Fluid Mechanics IB
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Summary of Vector Calculus

The following results apply to any (suitably differentiable) scalar field $\phi(\mathbf{x})$ and vector fields $\mathbf{u}(\mathbf{x})$ and $\boldsymbol{\omega}(\mathbf{x})$. They all have important applications in fluid mechanics.

Derivatives of a vector field:

$$\mathit{div}\,\mathbf{u} \equiv \nabla.\mathbf{u} = \frac{\partial u_i}{\partial x_i} \qquad \text{has one component and is a scalar}$$

$$\mathit{curl}\,\mathbf{u} \equiv (\nabla \wedge \mathbf{u})_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} \qquad \text{has three components and is a vector}$$

$$\mathit{grad}\,\mathbf{u} \equiv (\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j} \qquad \text{has nine components and is a 2nd-rank tensor}$$

Two identities:

$$\mathit{curl}\,\mathit{grad}\,\phi = \nabla \wedge \nabla\phi \equiv 0\ . \qquad \mathit{div}\,\mathit{curl}\,\mathbf{u} = \nabla.\nabla \wedge \mathbf{u} \equiv 0\ .$$

Two vector triple products:

$$\mathbf{u} \wedge (\nabla \wedge \mathbf{u}) = \tfrac{1}{2}\nabla u^2 - (\mathbf{u}.\nabla)\mathbf{u}$$

$$\nabla \wedge (\mathbf{u} \wedge \boldsymbol{\omega}) = (\boldsymbol{\omega}.\nabla)\mathbf{u} - (\mathbf{u}.\nabla)\boldsymbol{\omega} + \mathbf{u}(\nabla.\boldsymbol{\omega}) - \boldsymbol{\omega}(\nabla.\mathbf{u})$$

Divergence Theorem (Gauss): If V is a simply connected domain with surface S and outward normal $\mathbf{n}$ then

$$\int_V \frac{\partial f}{\partial x_i}\, dV = \int_S f n_i\, dA\ ,$$

where f can be a scalar, a vector or a general tensor. If f is the vector u_i, the LHS (with summation over i) is the *volume* integral of $\nabla.\mathbf{u}$ and the RHS is the *area* integral of $\mathbf{u}.\mathbf{n}$

Stokes' Theorem: If Γ is a closed curve spanned by the surface S then

$$\oint_\Gamma \mathbf{u}.\mathbf{dl} = \int_S (\nabla \wedge \mathbf{u}).\mathbf{dS}$$

Derivative in a direction: $\mathbf{u}.\nabla$ represents the rate of change in the direction of $\mathbf{u}$ of whatever it is applied to. Note that if $\mathbf{u}.\nabla$ is applied to a vector field in curvilinear coordinates then it cannot simply be applied to each coordinate of the vector field since the basis vectors $\mathbf{e}_i$ also vary with position.

Curl in curvilinear coordinates (ξ_1, ξ_2, ξ_3)**:**

$$\nabla \wedge \mathbf{u} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial \xi_1} & \dfrac{\partial}{\partial \xi_2} & \dfrac{\partial}{\partial \xi_3} \\ h_1 u_1 & h_2 u_2 & h_3 u_3 \end{vmatrix}\ ,$$

where: $h_1 = 1$, $h_2 = r$, $h_3 = 1$ in cylindrical polars (r, θ, z); and
$h_1 = 1$,$h_2 = r$, $h_3 = r\sin\theta$ in spherical polars (r, θ, φ).

Cylindrical coordinates (r, θ, z)

Grad, div and Laplacian:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla.\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

Volume and line elements, and normal to cylinder:

$$dV = r\,dr\,d\theta\,dz \qquad \mathbf{dl} = (dr, r\,d\theta, dz)$$

$$\mathbf{n} = (\cos\theta, \sin\theta, 0)$$

Stream function for 2D flow ($u_z = \partial/\partial z = 0$):

$$\mathbf{u} = \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}, -\frac{\partial\psi}{\partial r}, 0\right)$$

Stream function for axisymmetric flow ($u_\theta = \partial/\partial\theta = 0$):

$$\mathbf{u} = \left(-\frac{1}{r}\frac{\partial\Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial\Psi}{\partial r}\right)$$

Spherical coordinates (r, θ, φ)

Note that r and θ denote different quantities in cylindrical and spherical polars.

Grad, div and Laplacian:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\nabla.\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

Volume and line elements, and normal to sphere:

$$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi \qquad \mathbf{dl} = (dr, r\,d\theta, r\sin\theta\,d\varphi)$$

$$\mathbf{n} = (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$$

Stream function for axisymmetric flow ($u_\varphi = \partial/\partial\varphi = 0$):

$$\mathbf{u} = \left(\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}, 0\right)$$

Please email corrections/comments to N.G.Berloff@damtp.cam.ac.uk