Discrete Polynomial Optimization with Coherent Networks of Condensates and Complex Coupling Switching
Supplemental Material

Nikita Stroev[1] and Natalia G. Berloff[1,2*]
[1] *Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, bld.1, Moscow, 121205, Russian Federation and*
[2] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom*

1. DERIVATIONS OF RATE EQUATIONS FOR MINIMIZATION OF THE k-LOCAL HAMILTONIANS.

1.1. Rate equations from the complex Ginzburg-Landau model

In this section we provide the details of deriving Eq. (3) from Eq. (2) of the main text. Our starting point is Eq. (2) which is a generic laser model with a saturable nonlinearity

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\nabla^2\psi + \tilde{U}|\psi|^2\psi + i\left(\frac{P(\mathbf{r},t)}{1+b|\psi|^2} - \gamma_c\right)\psi, \tag{S1}$$

where $\psi(\mathbf{r},t)$ is the wavefunction of the system, $\tilde{U}$ is the strength of the delta-function interaction potential, γ_c is the rate of linear losses, b parametrizes the effective strength of nonlinear losses, $P(\mathbf{r},t)$ describes the gain mechanism that adds particles to the system. First of all, we assume that $b \ll U$, so that the saturation term can be replaced by its Taylor expansion:

$$\frac{P(\mathbf{r},t)}{1+b|\psi|^2} \approx P(\mathbf{r},t) - P(\mathbf{r},t)b|\psi|^2. \tag{S2}$$

We define by $p(\mathbf{r}) = P(\mathbf{r},t)$ the injection profile that gives rise to a single condensate centered at the origin that is described by a normalized wavefunction $\phi(\mathbf{r})$. Mathematically, $\phi(\mathbf{r})$ satisfies

$$\mu\phi = -\nabla^2\phi + \tilde{U}|\phi|^2\phi + i\left(p - pb|\phi|^2 - \gamma_c\right)\phi, \tag{S3}$$

$$\int_\Gamma |\phi(\mathbf{r})|^2\, d\mathbf{r} = 1, \tag{S4}$$

where μ is the chemical potential (the Lagrange multiplier) and Γ is the entire system space. Based on the wavefunction of the single isolated condensate, we can construct an approximation for N localized condensates noting that the well-separated condensates interact by the outflow of the particles from the positions where they were created [1, 2]. This is in a contrast with the conservative condensates, such as ultracold atomic Bose-Einstein condensates, where spatially separated condensates (with separation much larger than the condensate width) do not interact.

We shall assume that pumping $P(\mathbf{r},t)$ adds particles in N spatial locations centered at $\mathbf{r}_i$, $i = 1, ..., N$, so that $P(\mathbf{r},t) = \sum_i f_i(t)p_i(\mathbf{r})$, where f_i is the time-dependent part of the pumping at the position $\mathbf{r} = \mathbf{r}_i$ and $p_i(\mathbf{r}) \equiv p(\mathbf{r}-\mathbf{r}_i)$. If the distances between the neighbouring condensates are larger than the width of $p(\mathbf{r})$, we employ the tight binding approximation and write the wavefunction of the system as a linear superposition of the wavefunctions of the individual condensates $\psi(\mathbf{r},t) \approx \sum_{i=1}^N a_i(t)\phi_i(\mathbf{r})$, where $a_i(t)$ is the time-dependent complex amplitude and $\phi_i(\mathbf{r}) \equiv \phi(\mathbf{r}-\mathbf{r}_i)$. We substitute the expressions for P and ψ into Eq. (S1) with the Taylor expansion of the saturation term given by Eq. (S2), multiply by ϕ_j^* for $j = 1, ..., N$ and eliminate the spatial degrees of freedom by integrating in the entire plane Γ. The time evolution of the individual functions $a_i(t)$ will separate if we assume that the integrals that involve products of the wavefunctions of the separated condensates are negligible compared with the integrals of the products

