds just over 200 m/s, not far short of the airspeeds of subsonic passenger jet aircraft.)
The gravitational potential is $\Phi = gz$. Assume that the flow is irrotational, so that $\mathbf{u} = \nabla\phi$ with

$$\nabla^2\phi = 0 \ .$$

This is the only 'governing equation'; everything else is a matter of boundary conditions. At the free surface there are *two* boundary conditions: the pressure condition, and the kinematic boundary condition derived in §1.7.

As in §1.7, denote the elevation of the free surface by $z = \zeta(x, y, t)$. The kinematic boundary condition derived there is

$$\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} - w = 0 \qquad \text{on} \quad z = \zeta(x, y, t) \ , \tag{$*$}$$

signifying that fluid on the free surface stays on the free surface. The condition on the pressure field is, as usual, most conveniently expressed via the time-dependent, irrotational form of Bernoulli's theorem (§3.3), with $\Phi = gz$ and $\frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}|\nabla\phi|^2$:

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = \tilde{H}(t) \ ,$$

a function of time t alone. Apply this at the free surface $z = \zeta$, ignoring surface tension so that pressure p in the water, i.e., the p appearing in Bernoulli's equation above, is equal to $p_{\rm atm}$ at $z = \zeta$:

§4.2 Small-amplitude water waves

Consider a layer of water, of depth h when undisturbed:

$$\begin{aligned} \text{Full problem:}\quad & \nabla^2\phi = 0 \qquad \text{in} \qquad -h \le z \le \zeta(x,\,t) \\ & \frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h \\ & \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = \zeta\ , \quad \text{from } (*) \quad \text{with}\ \ u = \partial\phi/\partial x \\ & \frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + g\zeta = f(t) \quad \text{on} \quad z = \zeta\ , \quad \text{from } (**) \end{aligned}$$

Linearize: 1) neglect quadratic terms such as $\frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x}$ and $\frac{1}{2}|\nabla\phi|^2$;
2) use Taylor series to express the boundary condition at $z = \zeta$ in terms of quantities at $z = 0$,

e.g.

Hence the linearized equations and boundary conditions are:

To solve we can: (1) look for a separable solutions $\phi = \Phi(z)X(x)T(t)$ or (2) take a Fourier transform with respect to x to get ordinary differential equations in t with constant coefficients. In any case we are looking for a solution of the form $\phi = Re\left(\hat{\phi}(z)e^{ikx-i\sigma t}\right)$ and $\zeta = Re\left(\hat{\zeta}e^{ikx-i\sigma t}\right)$ where k, $\hat{\zeta}$ and σ are constants. Notice that, with such a solution, $f(t) = 0$ in the last-displayed boundary condition; for otherwise $f(t)$ would have to depend on x, which is a contradiction.

Notice furthermore that, with the form of solution just assumed, we can now replace $\partial/\partial x$ by ik, $\partial^2/\partial x^2$ by $-k^2$, and ∇^2 by $-k^2 + \partial^2/\partial z^2$.

The problem now becomes

The first of the above is an ordinary differential equation with constant coefficients. Its general solution can be written as a linear combination of real exponential functions, or equivalently as $A\cosh\{k(z-z_0)\}$, where A and z_0 are arbitrary constants. We can satisfy the bottom boundary condition $d\hat{\phi}/dz = 0$ at $z = -h$ if we choose $z_0 = -h$, i.e.,

Then from the first (the kinematic) boundary condition on $z = 0$ we see that

and from the second (pressure)

Regarding these as a system of two linear algebraic equations for the two unknown constants $\hat{\zeta}$ and A, we see that this system has nontrivial solutions if and only if its determinant vanishes, i.e. if and only if

This transcendental equation, relating the two constants σ and k, is usually called the *dispersion relation.*

Because the shape of the free surface is described by $Re\left(\hat{\zeta}e^{ikx-i\sigma t}\right)$, successive wavecrests are spaced apart at a distance $2\pi/k$: this distance $2\pi/k$ is called the wavelength. The dispersion relation says that the crests all travel with speed $c = \sigma/k$; and we have

Special cases

1) 'Deep water' case: $|kh| \gg 1$ (i.e., wavelength/$2\pi \ll h$)

Example: ocean swell (small-amplitude, low-frequency waves generated by distant storms, the waves responsible causing big 'breakers' on surf beaches like Malibu). The deep-water and small-amplitude approximations are excellent approximations throughout most of the waves' journey across the open ocean, though not near the beach. The ocean is about 5 km deep away from continental margins; and typical periods $\sim 15\,\mathrm{s}$, $\Rightarrow$ $|\sigma| \sim 2\pi/15\,\mathrm{s}$ $\simeq 0.4\,\mathrm{s}^{-1}$, $\Rightarrow$ $|k|^{-1} = g/\sigma^2 \simeq 63\,\mathrm{m}$, $\ll 5\,\mathrm{km}$. The wavelength $2\pi/|k|$ is nearly 400 m, and $|c| \simeq 25\,\mathrm{m\,s}^{-1}$.

