Solitary and Periodic Solutions of Nonlinear Nonintegrable Equations

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The singular manifold method and partial fraction decomposition allow one to find some special solutions of nonintegrable partial differential equations (PDE) in the form of solitary waves, traveling wave fronts, and periodic pulse trains. The truncated Painlevé expansion is used to reduce a nonlinear PDE to a multilinear form. Some special solutions of the latter equation represent solitary waves and traveling wave fronts of the original PDE. The partial fraction decomposition is used to obtain a periodic wave train solution as an infinite superposition of the "corrected" solitary waves.

1. Introduction

The existence of solitons and periodic wave trains is an important question in the study of nonlinear evolution equations. The methods of finding such solutions for integrable equations are well known: the solitary solutions can be found by the Hirota bilinear method [1, 2] and the periodic solutions can be represented by sums of equally spaced solitons represented by sech-function [3, 4]. We extend these methods to find the solitary and periodic solutions for some nonintegrable nonlinear equations by means of some generalization of the Painlevé expansion (Psi series [5]).

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Weiss et al. [5, 6] developed the singular manifold method to introduce the Painlevé property in the theory of partial differential equations (PDE). A PDE is said to possess the Painlevé property if its solutions are single valued about the movable singularity manifold [5]. To be more specific, if the singularity manifold is given by $F(z_1, z_2, \ldots, z_n) = 0$, then a solution of the PDE must have the expansion

$$u = F^{-\alpha} \sum_{j=0}^{\infty} u_j F^j, \tag{1.1}$$

where u_n are analytic functions in the neighborhood of the singularity manifold and α is a positive integer. Substitution of this expansion into the PDE determines the positive value of α (from the leading-order analysis) and defines the recursion relations for u_j.

Weiss [5, 6] truncated the expansion at the "constant term" level, i.e.,

$$u = u_0 F^{-\alpha} + u_1 F^{-\alpha+1} + \cdots + u_{m-1} F^{-1} + u_m. \tag{1.2}$$

Substituting back into the PDE, one obtains an overdetermined system of equations for F and u_j. The beauty of the singular manifold method is that this expansion for a nonlinear PDE contains a lot of information about this PDE. For an equation that possesses the Painlevé property the singular manifold method leads to the Bäcklund transformation, the Lax pair, and Muira transformations and makes connections to the Hirota bilinear method, Laplace–Darboux transformations, and the Toda lattice [5, 7].

Most nonlinear nonintegrable equations do not possess the Painlevé property; i.e., they are not free from "movable" critical singularities. For some equations it is still possible to obtain single-valued expansions by putting a constraint on the arbitrary function in the Painlevé expansion. Such equations are said to be partially integrable and Weiss [5] conjectured that these systems can be reduced to integrable equations. Another treatment of the partially integrable systems was offered by Hietarinta [2] by the generalization of the Hirota bilinear formalism for nonintegrable systems. He conjectured that all completely integrable PDEs can be put into a bilinear form. There are also nonintegrable equations that can be put into the bilinear form and then the partial integrability is associated with the levels of integrability defined by the number of solitons that can be combined to an N-soliton solution. Partial integrability then means that the equation allows a restricted number of multisoliton solutions. We suggest joining these treatments of the partial integrability and using the Painlevé expansion truncated *before* the "constant term" level as the transform for reducing a nonintegrable PDE to a *multilinear* equation. The partial integrability will be associated with the solvability of that equation.

We illustrate our method by applying it to the study of some nonlinear partial differential equations of the form

$$\frac{\partial u}{\partial t} + u^r \frac{\partial u}{\partial x} + a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^3 u}{\partial x^3} + c\frac{\partial^4 u}{\partial x^4} + d\frac{\partial^5 u}{\partial x^5} = 0. \tag{1.3}$$

This equation was first derived by Benney then by Kawahara and Lin and is an important general equation that describes the evolution of long waves in various problems in fluid dynamics. In purely dispersive form ($a = c = 0$, $r = 1$), (1.3) reduces to the Kawahara equation (generalized KdV equation with higher-order dispersion) that describes water waves with surface tension [8–10]. In the purely dissipative form ($b = d = 0$), (1.3) reduces to the Kuramoto–Sivashinsky equation that originates from a weakly nonlinear and long-wave simplification of the Navier–Stokes equation and has been used to describe different phenomena such as spatial patterns of the Belousov–Zhabotinsky reaction, surface-tension-driven convection in a liquid film, and unstable flame fronts. The dissipative–dispersive equation ($d = 0$) [11] is a generalized Kuramoto–Sivashinsky equation that describes the waves in the vertical and inclined falling film, in liquid films that are subjected to interfacial stress from adjacent gas flow, interfacial instability between two cocurrent viscous fluids, unstable drift waves in plasma, and phase evolution for the complex Ginzburg–Landau equation.

Such equations are the least integrable systems in the Hietarinta classification. For such equations we offer a simple method for finding the traveling wave solutions in the form of solitary waves and traveling wave fronts. Furthermore, if the soliton solution is found, the periodic wave train represented by the superposition of the solitons approximates the exact periodic solution as the spacing between pulses gets large. The partial fraction decomposition is used to characterize such solutions. The expressions for the speed of propagation and the constant of integration in terms of the period of the solutions is obtained. Such superposition is shown to satisfy the original equation plus some small correction term. Then we show how the exact solution can be obtained by taking into account the correction term and some small perturbation of the traveling wave coordinate. Our method gives an explicit expression for the velocity and amplitude of the periodic pulse train in terms of period.