*correspondence address: N.G.Berloff@damtp.cam.ac.uk

of the same condensates or with the integrals that involve the pumping profiles that overlap with ϕ_i, so that

$$\begin{aligned}
\int_\Omega \phi_i\phi_j^*\,d\mathbf{r} &\ll \int_\Omega |\phi|^2\,d\mathbf{r} = 1 \quad \text{for} \quad i\neq j,\\
\int_\Omega \phi_j^*\nabla^2\phi_i\,d\mathbf{r} &\ll \int_\Omega \phi^*\nabla^2\phi\,d\mathbf{r} \equiv d \quad \text{for} \quad i\neq j\\
\int_\Omega \phi_i\phi_j^*\,d\mathbf{r} &\ll \int_\Omega p_m\phi_i\phi_j^*\,d\mathbf{r} \quad \text{for} \quad i\neq j, m\in\{i,j\},\\
\int_\Omega \phi_i\phi_j^*\phi_k\phi_m^*\,d\mathbf{r} &\ll \int_\Omega p_m\phi_i\phi_j^*\phi_k\phi_l^*\,d\mathbf{r}, \quad m\in\{i,j,k,l\}, \quad \text{etc.}
\end{aligned}$$

The validity of these assumptions can be verified using asymptotics developed in [3] where it was shown that ϕ created with a Gaussian pump can be approximated by

$$\phi(r) = \sqrt{\frac{2}{\pi}}\beta\exp[-\beta r + ik_c r],$$

where k_c is the outflow velocity and β is the inverse characteristic width of the condensate [3]. The integrals $\chi_{ij} = \int \phi_i\phi_j^*\,d\mathbf{r}$ for $i\neq j$ can be evaluated using the elliptical coordinates in terms of the Bessel functions [4]

$$\chi_{ij} = 2\beta^2 l_{ij}\left[\frac{1}{\beta}J_0(k_c l_{ij})K_1(\beta l_{ij}) + \frac{1}{k_c}J_1(k_c l_{ij})K_0(\beta l_{ij})\right], \tag{S5}$$

where $l_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. We assumed that the condensates are well separated, $l_{ij}\beta \gg 1$, so that for $i\neq j$ we have $\chi_{ij}\ll 1$ as follows from Eq. (S5). The correctness of other assumptions can be established in a similar manner. Under these assumptions, the tight binding approximation of Eq. (S1) leads to N equations

$$\begin{aligned}
\frac{da_i}{dt}\int_\Gamma |\phi|^2\,d\mathbf{r} &= ia_i d - iU|a_i|^2 a_i\int_\Gamma |\phi|^4\,d\mathbf{r} + a_i(f_i\int_\Gamma p|\phi|^2\,d\mathbf{r} - \gamma_c\int_\Gamma |\phi|^2\,d\mathbf{r})\\
&+ \sum_{j,j\neq i} a_j f_j\int_\Gamma p_j\phi_j\phi_i^*\,d\mathbf{r} - b^{-1}\sum_{m\in\{i,j,k,l\},j,k,l} f_m\int_\Gamma p_m\phi_j\phi_k\phi_l^*\phi_i^*\,d\mathbf{r}\,a_j a_k a_l^*.
\end{aligned}$$

We use normalization Eq. (S4) and introduce $\Psi_i = a_i\exp(itd)$ to absorb the first term on the right hand side into the phase of Ψ_i. We also denote $\gamma_i = f_i\int_\Gamma p|\phi|^2\,d\mathbf{r} - \gamma_c$, $U = \tilde{U}\int_\Gamma |\phi|^4\,d\mathbf{r}$, $J_{ij} = f_j\int_\Gamma p_i\phi_j\phi_i^* d\mathbf{r}$ for $j\neq i$, $\sigma_i = bf_i\int_\Gamma p|\phi|^4\,d\mathbf{r}$ and $Q_{ijkl} = -b\sum_{m\in\{i,j,k,l\}} f_m\int_\Gamma p_m\phi_k\phi_l^*\phi_j\phi_i^*\,d\mathbf{r}$, where $i=j=k=l$ is excluded from the summation. With this notation, we obtain N equations of the form

$$\frac{d\Psi_i}{dt} = \Psi_i\left(\gamma_i - (\sigma_i + \mathrm{i}U)|\ \Psi_i\ |^2\right) + \sum_{j,j\neq i} J_{ij}\Psi_j + \sum_{\langle j,k,l\rangle} Q_{ijkl}\Psi_j\Psi_k\Psi_l^*. \tag{S6}$$

where $\langle i,j,k\rangle$ denotes the permutations of $\{j,k,l\}$ that exclude $j=k=l=i$.

