Mathematical Tripos Part IA

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Differential Equations A3

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DIFFERENTIAL EQUATIONS

Examples Sheet 1

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets

1. Show, from first principles, that $\dfrac{d}{dx}x^n = nx^{n-1}$.

2. Let $f(x) = u(x)v(x)$. Use the definition of the derivative of a function to show that
$$\frac{df}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v.$$

3. Calculate
   (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^2}\sin 2x\right)$,
   (ii) $\dfrac{\mathrm{d}^{12}}{\mathrm{d}x^{12}}(x\cos x)$ using (a) the Leibniz rule and (b) repeated application of the product rule,
   (iii) $\dfrac{\mathrm{d}^5}{\mathrm{d}x^5}(\log x)^2$.

4. (i) Write down or determine the Taylor series for $f(x) = e^{ax}$ about $x = 1$.
   (ii) Write down or determine the Taylor series for $\log(1 + x)$ about $x = 0$. Then show that
$$\lim_{k\to\infty} k\log(1 + x/k) = x$$
and deduce that
$$\lim_{k\to\infty}(1 + x/k)^k = e^x.$$
What property of the exponential function did you need?

5. Determine by any method the first three non-zero terms of the Taylor expansions about $x = 0$ of

(i) $(x^2 + a)^{-3/2}$,

(ii) $\ln(\cos x)$,

iii) $\exp\left\{-\dfrac{1}{(x-a)^2}\right\}$,

where a is a constant.

6. By considering the area under the curves $y = \ln x$ and $y = \ln(x-1)$, show that

$$N \ln N - N < \ln(N!) < (N+1)\ln(N+1) - N.$$

Hence show that

$$|\ln N! - N \ln N + N| < \ln\left(1 + \frac{1}{N}\right)^N + \ln(1+N).$$

7. Show that $y(x) = \int_x^\infty e^{-t^2} dt$ satisfies the differential equation $y'' + 2xy' = 0$.

*8. Let J_n be the indefinite integral

$$J_n = \int \frac{x^{-n}\,\mathrm{d}x}{(ax^2 + 2bx + c)^{\frac{1}{2}}} .$$

By integrating $\int x^{-n-1}(ax^2 + 2bx + c)^{\frac{1}{2}}\,\mathrm{d}x$ by parts, show that for $n \neq 0$,

$$ncJ_{n+1} + (2n-1)bJ_n + (n-1)aJ_{n-1} = -x^{-n}(ax^2 + 2bx + c)^{\frac{1}{2}}.$$

Hence evaluate

$$\int_1^2 \frac{\mathrm{d}x}{x^{5/2}(x+2)^{\frac{1}{2}}}.$$

*9. In a large population, the proportion with income between x and $x + \mathrm{d}x$ is $f(x)\mathrm{d}x$. Express the mean (average) income μ as an integral, assuming that any positive income is possible.

Let $p = F(x)$ be the proportion of the population with income less than x, and $G(x)$ be the mean (average) income earned by people with income less than x. Further, let $\theta(p)$ be the proportion of the total income which is earned by people with income less than x as a function of the proportion p of the population which has income less than x. Express $F(x)$ and $G(x)$ as integrals and thence derive an expression for $\theta(p)$, showing that

$$\theta(0) = 0, \quad \theta(1) = 1$$

and

$$\theta'(p) = \frac{F^{-1}(p)}{\mu}, \quad \theta''(p) = \frac{1}{\mu f(F^{-1}(p))} > 0.$$

Sketch the graph of a function $\theta(p)$ with these properties and deduce that unless there is complete equality of income distribution, the bottom (in terms of income) $100p\%$ of the population receive less than $100p\%$ of the total income, for all values of p.

10. For $f(x, y) = \exp(-xy)$, find $(\partial f/\partial x)_y$ and $(\partial f/\partial y)_x$. Check that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$. Find $(\partial f/\partial r)_\theta$ and $(\partial f/\partial \theta)_r$,

(i) using the chain rule,

(ii) by first expressing f in terms of the polar coordinates r, θ,

and check that the two methods give the same results.

[Recall: $x = r\cos\theta, \quad y = r\sin\theta$.]

11. If $xyz + x^3 + y^4 + z^5 = 0$ (an implicit equation for any of the variables x, y, z in terms of the other two), find

$$\left(\frac{\partial x}{\partial y}\right)_z, \quad \left(\frac{\partial y}{\partial z}\right)_x, \quad \left(\frac{\partial z}{\partial x}\right)_y$$

and show that their product is -1.

Does this result hold for an arbitrary relation $f(x, y, z) = 0$?

What about $f(x_1, x_2, \cdots, x_n) = 0$?

12. In thermodynamics, the pressure of a system, p, can be considered as a function of the variables V (volume) and T (temperature) or as a function of the variables V and S (entropy).

(i) By expressing $p(V,S)$ in the form $p(V,S(V,T))$ evaluate

$$\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S \text{ in terms of } \left(\frac{\partial S}{\partial V}\right)_T \text{ and } \left(\frac{\partial S}{\partial p}\right)_V .$$

(ii) Hence, using $TdS = dU + pdV$ (conservation of energy with U the internal energy), show that

$$\left(\frac{\partial \ln p}{\partial \ln V}\right)_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)_S = \left(\frac{\partial (pV)}{\partial T}\right)_V \left[\frac{p^{-1}(\partial U/\partial V)_T + 1}{(\partial U/\partial T)_V}\right] .$$

$\left[\text{Hint: } \left(\frac{\partial \ln p}{\partial \ln V}\right)_T = \frac{V}{p}\left(\frac{\partial p}{\partial V}\right)_T\right]$

13. By differentiating I with respect to λ, show that

$$I(\lambda,\alpha) = \int_0^\infty \frac{\sin \lambda x}{x} e^{-\alpha x} dx = \tan^{-1}\frac{\lambda}{\alpha} + c(\alpha).$$

Show that $c(\alpha)$ is constant (independent of α) and hence, by considering the limits $\alpha \to \infty$ and $\alpha \to 0$, show that, if $\lambda > 0$,

$$\int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}.$$

14. Let $f(x) = \left[\int_0^x e^{-t^2} \mathrm{d}t\right]^2$ and let $g(x) = \int_0^1 [e^{-x^2(t^2+1)}/(1+t^2)]\mathrm{d}t$.

Show that

$$f'(x) + g'(x) = 0.$$

Deduce that

$$f(x) + g(x) = \pi/4,$$

and hence that

$$\int_0^\infty e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

Comments and corrections may be sent by email to N.G.Berloff@damtp.cam.ac.uk