2. Solitary solutions of the generalized Kuramoto–Sivashinsky equations

We consider the generalized Kuramoto–Sivashinsky equation (GKSE) in the form

$$u_t + 2uu_x + u_{xx} + \sigma u_{xxx} + u_{xxxx} = 0. \tag{2.1}$$

We take the transform (1.2) truncated at the term before the "constant term" level u_m:

$$u = u_0 F^{-\alpha} + u_1 F^{-\alpha+1} + \cdots + u_{m-1} F^{-1}. \tag{2.2}$$

The analysis of the leading-order terms gives $\alpha = 3$. By substituting the expansion (2.2) into (2.1) in the special case $\sigma = 4$ and equating the coefficients of the highest powers of F to zero one obtains expressions for u_0, u_1, u_2 in terms of F that lead to the transform

$$u = 30(\ln F)_{xxx} + 30(\ln F)_{xx}. \tag{2.3}$$

This transform implemented to

$$u_t + 2uu_x + u_{xx} + 4u_{xxx} + u_{xxxx} = 0 \tag{2.4}$$

leads after one integration in x to the following equation, which is *trilinear* in F, meaning that each term contains three functions F or its derivatives:

$$\begin{aligned} &- FF_tF_x + 2F_tF_x^2 + 2F_x^3 + F^2F_{xt} - 2FF_xF_{xt} - FF_tF_{xx} - 3FF_xF_{xx} + 15FF_{xx}^2 \\ &- 20F_xF_{xx}^2 + 30F_{xx}^3 + F^2F_{txx} + F^2F_{xxx} - 20FF_xF_{xxx} + 40F_x^2F_{xxx} \\ &+ 10FF_{xx}F_{xxx} - 60F_xF_{xx}F_{xxx} + 20FF_{xxx}^2 + 5F^2F_{xxxx} - 25FF_xF_{xxxx} \\ &+ 30F_x^2F_{xxxx} - 15FF_{xx}F_{xxxx} + 5F^2F_{xxxxx} - 6FF_xF_{xxxxx} + F^2F_{xxxxxx} \\ &= B. \end{aligned} \tag{2.5}$$

Substituting $F = 1 + e^{kx+\omega t - x_0}$ into this equation and equating the coefficients of different powers of e to zero we get the solution $F = 1 + e^{x-6t-x_0}$, where the constant of integration $B = 0$. This solution corresponds to the homoclinic orbit−one-soliton solution of (2.4):

$$u(x) = \frac{60e^{x-6t-x_0}}{(1+e^{x-6t-x_0})^3}. \tag{2.6}$$

The transform (2.2) for (2.1) and arbitrary σ has the form

$$u = 30(\ln F)_{xxx} + \frac{15}{2}\sigma(\ln F)_{xx} + \frac{15}{152}(16-\sigma^2)(\ln F)_x, \tag{2.7}$$

the same as in [12] omitting the "constant term" level. One can use this transform to find the solutions of (2.1) for the following values of the parameter σ with corresponding values of k and ω

σ	ω	k
± 4	-6	± 1
0	$-\frac{330}{361}$	$\pm\sqrt{\frac{11}{19}}$
$\pm\frac{12}{\sqrt{47}}$	$-\frac{60}{2209}$	$\pm\frac{1}{\sqrt{47}}$
$\pm\frac{16}{\sqrt{73}}$	$-\frac{90}{5329}$	$\pm\frac{1}{\sqrt{73}}$

With other parameter values, solutions F of the above form are not found [12].

Furthermore, since the formal Painlevé expansion applied to this equation, which does not possess the Painlevé property, gives the exact solution of the equation under consideration we can try to use this expansion even for equations with noninteger α. Let us consider the GKSE with the cubic nonlinear term

$$u_t + u^2 u_x + u_{xx} + \sigma u_{xxx} + u_{xxxx} = 0. \tag{2.8}$$

The leading term analysis gives $\alpha = \frac{3}{2}$. So we can consider the *generalized* Painlevé expansion

$$u(x,t) = \frac{u_0(x,t)}{h(x,t)^{3/2}} + \frac{u_1(x,t)}{h(x,t)^{1/2}}. \tag{2.9}$$

By equating the coefficients at the same powers of $h(x,t)$ we obtained the expressions for $u_0(x,t)$ and $u_1(x,t)$ in terms of h, leading to the transform

$$u(x,t) = -\frac{3\sqrt{35}}{2\sqrt{2}}\left(\frac{h_x}{h}\right)^{3/2} + \frac{3\sqrt{70}}{80}\,\frac{2\sigma h_x + 15h_{xx}}{\sqrt{h_x h}}. \tag{2.10}$$

Then (2.10) transforms (2.8) to the *octalinear* equation with 56 terms, which resembles (2.5). Despite the seeming complexity the one-soliton solution can be found for some values of the parameter σ:

$$\sigma = \frac{5}{\sqrt{2}}, \qquad \pm\frac{15}{\sqrt{71}}, \qquad \pm\sqrt{\frac{5}{11}}, \qquad \pm\frac{45}{\sqrt{374}}.$$

For example, if $\sigma = 5/\sqrt{2}$, the one-soliton solution was found as

$$u(x,t) = \frac{3\sqrt{35}\sqrt[4]{2}}{2}\sqrt{\frac{e^{\sqrt{2}x-4t}}{\left(1+e^{\sqrt{2}x-4t}\right)^3}}. \tag{2.11}$$

Some other solutions were found for the following values of the parameters:

σ	ω	k
$\pm\sqrt{\frac{5}{11}}$	$-\frac{210}{121}$	$\pm 2\sqrt{\frac{5}{11}}$
$\pm\frac{15}{\sqrt{71}}$	$-\frac{210}{5041}$	$\mp\frac{2}{\sqrt{71}}$
$\pm\frac{45}{\sqrt{374}}$	$-\frac{210}{34969}$	$\pm\sqrt{\frac{2}{187}}$

All of these latter solutions are "fronts." The graphs of (2.6) and (2.11) look qualitatively like "solitons," but are not quite symmetric front-to-back.

3. Periodic pulse train

Whitham [3, 4] originally suggested the representation of periodic waves as sums of equally spaced solitons for the KdV, modified KdV, and the Boussinesq equations.