1.2. Physics-inspired gain-dissipative algorithm.

To formulate the physics-inspired algorithm, we extend Eq.(3) to a general tensor minimisation by replacing the four-way interaction Q_{ijkl} with k-way interactions given by $A_{i_1,\ldots i_k}$. Here $A_{i_1,\ldots i_k}$ is an arbitrary super-symmetric real-valued tensor. We drop the quadratic coupling terms J_{ij} as they can always be incorporated by the higher order terms by introducing squares of the spins, e.g. the term $J_{12}x_1x_2$ can be written as $J_{1233}x_1x_2x_3^2$ where as before, $x_i = \cos\theta_i$. We also simplify Eq. (S6) by rescaling $\rho_i\to\rho_i/\sigma_i$ and setting $U=0$, as the main effect of U is to introduce the natural frequencies to individual oscillators (see below the discussion of the meaning of the natural frequencies in the Kuramoto system) and their absence only helps synchronisation. Finally, we drive the occupancy of the oscillators to the same a priori known value $\rho_{\rm th}$ by introducing the feedback for the pumping adjustments

$$\dot{\gamma_i} = \epsilon(\rho_{\rm th} - \rho_i), \tag{S7}$$

where ϵ characterizes how fast γ_i adjusts to changes in ρ_i. These arguments lead to Eq.(6-7) of the main text.

2. HIGHER-ORDER KURAMOTO OSCILLATORS

In this section we elucidate the relationship between Eq.(5) of the main text and the higher-order Kuramoto model. The paradigmatic Kuramoto model for the dissipative collective dynamics of N phase oscillators coupled with strengths $\tilde{J}_{ij}$ described by $\theta_i(t)$ with natural frequencies ω_i is [5]

$$\dot{\theta}_i(t) = \omega_i + \sum_{j,j\neq i} \tilde{J}_{ij} \sin(\theta_j - \theta_i), \quad i = 1, 2, \cdots, N. \tag{S8}$$

Such (standard) Kuramoto model describes self-sustained phase oscillators coupled through the sine of their phase differences. This model exhibits a phase transition at a critical coupling, beyond which a collective behavior is achieved. It is commonly used as a toy model to describe ubiquitous synchronization phenomena in various driven-dissipative systems with examples ranging from chemical reactions, laser arrays, optomechanical systems, neurons to heart cells. A conservative Hamiltonian systems may exhibit a family of invariant tori on which the dynamics is identical to that of the Kuramoto model [6, 7].The higher-order Kuramoto models have been proposed recently to model higher-order interactions between neurological dynamical units, i.e., including four-way interactions G_{ijkl} in addition to pairwise interactions [8]

$$\dot{\theta}_i(t) = \omega_i + \sum_{j,j\neq i} \tilde{J}_{ij} \sin(\theta_j - \theta_i) + \sum_{\langle i,j,k\rangle} G_{ijkl} \sin(\theta_j + \theta_k - \theta_l - \theta_i). \tag{S9}$$

The higher-order couplings come directly from the higher order terms that emerge from phase-reductions of limit-cycle oscillators [9].

By using the Madelung transformation $\Psi_i = \sqrt{\rho_i}\exp[\mathrm{i}\theta_i]$ in Eq. (3) of the main text (Eq. (S6) above) we obtain the equation on the real part of Eq.(3) as Eq. (5) of the main text:

$$\dot{\theta}_i(t) = -U\rho_i + \sum_{j,j\neq i} J_{ij} \frac{\sqrt{\rho_j}}{\sqrt{\rho_i}} \sin(\theta_j - \theta_i) + \sum_{\langle j,k,l\rangle} Q_{ijkl} \frac{\sqrt{\rho_j\rho_k\rho_l}}{\sqrt{\rho_i}} \sin(\theta_j + \theta_k - \theta_l - \theta_i). \tag{S10}$$

This coincides with the higher-order Kuramoto model Eq. (S9) if we allow for density-dependent (and therefore time-dependent) coupling tensors $\tilde{J}_{ij} = J_{ij}\frac{\sqrt{\rho_j(t)}}{\sqrt{\rho_i(t)}}$ and $G_{ijkl} = Q_{ijkl}\frac{\sqrt{\rho_j(t)\rho_k(t)\rho_l(t)}}{\sqrt{\rho_i(t)}}$. When the feedback given by Eq. (S7) is implemented, near the fixed point where $\rho_i \approx \rho_{\mathrm{th}}$ Eq. (S10) represents the system of N almost identical higher-order Kuramoto oscillators with $\omega_i \approx -U\rho_{\mathrm{th}}$ with couplings $J_{ij} = \tilde{J}_{ij}$ and $Q_{ijkl} = G_{ijkl}/\rho_{\mathrm{th}}$.

