§2.6 Vorticity

Definition: Vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. A **vortex line**, at a particular time t, is a curve wich has the same direction as the vorticity vector $\boldsymbol{\omega}$.

Example (1): Rigit body rotation $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}$

Interpretation of vorticity in 2D flow

Consider two short fluid line elements AB and AC which are perpendicular at a certain instant of time

Note that the $y-$component of velocity at B exceeds that at A by

so that $\partial v/\partial x$ represents the instantaneous angular velocity of the fluid line element AB. Likewise, $-\partial u/\partial y$ represents the instantaneous angular velocity of AC. Thus at any point of the flow field $\frac{1}{2}\omega = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ represents the **average angular velocity** of two short fluid line elements that happen, at that instant, to be mutually perpendicular. In this precise sense the vorticity ω acts as a measure of the local rotation, or spin, of fluid elements.

Example (2): Shear flow $\mathbf{u} = (\alpha y, 0, 0)$

Example (3): Line vortex $\mathbf{u} = \frac{k}{r}\mathbf{e}_\theta$ in cylindrical coordinates (r, θ, z). k is a constant.

The vorticity is zero except at $r = 0$, where neither $\mathbf{u}$ nor $\nabla \times \mathbf{u}$ is defined.

Examples (2) and (3) illustrate an important distinction. Rotation, as specified by vorticity, corresponds to changing orientation in space of the fluid particle and *not* to motion round a circular path.

Let's compare the vorticity field with the velocity field:

velocity $\mathbf{u}$	vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$
stream line	
$\nabla \cdot \mathbf{u} = 0$ (if incompressible)	
$\int_S \mathbf{u} \cdot \mathbf{n} = 0$ (integral form)	

The vortex lines which pass through some simple closed curve in space are said to form the boundary of a **vortex tube**.

We say that 'stretching amplfies vorticity'. It is also called the 'ballerina effect'. While spinning on your toes, you pull your arms in and spin faster.
This is essentially how the familiar 'bathtub vortex' works:

Let's derive an equation for vorticity of inviscid, incompressible fluid under action of conservative volume force:

Recall the vector identities: **(1)** $(\mathbf{u}\cdot\nabla)\mathbf{u} = \boldsymbol{\omega}\times\mathbf{u} + \nabla(\frac{1}{2}|\mathbf{u}|^2)$ **(3)** $\nabla\cdot(\nabla\times\mathbf{u}) = 0$

(2) $\nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - \mathbf{u}(\nabla\cdot\boldsymbol{\omega}) + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ **(4)** $\nabla\times\nabla\phi = 0$

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}.$$

Therefore, watching a given fluid particle/parcel we see its $\boldsymbol{\omega}$ value changing at the rate $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$.

Recall equation for a material line element: $\frac{D\delta\mathbf{l}}{Dt} = (\delta\mathbf{l}\cdot\nabla)\mathbf{u}$. Note: we can think of the d/dt as D/Dt (of $\delta\mathbf{l}$ regarded as a field, i.e. expressed as a function of $\mathbf{x}$ and t).

In component form:

Hence the tensor $\partial u_i/\partial x_j$ determines the local rate of deformation of line element.

The local motion of line elements due to the second term is $\frac{1}{2}\epsilon_{jik}\delta l_j\omega_k = (\frac{1}{2}\boldsymbol{\omega}\times\delta\mathbf{l})_i$

Local motion due to the first term, called the **pure strain**, gives zero angular velocity when averaged over all orientations of $\delta\mathbf{l}$.
So: *Vortex lines move as if they were material lines.* Or, vortex tubes rotate and stretch just like the material line elements. This is another statement of conservation of angular momentum.

§2.7 Kelvin's circulation theorem and the persistence of irrotationality

Circulation is the integral counterpart of vorticity; Kelvin's circulation theorem (sometimes just called 'the circulation theorem') is the integral counterpart of the vorticity equation. Define the *circulation*, $\mathcal{C}$, around a closed curve Γ by

$$\mathcal{C} = \oint_\Gamma \mathbf{u}\cdot d\mathbf{l} = \int_S \boldsymbol{\omega}\cdot\mathbf{n}\,dS \qquad \text{where} \quad S \quad \text{spans} \;\; \Gamma$$

(Stokes' theorem).

Let Γ be a *material* curve (this is crucial):

This is Kelvin's circulation theorem. In words: *for* **inviscid** *(frictionless) fluid of* **uniform density** *with* **conservative forces**, *the circulation around a closed* **material** *curve remains constant.*

Definition: *irrotational flow*, or *irrotational fluid motion* $\Leftrightarrow$ $\boldsymbol{\omega} = 0$ everywhere.

Corollary of the circulation theorem: *irrotational flow remains irrotational.*
Proof for smooth $\boldsymbol{\omega}(\mathbf{x},t)$: Initially irrotational $\Rightarrow$ circulation around all (arbitrary) material circuits initially zero $\Rightarrow$ circulation around all material circuits remains zero $\Rightarrow$ flow remains irrotational.