We use this idea to apply the direct partial fraction decomposition to find the periodic solution for the GKSE

$$u_t + 2uu_x + u_{xx} + 4u_{xxx} + u_{xxxx} = 0. \tag{3.1}$$

The traveling form, $u = u(x-ct) = u(\xi)$, in this gives

$$-cu' + 2uu' + u'' + 4u''' + u'''' = 0, \tag{3.2}$$

or after we integrate once

$$-cu + u^2 + u' + 4u'' + u''' = B. \tag{3.3}$$

In view of (2.6) the solution of (3.9) for $c = 6$ and $B = 0$ is

$$u_s(\xi) = \frac{60e^{\xi}}{(1+e^{\xi})^3}. \tag{3.4}$$

We consider the solution of (3.3) in the form

$$u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta). \tag{3.5}$$

$u(\xi)$ is a solution of (3.3) if one of the following identity holds true:

$$\left(\sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta)\right)^2 - \sum_{m=-\infty}^{+\infty} u_s^2(\xi - 2m\delta)$$

$$= B - A \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \tag{3.6}$$

or

$$2 \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \sum_{j=1}^{+\infty} u_s(\xi - 2(m+j)\delta) = B - A \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta). \tag{3.7}$$

Introducing $w = e^{-2\delta}$ and $z = e^{-\xi}$ this becomes

$$2 \cdot 60^2 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{z^4 w^{2m+j}}{(z+w^m)^3 (z+w^{m+j})^3} = B - A \sum_{m=-\infty}^{+\infty} \frac{z^2 w^m}{(z+w^m)^3}. \tag{3.8}$$

Using the partial fraction decomposition for

$$\frac{z^4 w^{2m+j}}{(z+w^m)^3 (z+w^{m+j})^3}$$

and the identities

$$\frac{w^{kn}}{(z+w^n)^k} = \sum_{i=0}^{k-1} (-1)^i C_{k-1,i} \frac{z^i w^n}{(z+w^n)^{i+1}} \tag{3.9}$$

it can be shown that

$$\begin{aligned}
&\frac{z^4 w^{2m+j}}{(z+w^m)^3 (z+w^{m+j})^3} \\
&\quad = \frac{w^j}{(w^j-1)^3} \frac{z^2 w^m}{(z+w^m)^3} - \frac{w^{2j}}{(w^j-1)^3} \frac{z^2 w^{m+j}}{(z+w^{m+j})^3} \\
&\qquad + \frac{2w^{2j}+w^j}{(w^j-1)^4} \frac{z w^m}{(z+w^m)^2} + \frac{2w^{2j}+w^{3j}}{(w^j-1)^4} \frac{z w^{m+j}}{(z+w^{m+j})^2} \\
&\qquad + \frac{3w^{2j}(1+w^j)}{(w^j-1)^5} \left(\frac{w^m}{(z+w^m)} - \frac{w^{m+j}}{(z+w^{m+j})} \right).
\end{aligned} \tag{3.10}$$

When we sum over m and j the series corresponding to the last term in the (3.10) tend to 0 for $m \to +\infty$, but for $m \to -\infty$ they do not (individually) since the terms of the m-series tend to 1. But this part has the form

$$\begin{aligned}
&\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} a(j)(b(m) - b(m+j)) \\
&\quad = \sum_{m=0}^{+\infty} \sum_{j=1}^{+\infty} a(j) b(m) - \sum_{m=0}^{+\infty} \sum_{j=1}^{+\infty} a(j) b(m+j) \\
&\qquad + \sum_{m<0} \sum_{j=1}^{+\infty} a(j)(b(m)-1) - \sum_{m<0} \sum_{j=1}^{+\infty} a(j)(b(m+j)-1) \\
&\quad = \sum_{j=1}^{+\infty} a(j) \sum_{m=0}^{+\infty} b(m) - \sum_{j=1}^{+\infty} a(j) \sum_{m'=j}^{+\infty} b(m') + \sum_{j=1}^{+\infty} a(j) \sum_{m<0} (b(m)-1) \\
&\qquad - \sum_{j=1}^{+\infty} a(j) \sum_{m'<j}^{+\infty} (b(m')-1) \\
&\quad = \sum_{j=1}^{+\infty} a(j) \sum_{m=0}^{j-1} b(m) - \sum_{j=1}^{+\infty} a(j) \sum_{m'=0}^{j-1} (b(m')-1) = \sum_{j=1}^{+\infty} a(j) j.
\end{aligned} \tag{3.11}$$

By taking the sum in m and j and using (3.11) in (3.10) we obtain that

$$\sum_{m=-\infty}^{+\infty}\sum_{j=1}^{+\infty}\frac{z^4 w^{2m+j}}{(z+w^m)^3(z+w^{m+j})^3}$$

$$=\sum_{j=1}^{+\infty}\frac{w^j-w^{2j}}{(w^j-1)^3}\sum_{m=-\infty}^{+\infty}\frac{z^2w^m}{(z+w^m)^3}$$

$$+\sum_{j=1}^{+\infty}\frac{w^j+4w^{2j}+w^{3j}}{(w^j-1)^4}\sum_{m=-\infty}^{+\infty}\frac{zw^m}{(z+w^m)^2}+\sum_{j=1}^{+\infty}\frac{3w^{2j}(1+w^j)}{(w^j-1)^5}j. \tag{3.12}$$

Since

$$\frac{zw^m}{(z+w^m)^2}=\frac{z^2w^m}{(z+w^m)^3}+\frac{zw^{2m}}{(z+w^m)^3}, \tag{3.13}$$

the left-hand side of (3.8) can be written as

$$2\cdot 60^2\sum_{m=-\infty}^{+\infty}\sum_{j=1}^{+\infty}\frac{z^4 w^{2m+j}}{(z+w^m)^3(z+w^{m+j})^3}$$

$$=2\cdot 60^2\left(\sum_{j=1}^{+\infty}\frac{6w^{2j}}{(1-w^j)^4}\sum_{m=-\infty}^{+\infty}\frac{z^2w^m}{(z+w^m)^3}\right.$$

$$\left.+\sum_{j=1}^{+\infty}\frac{w^j+4w^{2j}+w^{3j}}{(w^j-1)^4}\sum_{m=-\infty}^{+\infty}\frac{zw^{2m}}{(z+w^m)^3}\sum_{j=1}^{+\infty}\frac{3w^{2j}(1+w^j)}{(w^j-1)^5}j\right). \tag{3.14}$$