3. K TENSORS

Here we present the detailed description of K tensors for both cases considered in the main text: the small-scale toy-model Hamiltonian and the large-scale problem for testing the proposed TGD algorithm.

3.1. Toy-model Hamiltonian

The purpose of the toy-model Hamiltonian is to present an instance K tensor to demonstrate the algorithmic effectiveness and to introduce the idea of the complex-coupling switching. The tensor rank $k = 3$ is different from the quadratic problems, which were extensively studied before. We choose K elements so that the minimization of the corresponding tensor Hamiltonian is non trivial and has not only the global but also several local minima with a large basin of attraction.

The toy Hamiltonian considered in the main text is

$$H_{\mathrm{test}}(\mathbf{x}) = -8x_1x_2x_3 - 4x_1x_2x_4 - 2x_2x_3x_4 - x_1x_3x_4, \tag{S11}$$

with variables $x_i \in \{\pm 1\}$ and the corresponding tensor K has nonzero entries $K_{123} = 1, K_{124} = 4, K_{234} = 2$, and $K_{134} = 1$. Tensor K is assumed to be super-symmetric, so that the permutations of indexes give the same values of K, for instance, $K_{213} = K_{312} = 1$. There are 2^4 possible minimizers (the possible values of variables x_i) while the Hamiltonian H_{test} has 2^4 possible distinct values $\{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15\}$. The coefficients before each of the triplet of Eq. (S11) is a certain power of 2, which allows one to classify the solution by the energy value in

the binary tree and excludes the possibility of the degenerate solutions with the same energy. There are three local minima with $H_1 = -9, H_2 = -11, H_3 = -13$ and the global minimum $H_4 = -15$, that can be accessed during the time evolution of the system. To understand the basins of attraction for these stationary points we numerically integrate the corresponding differential equations Eqs. (6-7) from the main article. It appeared that the basins of attraction of the local minima for the dynamics of Eqs. (6-7) in total is larger than the basin of attraction of the global minimum with the statistics shown in Fig. 1(a) of the main text.

The system of Eqs. (6) from the main text becomes

$$\begin{aligned}
\frac{d\Psi_1}{dt} &= \Psi_1\left(\gamma_1 - |\Psi_1|^2\right) + 8\Psi_2\Psi_3^* + 4\Psi_2\Psi_4^* + \Psi_3\Psi_4^*,\\
\frac{d\Psi_2}{dt} &= \Psi_2\left(\gamma_2 - |\ \Psi_2\ |^2\right) + 8\Psi_1\Psi_3^* + 4\Psi_1\Psi_4^* + 2\Psi_3\Psi_4^*,\\
\frac{d\Psi_3}{dt} &= \Psi_3\left(\gamma_3 - |\ \Psi_3\ |^2\right) + 8\Psi_1\Psi_2^* + 2\Psi_2\Psi_4^* + \Psi_1\Psi_4^*,\\
\frac{d\Psi_4}{dt} &= \Psi_4\left(\gamma_4 - |\ \Psi_4\ |^2\right) + 4\Psi_1\Psi_2^* + 2\Psi_2\Psi_3^* + \Psi_1\Psi_3^*.
\end{aligned}$$

These equations are integrated together with Eq. (S7). The system with always reach a steady state with $\rho_i = \rho_{\rm th}$ for all $i = 1, 2, 3, 4$. At the steady state, from the imaginary part of the equations above we have