2) 'Shallow water' or 'long wave' case: $|kh| \ll 1$ (i.e., wavelength$/2\pi \gg h$)

E.g. flood waves in river: say $h = 2\,\mathrm{m}$ $\Rightarrow$ $|c| = (gh)^{1/2} \simeq 4.5\,\mathrm{m\,s}^{-1}$ or 16 km per hour.

Trans-Pacific propagation of tsunamis: The longest wavelengths travel the fastest; and the longest wavelengths, for this purpose, are certainly far greater than 2π times the typical ocean depth, 5 km. So this (maximal) wave speed is

$|c| = (gh)^{1/2} \sim (10\,\mathrm{m\,s}^{-2} \times 5000\,\mathrm{m})^{1/2} = 224\,\mathrm{m\,s}^{-1}$.

I.e., as stated earlier, these waves travel almost as fast as a subsonic passenger jet aircraft.

Velocity field

From the first of $(\star)$, we have $A = -i\sigma\hat{\zeta}/\big(k \sinh kh\big)$, hence

$$\phi(x,z,t) \;=\; Re\left\{-\frac{i\sigma\hat{\zeta}}{k\sinh kh}e^{i(kx-\sigma t)}\cosh k(z+h)\right\}$$

for the velocity potential, hence

Particle paths

We solve for the paths of particles in the fluid below the free surface, using the above expression for $\mathbf{u}(x,t)$. To find the particle path we need to integrate

We'll assume the displacement from the equilibrium position $\mathbf{X} - \mathbf{X}_0$ is small, so that the velocity can be linearized about the fixed point $\mathbf{X}_0$. Thus we need only solve

This says that the particle paths are ellipses with their major axes horizontal, their minor axes vertical, and their aspect ratios $|\tanh k(Z_0 + h)|$.

Free-surface modes in a container

We now solve the surface wave problem in a different geometry, relevant to understanding the behaviour of harbours and lakes and to industrial problems involving fluid containment and associated hazards. It may be important to know the frequencies of oscillation of the modes of free surface oscillation within such a container.

Take the case of a rectangular container partly filled with fluid to depth h:

$$0 < x < a, \quad 0 < y < b, \quad -h < z < \zeta(x, y, t) \qquad \text{(height of free surface)}$$

The undisturbed configuration is $\zeta(x, y, t) = 0$ with the fluid at rest everywhere. The linearized equations and boundary conditions will be valid if ζ is sufficiently small; they are

We seek solutions of the form

The boundary conditions at $z = 0$ become

We solve the problem by separation of variables. To satisfy the boundary conditions in x and y, try

This solves $\nabla^2\hat{\phi} = 0$ provided that $\hat{\phi}_{mn}(z)$ satisfies

The boundary conditions in z will be satisfied if

$$\left.\frac{d\hat{\phi}_{mn}}{dz}\right|_{z=-h} = 0 \ ,$$

$$\left.\frac{d\hat{\phi}_{mn}}{dz}\right|_{z=0} = \frac{\sigma^2}{g}\hat{\phi}_{mn}(0) \ ,$$

implying in turn that

$$\hat{\phi}_{mn}(z) \ \propto \ \cosh\{k_{mn}(z+h)\}$$

a nontrivial solution provided that one or both of the integers m and n are nonzero so that $k_{mn} \neq 0$, and provided also that $(*)$ is satisfied. This requires $\sigma = \sigma_{mn}$ where

$$\sigma_{mn}^2 = g\,k_{mn}\tanh(k_{mn}h) \ .$$

For a simple illustration, take the case $h = \infty$, $m = 2$, $n = 0$, $\Rightarrow$ $k_{mn} = 2\pi/a = k$, say, a purely two-dimensional standing wave in which the value of b is irrelevant. Then $\hat{\phi} = \hat{\phi}(x,z) \propto \exp(kz)\cos(kx)$, equivalent to a single pair of progressive deep-water waves travelling in the $\pm x$ directions.

Rayleigh–Taylor instability: (a simple extension of the above)
Suppose that the fluid is above the free surface rather than below. The above analysis still applies if we take g to be negative! Then

$$\sigma_{mn}^2 = -\,g\,k_{mn}\tanh(k_{mn}h) \ ,$$

and σ_{mn} is a pure imaginary number. Hence $e^{-i\sigma_{mn}t} = e^{\pm|\sigma_{mn}|t}$, corresponding to growing and decaying solutions rather than to oscillations.

Implication: the steady solution with water *above* air and the free surface horizontal is — not surprisingly — unstable, with small displacements growing exponentially. Growth rate increases as m and n increase, implying that smaller-scale structures grow faster.