So instead of identity (3.6) we have

$$\left(\sum_{m=-\infty}^{+\infty}u_s(\xi-2m\delta)\right)^2-\sum_{m=-\infty}^{+\infty}u_s^2(\xi-2m\delta)$$

$$=B-A\sum_{m=-\infty}^{+\infty}u_s(\xi-2m\delta)+\epsilon\sum_{m=-\infty}^{+\infty}u_s(-(\xi-2m\delta)), \tag{3.15}$$

where the constants A, B, and ϵ are defined as infinite series of hyperbolic functions

$$A(\delta) = 90 \sum_{j=1}^{+\infty} \operatorname{csch}^4(\delta j), \tag{3.16}$$

$$B(\delta) = 45/2 \sum_{j=1}^{+\infty} j \frac{\operatorname{csch}^4(\delta j)}{\tanh(\delta j)}, \tag{3.17}$$

$$\epsilon(\delta) = 45 \sum_{j=1}^{+\infty} \operatorname{csch}^4(\delta j) + 30 \sum_{j=1}^{+\infty} \operatorname{csch}^2(\delta j). \tag{3.18}$$

So $u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta)$ satisfies the equation

$$-(6-A)u(\xi) + u^2(\xi) + u'(\xi) + 4u''(\xi) + u'''(\xi) = B + \epsilon u(-\xi). \tag{3.19}$$

For δ large enough the periodic pulse train $u(\xi)$ defined by (3.5) approaches the exact periodic solution of GKSE, since the ϵ given by (3.18) is small.

4. Solitary and approximate periodic solutions for the Kawahara equation

The Kawahara equation describing nonlinear wave processes in a dispersive system [10] has form

$$u_t + \alpha u u_x + \beta u_{xxx} = u_{xxxxx}. \tag{4.1}$$

This equation can be easily normalized by rescaling $u = (2\beta^{3/2}/\alpha)u'$, $x = \sqrt{(1/\beta)}\, x'$, $t = (1/\beta)^{5/2} t'$, so that

$$u_t + 2uu_x + u_{xxx} = u_{xxxxx}. \tag{4.2}$$

For this equation the transform (2.2) is

$$u = \frac{140}{13}(\ln F)'' - 140(\ln F)'''' . \tag{4.3}$$

This transform implemented to the Kawahara equation gives an equation consisting of 59 terms *quintilinear* in F. If one substitutes $F = 1 + \epsilon^{kx + \omega t - x_0}$ into that equation and equates the coefficients of the same powers of e, one gets the solution

$$F = 1 + e^{\pm\sqrt{(1/13)}x \mp (36/169\sqrt{13})t - x_0}. \tag{4.4}$$

So the solitary solution of Kawahara equation (4.2) is

$$\frac{840}{169}\frac{e^{\pm 2\sqrt{(1/13)}x \mp (72/169\sqrt{13})t - x_0}}{\left(1+e^{\sqrt{(1/13)}x - (36/169\sqrt{13})t - x_0}\right)^4}. \tag{4.5}$$

Furthermore, the Kawahara equation written in the traveling wave form

$$u'''' - u'' - u^2 + c_0 u = B \tag{4.6}$$

has a solitary wave solution

$$u(\xi) = \frac{840}{169}\frac{e^{\pm(2/\sqrt{13})\xi}}{\left(1+e^{\pm(1/\sqrt{13})\xi}\right)^4}, \tag{4.7}$$

with $c_0 = 36/169$ and $B = 0$. This solution was repeatedly found in a number of papers [9] in the form of the hyperbolic function

$$u(\xi) = \frac{105}{338}\cosh^{-4}\left(\frac{1}{2}\xi\sqrt{13}\right). \tag{4.8}$$

Next, we look for a periodic solution of (4.6) in the form

$$u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta), \tag{4.9}$$

where

$$u_s(\xi) = \frac{840}{169}\frac{e^{2k\xi}}{(1+e^{k\xi})^4}, \tag{4.10}$$

with $|k| = 1/\sqrt{13}$.

As before we consider the difference

$$\left(\sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta)\right)^2 - \sum_{m=-\infty}^{+\infty} u_s^2(\xi - 2m\delta)$$
$$= 2\sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta)\sum_{j=1}^{+\infty} u_s(\xi - 2(m+j)\delta). \tag{4.11}$$

Introducing $w = e^{-2\delta}$ and $z = e^{-k\xi}$ this becomes

$$2\left(\frac{840}{169}\right)^2 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{z^4 w^{4m+2j}}{(z+w^m)^4 (z+w^{m+j})^4}. \tag{4.12}$$

Using the same arguments as in the previous section, it can be shown that

$$\begin{aligned}
&\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{z^4 w^{4m+2j}}{(z+w^m)^4 (z+w^{m+j})^4} \\
&\quad = -\sum_{j=1}^{+\infty} \frac{4w^{2j}+12w^{3j}+4w^{4j}}{(w^j-1)^6} \sum_{m=-\infty}^{+\infty} \frac{z^3 w^m + z w^{3m}}{(z+w^m)^4} \\
&\qquad -2\sum_{j=1}^{+\infty} \frac{3w^{2j}+14w^{3j}+3w^{4j}}{(w^j-1)^6} \sum_{m=-\infty}^{+\infty} \frac{z^2 w^{2m}}{(z+w^m)^4} \\
&\qquad -\sum_{j=1}^{+\infty} \frac{w^{2j}+9w^{3j}+9w^{4j}+w^{5j}}{(w^j-1)^7} j.
\end{aligned} \tag{4.13}$$

So, $u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta)$ satisfies the equation

$$(c_0 - A)u(\xi) - u^2(\xi) - u''(\xi) + u''''(\xi) = B + \epsilon g(\xi), \tag{4.14}$$

where

$$\begin{aligned}
\epsilon &= \left(\frac{840}{169}\right)^2 \sum_{j=1}^{+\infty} \frac{1}{2}\coth^2(j\delta)\operatorname{csch}^4(j\delta) + \frac{1}{8}\operatorname{csch}^6(j\delta), \\
A &= \left(\frac{840}{169}\right)^2 \sum_{j=1}^{+\infty} \frac{3}{4}\coth^2(j\delta)\operatorname{csch}^4(j\delta) + \frac{1}{2}\operatorname{csch}^6(j\delta), \\
B &= \left(\frac{840}{169}\right)^2 \sum_{j=1}^{+\infty} \frac{1}{8}\coth^3(j\delta)\operatorname{csch}^4(j\delta) j + \frac{3}{8}\coth(j\delta)\operatorname{csch}^6(j\delta) j,
\end{aligned} \tag{4.15}$$

and $g(\xi) = \sum_{m=-\infty}^{+\infty} g_s(\xi - 2m\delta)$ with $g_s(\xi) = (e^{k\xi} + e^{3k\xi})/(1+e^{k\xi})^4$, which is the solitary wave of amplitude 0.125. So the term $\epsilon g(\xi)$ gets exponentially small as δ gets large and $u(\xi)$ approximates the exact solution. The comparison of the approximate periodic solution with the exact one indicates that these two solutions are very close to each other even for not very large values of δ.