$$\begin{aligned}
\rho_{\rm th} &= \gamma_1 + 8\sqrt{\rho_{\rm th}}\cos(\theta_2 - \theta_3 - \theta_1) + 4\sqrt{\rho_{\rm th}}\cos(\theta_2 - \theta_3 - \theta_1) + \sqrt{\rho_{\rm th}}\cos(\theta_3 - \theta_4 - \theta_1),\\
\rho_{\rm th} &= \gamma_2 + 8\sqrt{\rho_{\rm th}}\cos(\theta_1 - \theta_3 - \theta_2) + 4\sqrt{\rho_{\rm th}}\cos(\theta_1 - \theta_4 - \theta_2) + 2\sqrt{\rho_{\rm th}}\cos(\theta_3 - \theta_4 - \theta_2),\\
\rho_{\rm th} &= \gamma_3 + 8\sqrt{\rho_{\rm th}}\cos(\theta_1 - \theta_2 - \theta_3) + 2\sqrt{\rho_{\rm th}}\cos(\theta_2 - \theta_4 - \theta_3) + \sqrt{\rho_{\rm th}}\cos(\theta_1 - \theta_4 - \theta_3),\\
\rho_{\rm th} &= \gamma_4 + 4\sqrt{\rho_{\rm th}}\cos(\theta_1 - \theta_2 - \theta_4) + 2\sqrt{\rho_{\rm th}}\cos(\theta_2 - \theta_3 - \theta_4) + \sqrt{\rho_{\rm th}}\cos(\theta_1 - \theta_3 - \theta_4),
\end{aligned}$$

while from the real part we get

$$\begin{aligned}
0 &= 8\sin(\theta_2 - \theta_3 - \theta_1) + 4\sin(\theta_2 - \theta_3 - \theta_1) + \sin(\theta_3 - \theta_4 - \theta_1),\\
0 &= 8\sin(\theta_1 - \theta_3 - \theta_2) + 4\sin(\theta_1 - \theta_4 - \theta_2) + 2\sin(\theta_3 - \theta_4 - \theta_2),\\
0 &= 8\sin(\theta_1 - \theta_2 - \theta_3) + 2\sin(\theta_2 - \theta_4 - \theta_3) + \sin(\theta_1 - \theta_4 - \theta_3),\\
0 &= 4\sin(\theta_1 - \theta_2 - \theta_4) + 2\sin(\theta_2 - \theta_3 - \theta_4) + \sin(\theta_1 - \theta_3 - \theta_4).
\end{aligned}$$

The system has $U(1)$ symmetry, so we can set $\theta_1 = 0$ and notice that the only possible solution of the last four equations is when $\theta_i = \{0, \pi\}$ in which case $\cos(\theta_i) = \{\pm 1\}$ and $\cos(\theta_i - \theta_j - \theta_k) = \cos(\theta_i)\cos(\theta_j)\cos(\theta_k) = x_i x_j x_k$. Finally, when we add the equations from the imaginary part together we get

$$4\rho_{\rm th} = \sum_{i=1}^{4}\gamma_i - 3\sqrt{\rho_{\rm th}}H_{\rm test}.$$

The left-hand side represents the total number density in the system that is a priori known. The first term on the right-hand side is the total effective injection that we assume is the minimum possible, therefore, $H_{\rm test}$ is minimised at the steady state of the system.

Complex coupling switching parameters.

The presence of the complex part of the coupling introduces the phase lag in the system. To see this we consider a one tensor element in Eq. (S6) and replace the real coupling Q_{ijkl} with the complex coupling $Q_{ijkl} + i\hat{Q}_{ijkl}$. The corresponding term in the Eq. (S6) is replaced with $a_{ijkl} = -Q_{ijkl}\sin\theta_{ijkl} + \hat{Q}_{ijkl}\cos\theta_{ijkl}$ where $\theta_{ijkl} = \theta_i + \theta_l - \theta_k - \theta_j$. We denote $\tan\delta_{ijkl} = \hat{Q}_{ijkl}/Q_{ijkl}$ so that $a_{ijkl} = -Q_{ijkl}(\cos\delta)^{-1}\sin(\theta_{ijkl} - \delta_{ijkl})$. The meaning and the role played by the phase lag δ_{ijkl} is well-understood in the Kuramoto systems [5]. It leads to either shift of the stationary points that always exist when lag is zero or to the destabilization of it by the creation of a saddle point. In the latter case, if the complex part of the coupling is turned on, the system trajectory leaves the neighborhood of the previous stationary point along the fastest direction. Including this switching dynamics into our system facilitates the search for the true global minimum by allowing fuller exploration of the phase space.

