## §4.3 River flows and weirs

Here we consider free-surface potential flows dominated by horizontal accelerations.

Assume that:

1) the river has vertical sides and constant width;

2) it varies only slowly in the $x-$ direction, so that the flow is horizontal to a good approximation and uniform across any vertical section (since $\partial u/\partial z \approx$ (vorticity)$_y$=0);

3) the flow is steady;

4) pressure $p$ on the free surface is constant, $p = p_{\text{atm}}$.

Bernoulli's streamline theorem, applied along the free-surface streamline, says that

So there are just two 'Bernoulli possibilities', wherever, for one reason or other, the flow properties are changing with $x$:

We are usually interested in cases where the flow is uniform upstream, with $\zeta = 0$, $U = U_\infty$, and $h = h_\infty$ , say. Then Bernoulli gives

Mass conservation:

**§4.3.1 First example: Long waves brought to rest** Take a water wave in the shallow-water limit $|k|h \ll 1$, and bring it to rest by adopting the frame of reference travelling with the wave.

rises and the flow slows down. Toward a wave trough, the surface dips and the flow speeds up. Indeed this, plus mass conservation, is what determines the wave speed.

We can verify the last statement (i.e. rederive the dispersion relation) by linearizing the Bernoulli and mass-conservation relations. Writing $U = U_\infty + u$, we have

Linearizing (neglecting squares of $u$ and its products with $\zeta$), we have

which has nontrivial solutions $\big(u(x),\, \zeta(x)\big)$ if and only if the determinant vanishes:

### §4.3.2 Second example: Free-surface flow over a gentle bump

Does the free surface go up or down?

Mass conservation:

Bernoulli's streamline theorem, applied along the free-surface streamline

This equation for $\zeta(x)$ may be approximated, since $\zeta$ and $Z$ are both small compared to $h_\infty$, as

This rather simple result determines the depression $\zeta(x)$ in terms of the obstacle shape $Z(x)$. The important parameter in the result is $gh_\infty/U_\infty^2$, because this determines whether $\zeta$ has the same sign as $Z$ or the opposite sign. It is conventional to call $U/(gh)^{1/2}$ the Froude number $F$. Thus we have

### §4.3.3 Third example: Flow out of a reservoir over a broad weir

Question: How fast is the outflow as a function of the minimum of $h(x)$?

$h_\infty$ is very large, $U_\infty$ negligibly small. Assume *volume flux or flow rate* $Q = h_\infty U_\infty$ is finite.

Mass conservation:

Bernoulli's streamline theorem, applied along the free-surface streamline:

$$\tfrac{1}{2}U^2 + \frac{gQ}{U} \quad = \quad gh(x),$$

which is a cubic for unknown $U(x)$ given $h(x)$.

The flow starts in the reservoir with $h(x)$ large and $U(x)$ small, i.e. high on the LH branch, but comes out of weir with $U$ large, i.e. on the RH branch.

We can change branch smoothly only if minimum $h$ (at the crest of the weir), $h_{\text{min}}$ say, corresponds to minimum $F(U) = \frac{1}{2}U^2 + gQU^{-1}$ $(Q = h_\infty U_\infty)$. Hence

So, at the crest of the weir

## §4.4 *Bores and hydraulic jumps*

[‘bore’ as in ‘drill’, or ‘penetrate’. E.g. the famous ‘Severn bore’.] By far the biggest bore in the World is the bore propagating up the Qiantang River near Hangchow, China. At spring tides the wave attains a height of up to 7.5 m and a speed of 24-27 km/h. It is heard advancing at a range of 22 km.

reference: an abrupt change in velocity $U$ and surface elevation $\zeta$, propagating at constant speed, $V$ say, relative to the fluid ahead.

This picture, with the fluid ahead at rest, is of what would usually be called a ‘bore’. (It ‘bores along’ toward the right ($V > 0$).) In another frame in which $V = 0$, the same thing would usually be called a ‘hydraulic jump’.

Turbulent energy loss in the transition region can be so strong, in fact, that Bernoulli cannot be used, even as a rough approximation.

Apply mass conservation to *fixed* box containing the bore:

$$\text{mass flux out} \;=\; -\frac{d}{dt}(\text{ mass in box}) \;;$$

mass in (fixed) box is increasing. Therefore

$$-\rho h_2 U_2 \;=\; -V\rho(h_2 - h_1) \;,$$

i.e.

$$h_2 U_2 \;=\; V(h_2 - h_1) \;. \qquad (1)$$

(Note that a segment of the box of length $V\delta t$ changes from being ‘ahead’ of the bore to being ‘behind’ the bore in time $\delta t$.)

Similarly apply momentum conservation to the box:
(force exerted by fluid in box on fluid outside)
+ (advective momentum flux out) = $-\frac{d}{dt}$ (momentum in box):

$$\int_0^{h_1} p_1(z)dz + \int_{h_1}^{h_2} p_{\mathrm{atm}}\,dz - \int_0^{h_2} p_2(z)dz - \rho h_2 U_2^2 \;=\; -\rho V h_2 U_2 \;.$$

The pressure distributions $p_1(z)$ and $p_2(z)$ are hydrostatic and given by

$$p_1(z) = p_{\mathrm{atm}} + \rho g(h_1 - z) \qquad (0 < z < h_1)$$
$$p_2(z) = p_{\mathrm{atm}} + \rho g(h_2 - z) \qquad (0 < z < h_2)$$

Hence the momentum balance equation simplifies to

$$\tfrac{1}{2}gh_1^2 - \tfrac{1}{2}gh_2^2 - h_2U_2^2 = -Vh_2U_2 \;. \qquad (2)$$

We now have two equations, with unknowns $V$ and $U_2$ (if regard $h_2$ as given).
Eliminate $U_2$:

$$\tfrac{1}{2}g(h_1^2 - h_2^2) \;=\; -h_2\frac{V(h_2 - h_1)}{h_2}\,\frac{Vh_1}{h_2} \;.$$

$$\text{speed of bore} \qquad V = \left( g \frac{(h_1 + h_2) h_2}{2h_1} \right)^{1/2} ,$$

$$\text{flow speed behind bore} \qquad U_2 = \frac{h_2 - h_1}{h_2} V \, .$$

Note that $V \to \big(gh_1\big)^{1/2}$ if $h_2 \to h_1$, again checking consistency with small-amplitude wave theory, whereas, at finite amplitude, $h_2 > h_1$, we have

$$V > \big(gh_1\big)^{1/2} \, .$$

So the bore travels faster than, and will catch up with, any gravity waves ahead of it. But $V - U_2 = \big(h_1/h_2\big)V < \big(gh_2\big)^{1/2}$; i.e.

$$V < U_2 + \big(gh_2\big)^{1/2} \, ,$$

which says that waves upstream can catch up with the bore.

*Exercise* (last question on Ex. Sheet 3): Show that the rate of energy dissipation in the bore is positive if $h_2 > h_1$; this must be so if the model is to be physically reasonable.