5. Exact periodic solutions of the GKSE

The presence of the $u(-\xi)$ term in (3.19) suggests a possible correction to the solitary pulse that can lead to the exact periodic pulse solution of (3.1). We can seek the solution of (3.3) in the form

$$u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta), \tag{5.1}$$

where

$$u_s(\xi) = A\frac{\beta_1 e^{a\xi} + \beta_2 e^{2a\xi}}{(1+e^{a\xi})^3}. \tag{5.2}$$

$u(\xi)$ satisfies (3.3) if we can find constants c, δ, and B such that

$$\sum_{m=-\infty}^{+\infty} (u_s''' + 4u_s'' + u_s' - cu_s) + \left(\sum_{m=-\infty}^{+\infty} u_s\right)^2 = B, \tag{5.3}$$

which can be written as

$$\sum_{m=-\infty}^{+\infty} \left(u_s''' + 4u_s'' + u_s' - cu_s + u_s^2\right)$$
$$+ 2\sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \sum_{j=1}^{+\infty} u_s(\xi - 2(m+j)\delta) = B. \tag{5.4}$$

Using the substitution $z = e^{-a\xi}$ and $w = e^{-2\delta a}$ the last term on the left-hand side of the previous expression can be written as

$$2\sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \sum_{j=1}^{+\infty} u_s(\xi - 2(m+j)\delta)$$
$$= 2A^2 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{\left(\beta_1 w^m z^2 + \beta_2 w^{2m} z\right)\left(\beta_1 w^{m+j} z^2 + \beta_2 w^{2m+2j} z\right)}{(z+w^m)^3 (z+w^{m+j})^3}. \tag{5.5}$$

The partial fraction decomposition along with the principles outlined in previous sections gives the following expression for this term

$$
\begin{aligned}
2A^2\Bigg(& \beta_1^2 \sum_{j=1}^{+\infty} \frac{6w^{2j}}{(1-w^j)^4} \sum_{m=-\infty}^{+\infty} \frac{z^2 w^m}{(z+w^m)^3} \\
& + \beta_1^2 \sum_{j=1}^{+\infty} \frac{w^j+4w^{2j}+w^{3j}}{(1-w^j)^4} \sum_{m=-\infty}^{+\infty} \frac{zw^{2m}}{(z+w^m)^3} - \beta_1^2 \sum_{j=1}^{+\infty} \frac{3w^{2j}(1+w^j)}{(1-w^j)^5} j \\
& - \beta_1\beta_2 \sum_{j=1}^{+\infty} \frac{3(w^j+2w^{2j}+w^{3j})}{(1-w^j)^4} \sum_{m=-\infty}^{+\infty} \frac{z^2 w^m}{(z+w^m)^3} \\
& - \beta_1\beta_2 \sum_{j=1}^{+\infty} \frac{3(w^j+2w^{2j}+w^{3j})}{(1-w^j)^4} \sum_{m=-\infty}^{+\infty} \frac{zw^{2m}}{(z+w^m)^3} \\
& + \beta_1\beta_2 \sum_{j=1}^{+\infty} \frac{w^j+5w^{2j}+5w^{3j}+w^{4j}}{(1-w^j)^5} j \\
& + \beta_2^2 \sum_{j=1}^{+\infty} \frac{w^j+4w^{2j}+w^{3j}}{(1-w^j)^4} \sum_{m=-\infty}^{+\infty} \frac{z^2 w^m}{(z+w^m)^3} \\
& + \beta_2^2 \sum_{j=1}^{+\infty} \frac{6w^{2j}}{(1-w^j)^4} \sum_{m=-\infty}^{+\infty} \frac{zw^{2m}}{(z+w^m)^3} - \beta_2^2 \sum_{j=1}^{+\infty} \frac{3w^{2j}(1+w^j)}{(1-w^j)^5} j \Bigg). \qquad (5.6)
\end{aligned}
$$

So that

$$
\begin{aligned}
& 2 \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \sum_{j=1}^{+\infty} u_s(\xi - 2(m+j)\delta) \\
& \quad = 2A^2 \sum_{m=-\infty}^{+\infty} \frac{P_1(a,\delta) e^{a\xi - 2am\delta} + P_2(a,\delta) e^{2a\xi - 4am\delta}}{(1+e^{a\xi-2am\delta})^3} + 2A^2 C(a,\delta), \qquad (5.7)
\end{aligned}
$$

where we used the notation

$$P_1(a,\delta) = \beta_1^2 r_1 + \beta_1\beta_2 r_2 + \beta_2^2 r_3, \qquad (5.8)$$

$$P_2(a,\delta) = \beta_1^2 g_1 + \beta_1\beta_2 g_2 + \beta_2^2 g_3, \qquad (5.9)$$

$$
\begin{aligned}
C(a,\delta) = & -(\beta_1^2 + \beta_2^2) \sum_{j=1}^{+\infty} \frac{3w^{2j}(1+w^j)}{(1-w^j)^5} j \\
& + \beta_1\beta_2 \sum_{j=1}^{+\infty} \frac{w^j+5w^{2j}+5w^{3j}+w^{4j}}{(1-w^j)^5} j, \qquad (5.10)
\end{aligned}
$$

$$r_1 = g_3 = \sum_{j=1}^{+\infty} \frac{6w^{2j}}{(1-w^j)^4}, \tag{5.11}$$

$$r_2 = g_2 = g_3 - 3g_1 = -\sum_{j=1}^{+\infty} \frac{3(w^j + 2w^{2j} + w^{3j})}{(1-w^j)^4}, \tag{5.12}$$

$$r_3 = g_1 = \sum_{j=1}^{+\infty} \frac{w^j + 4w^{2j} + w^{3j}}{(1-w^j)^4}. \tag{5.13}$$