By varying the coupling elements to be switched, the duration of the switching in time and the amplitude of the imaginary coupling we allow the system to explore the phase space in the search for global minimum. In our test example, we implemented the switching of the coupling coefficients K_{123} and K_{124} (and their super-symmetric counterparts) according to $K_{123}(t) = 8(1+4i), K_{124}(t) = 4(1-10i), t \in [t_1, t_1+160]\cup[t_2, t_2+160]\cup[t_3, t_3+280]$ while

keeping $K_{123}(t) = 8, K_{124}(t) = 4$ otherwise, which allows *every* trajectory irrespective of its initial state to arrive to the global minimum. Here t_1, t_2, t_3 are times at which the system settles to a steady state after switching the complex part of the couplings off. Turning on the complex part of $K_{123}(t) = 8(1+4i)$ and $K_{124}(t) = 4(1-10i)$ for $t = T$ after the steady states are reached leads to the following permutation of the states: a)$(1,2,3,4) \to (2,3,3,4)$ if $T = 80$; b)$(1,2,3,4) \to (1,1,1,2)$ if $T = 160$; c)$(1,2,3,4) \to (4,3,3,4)$ if $T = 280$. Clearly the switching protocol b)b)c) brings all trajectories to the global minimum.

3.2. Large-scale simulations

We adapt the idea of the complex couplings switching for the large scale simulations and illustrate the benefits of such approach on 20 dense and 20 sparse tensor sets K_{ijk} of 3^d rank of size 10^6 over 500 runs. The elements of the super-symmetric tensor K_{ijk} are generated are uniformly randomly chosen from $[-1, 1]$ for each unique cluster of triplets $(ijk, i < j < k)$ independently. The same value is assigned to K_{ijk} with any permutation of indices ijk. To generate sparse tensors we take the generated dense tensors with elements from the uniform distribution on $[-1, 1]$ and randomly set $9/10^{th}$ of all elements to zero. Both types of tensors lead to the spin glass problems for which it is hard (highly likely NP-hard) to find the ground state.

Complex coupling switching parameters.

To implement the complex coupling switching (TGD+CC) method on large N, as soon as the system reaches the steady state we randomly choose $N/50$ of the coupling strengths K_{ijk} (and their corresponding elements with all possible permutations of the indexes) and modify them by adding $\chi_{ijk} = 3iK_{ijk}$. This destabilises the system and forces the trajectory to leave along a certain orbit. After that we let $\chi_{ijk} = 0$ and allow the original system to relax to a new steady state.

4. DETAILS OF THE NUMERICAL SIMULATIONS.

The systems of differential equations described in the main text such as Eqs. (6-7) with and without the complex switching, Eq.(9) and Eqs.(10) were numerically integrated using the Euler integration scheme. All systems were allowed to run for the same amount of time which favoured the comparison methods (Eq.(9) and Eqs.(10)) as they require 4 times fewer equations than TGD and TGD+CC. Nevertheless, TGD+CC outperformed all these methods. We used the time step $dt = 0.001 - 0.02$ in all runs. The parameters were optimised for each method independently taking into the account 1) stability of the scheme (using the Runge-Kutta scheme would allow for a larger time step) and the sufficiently slow approach to the minimum to allow the system to search the system hyperspace. While integrating Eqs. (6-7) we used $\epsilon = 0.01M$, $\gamma_l(t = 0) = -0.2M$, $\rho_{\rm th} = 0.1M$, the diffusion coefficient of the white noise $D = 100 \max(1 - \rho/\rho_{\rm th}, 0)$, $dt = 0.001$, $M = \max_{1\le l\le N} \sum_{\bar{\Omega}} |A^k_{l,i_1,i_2,\cdots,i_k}|$. The elements of the dense (sparse) tensors were scaled by $5/6N$ $(15/6N)$ to slow the dynamics down. To numerically integrate Eq. (9) we used the initial conditions $x_l(t = 0)$ that were uniformly randomly distributed in $[-0.5, 0.5]$, and $h_{j+1} = 1.2h_j$ was updated each time $\sum_{l=1}^{N} |\frac{dx_l}{dt}| < 0.001$ was satisfied. To numerically integrate Eqs.(10) we used the initial conditions $x_l(t = 0)$ that were randomly distributed in $[-5, 5]$, $\beta(t = 0) = 10$, while $\beta_{j+1} = 0.8\beta_j$ was updated each time $\sum_{l=1}^{N} |\frac{dx_l}{dt}| < 0.001$ was satisfied.

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