The substitution of (5.7) into (5.4) indicates that the constant of integration B must be chosen as

$$B = 2A^2 C(a, \delta), \tag{5.14}$$

and then we also get the identity that must be satisfied for some values of parameters a, δ, and c. By equating the coefficients of equal powers of $e^{a\xi - 2ma\delta}$ to zero we get a system of five algebraic equations, compatible if $\beta_1 = 1 + a$ and $\beta_2 = 1 - a$, so that the velocity of propagation c and the amplitude A can be expressed in terms of a, r_1, g_1, g_3:

$$A = 30a^2, \tag{5.15}$$

$$c = \frac{2a^2 - 1 + 11a^4 + 240a^4 g_3}{2a^2}, \tag{5.16}$$

and a must satisfy

$$a^4 - 1 + 240a^4 g_1 = 0. \tag{5.17}$$

The latter can be written as

$$\delta = (a\delta)(1 + 240 g_1)^{1/4}. \tag{5.18}$$

The coefficients g_1 and g_3 depend on a and δ in the following way:

$$g_1 = \sum_{j=1}^{+\infty} \frac{e^{6aj\delta} + 4e^{4aj\delta} + e^{2aj\delta}}{(e^{2aj\delta} - 1)^4}, \tag{5.19}$$

$$g_3 = \sum_{j=1}^{+\infty} \frac{6e^{4aj\delta}}{(e^{2aj\delta} - 1)^4}. \tag{5.20}$$

Table 1

$a\delta$	δ	a	c	A
10	10.00000124	0.999999876	5.999998516	29.999999
8	8.00054016	0.99999325	5.999918976	29.9996
6	6.0022108	0.9996316	5.99558056	29.9779
4	4.078418	0.9807724339	5.7708114825	28.8574
3	3.37840207	0.88799645759	4.7063654137	23.656
2	3.15130168	0.63465837	2.0788004	12.0837
1.5	3.14195585	0.477409635	0.55978631	6.8376
1.3	3.1416462	0.413796534	−0.04841128	5.13683
1	3.141593157	0.318309835	−0.96036354	3.3963
0.1	$\pi+2\times10^{-11}$	0.031831	−3.6960364	0.3039
0.01	$\pi+10^{-12}$	0.0031831	−3.969603	0.000303
0.001	$\pi+10^{-14}$	0.000318	−3.99696	3.039×10^{-6}

Therefore, we can fix the product $a\delta$, calculate δ using (5.18), and find a; then the expressions (5.15) and (5.16) give the corresponding values for the amplitude and velocity. The sketch of the dependence of the left-hand side of (5.17) on a for different values of δ and the form of (5.18) and (5.19) suggest that a increases with δ and approaches 1, so that each function (5.2) in superposition (5.1) approaches the one-soliton solution of (3.7).

Table 1 and Figures 1–3 give the values of a, velocity c, and amplitude A for different values of δ.

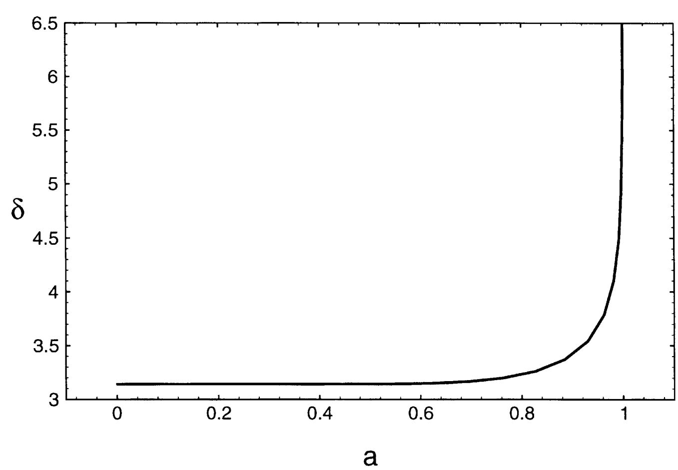

Figure 1. The dependence of the correction of the traveling variable a on the half period δ of the exact periodic solution of the GKSE.

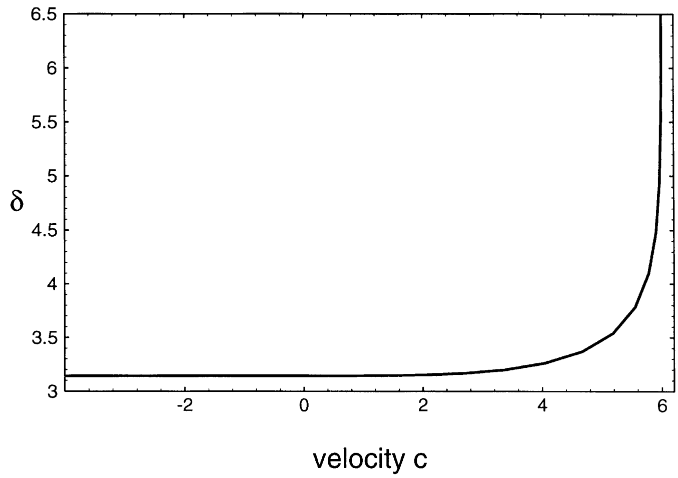

Figure 2. The dependence of the velocity of propagation c on the half period δ of the exact periodic solution of the GKSE.

For the specific value of $\delta_{st} \approx 3.14165$ one of the periodic solutions represents a wave that propagates with zero velocity. This wave has the amplitude $A \approx 5.26340267$ and $a \approx 0.418863648911$. We can see from the results of the calculations that as $a \to 0$, $c \to -4$ and amplitude $A \to 0$.

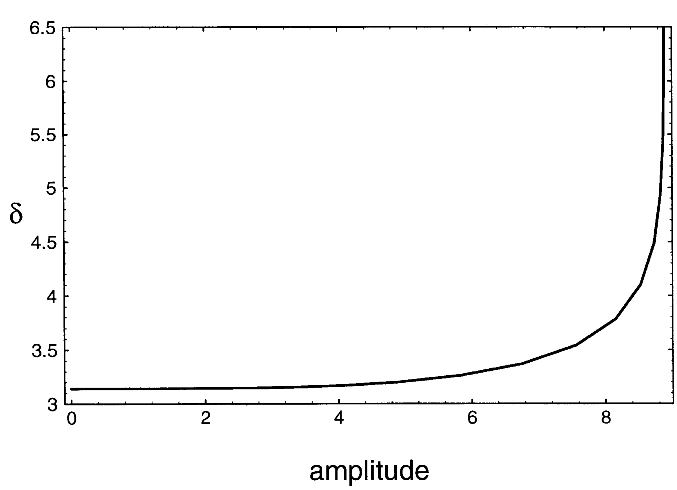

Figure 3. The dependence of the amplitude of the single solitary wave in the superposition on the half period δ of the exact periodic solution of the GKSE.

6. Exact periodic solutions of the Kawahara equation

The traveling wave ODE for the Kawahara equation has form

$$u'''' - u'' - u^2 + c_0 u = B. \tag{6.1}$$

We look for a solution of the form

$$u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta), \tag{6.2}$$

where

$$u_s(\xi) = \frac{\beta_1 e^{a\xi} + \beta_2 e^{2a\xi} + \beta_3 e^{3a\xi}}{(1+e^{a\xi})^4}. \tag{6.3}$$

The periodic pulse train (6.2) satisfies (6.1) if the following identity is true:

$$\sum_{m=-\infty}^{+\infty} \left(u''''_{\,s} - u''_s + c_0 u_s - u_s^2\right)$$
$$- 2\sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \sum_{j=1}^{+\infty} u_s(\xi - 2(m+j)\delta) = B. \tag{6.4}$$

The partial fraction decomposition yields

$$2\sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \sum_{j=1}^{+\infty} u_s(\xi - 2(m+j)\delta)$$
$$= 2\sum_{m=-\infty}^{+\infty}\sum_{j=1}^{+\infty} \frac{\left(\beta_1 w^m z^3 + \beta_2 w^{2m} z^2 + \beta_3 w^{3m} z\right) \times \left(\beta_1 w^{m+j} z^3 + \beta_2 w^{2m+2j} z^2 + \beta_3 w^{3m+2} z\right)}{(z+w^m)^4 (z+w^{m+j})^4}$$
$$= 2\sum_{m=-\infty}^{+\infty} \frac{Q_1(a,\delta) e^{a\xi - 2am\delta} + Q_2(a,\delta) e^{2a\xi - 4am\delta} + Q_3(a,\delta) e^{3a\xi - 6am\delta}}{(1+e^{a\xi - 2am\delta})^4}$$
$$+ 2C(a,\delta), \tag{6.5}$$

with

$$Q_1(a,\delta) = \beta_1^2 r_1 + \beta_2^2 r_2 + \beta_3^2 r_3 + \beta_1\beta_2 r_{12} + \beta_1\beta_3 r_{13} + \beta_2\beta_3 r_{23},$$
$$Q_2(a,\delta) = \beta_1^2 g_1 + \beta_2^2 g_2 + \beta_3^2 g_3 + \beta_1\beta_2 g_{12} + \beta_1\beta_3 g_{13} + \beta_2\beta_3 g_{23}, \tag{6.6}$$
$$Q_3(a,\delta) = \beta_1^2 p_1 + \beta_2^2 p_2 + \beta_3^2 p_3 + \beta_1\beta_2 p_{12} + \beta_1\beta_3 p_{13} + \beta_2\beta_3 p_{23},$$

where we used the notation

$$C(\alpha,\delta) = \sum_{j=1}^{+\infty} j\Big(\big(10(w^{3j}+w^{4j})(\beta_1^2+\beta_3^2)+(w^{2j}+9w^{3j}+9w^{4j}+w^{5j})\beta_2^2$$

$$-4w^{2j}(1+4w^j+4w^{2j}+w^{3j})(\beta_1\beta_2+\beta_2\beta_3)$$

$$-(w^j+9w^{2j}+10w^{3j}+10w^{4j}+9w^{5j}+w^{6j})\beta_1\beta_3\big)\Big)/(1-w^j)^7,$$

$$r_1 = -\sum_{j=1}^{+\infty}\frac{20w^{3j}}{(1-w^j)^6}$$

$$g_1 = \sum_{j=1}^{+\infty}\frac{w^j-6w^{2j}-30w^{3j}-6w^{4j}+w^{5j}}{(1-w^j)^6},$$

$$p_1 = -\sum_{j=1}^{+\infty}\frac{4w^{2j}+12w^{3j}+4w^{4j}}{(1-w^j)^6},$$

and identities

$$r_2 = p_1,\quad g_2 = \tfrac{3}{2}p_1+\tfrac{1}{2}r_1,\quad p_2 = p_1,\quad r_3 = p_1,\quad g_3 = g_1,\quad p_3 = r_1$$

$$r_{12} = -\tfrac{5}{2}p_1+\tfrac{1}{2}r_1,\quad g_{12} = 2r_{12},\quad p_{12} = g_1-4p_1,\quad r_{13} = -4g_1+9p_1+r_1,$$

$$g_{13} = -6g_1+16p_1,\quad p_{13} = r_{13},\quad r_{23} = p_{12},\quad g_{23} = g_{12},\quad p_{23} = r_{12}.$$

The constant of integration B must be chosen as

$$B = 2C(a,\delta).$$

If we substitute (6.5) into (6.4) we get the identity that must be satisfied for some values of parameters a, δ, and c_0. By equating the coefficients of equal powers of $e^{a\xi-2ma\delta}$ to zero we get a system of seven algebraic equations, compatible if $\beta_1=\beta_3=(140a^2/13)(1-13a^2)$ and $\beta_2=(280a^2/13)(1+26a^2)$, so that the velocity of propagation c_0 can be expressed in terms of a, r_1, g_1, p_1:

$$c_0 = \big(-31+4836a^2+145509a^4$$

$$+206518a^6+5110560a^4r_1-2555280a^4g_1+33218640a^6g_1\big)/79092a^2$$

Table 2

δ	a	c_0
18.0277	0.277351	0.213018
14.4208	0.277377	0.213052
10.7456	0.279185	0.215457
9.9198594	0.282262	0.21971
8.0085	0.319056	0.278046
5.452220	0.513552	0.803223
3.51689867	0.853024	3.27584

and a must satisfy

$$\begin{aligned} &-31 + 3549a^4 + 21970a^6 - 1703520a^4p_1 \\ &\quad + 88583040a^6p_1 + 851760a^4g_1 - 11072880a^6g_1 = 0. \end{aligned} \tag{6.7}$$

Or, if we multiply by δ^6, (6.7) can be written as

$$\begin{aligned} &-31\delta^6 + (3549 - 1703520p_1 + 851760g_1)(a\delta)^4\delta^2 \\ &\quad + (21970 + 88583040p_1 - 11072880g_1)(a\delta)^6 = 0, \end{aligned} \tag{6.8}$$

which is a cubic equation in δ^2 for any fixed value of $a\delta$. The numerical analysis of (6.8) shows that (6.8) has a real solution only if $a\delta \geq 2.5551572$. Table 2 and Figures 4–6 give the parameters of the different periodic solutions of (6.1).

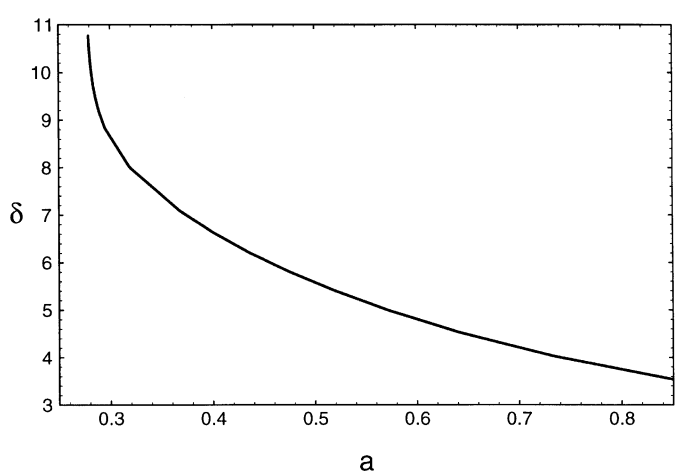

Figure 4. The dependence of the correction of the traveling variable a on the half period δ of the exact periodic solution of the Kawahara equation.

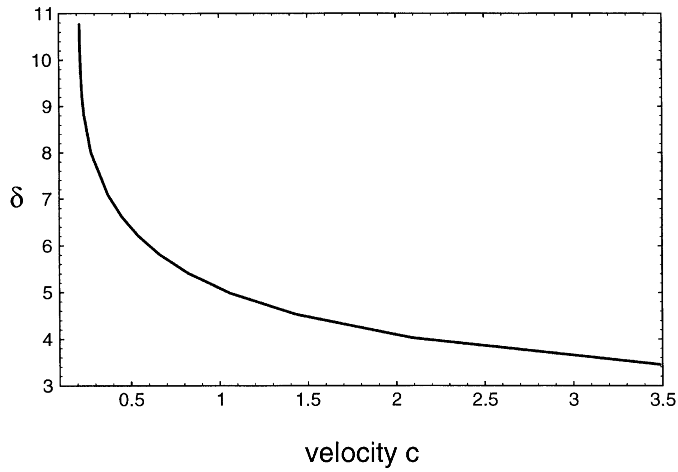

Figure 5. The dependence of the velocity of propagation c on the half period δ of the exact periodic solution of the Kawahara equation.

As one can see, as the half period δ gets larger each function of the superposition (6.2) approaches the one-soliton solution of (6.1) given by (4.9), since c_0 approaches $36/169 = 0.213018$, a approaches $1/\sqrt{13} \approx 0.27735$, β_2 approaches $840/169 = 4.97041$, and $\beta_1 = \beta_3$ approaches zero.

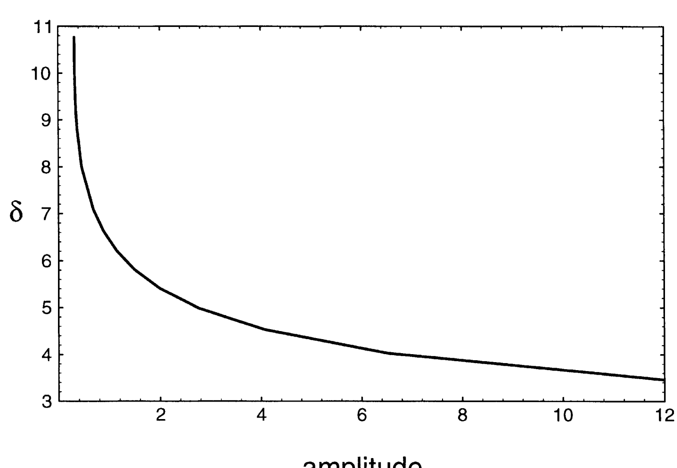

Figure 6. The dependence of the amplitude of the single solitary wave in the superposition on the half period δ of the exact periodic solution of the Kawahara equation.

7. Discussion

The application of the truncated Painlevé expansion to an integrable equation leads to the Hirota bilinear form as a special case of the "multilinear" equation that we obtained for nonintegrable PDEs. In the proposed approach the Painlevé expansion has been used to find only the solitary and traveling wave front solutions. After such solutions are found the periodic pulse train solutions are the result of superposition and partial fraction decompositions. Surprisingly, the Painlevé expansion bears information about such solutions as well. Let us illustrate this idea. The exact periodic solution for the generalized Kuramoto–Sivashinsky equation

$$u_t + 2uu_x + u_{xx} + 4u_{xxx} + u_{xxxx} = 0 \tag{7.1}$$

was found as

$$u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) \tag{7.2}$$

with

$$u_s(\xi) = 30a^2 \frac{(1+a)e^{a\xi} + (1-a)e^{2a\xi}}{(1+e^{a\xi})^3}. \tag{7.3}$$

The Painlevé expansion has the form

$$u = 30(\ln F)_{xxx} + 30(\ln F)_{xx}, \tag{7.4}$$

which coincides with (7.3) if we take $F = 1 + e^{a\xi}$, $\xi = x - ct$. Also, in view of (7.2)

$$F = \prod_{m=0}^{\infty} (1 + e^{a\xi - 2am\delta}) \prod_{m=1}^{\infty} (1 + e^{-a\xi - 2am\delta}) \tag{7.5}$$

is the solution of the *trilinear* equation (2.5).

A similar result is obtained for the Kawahara equation with Painlevé expansion

$$u = \frac{140}{13}((\ln F)_{xx} - 13(\ln F)_{xxxx}), \tag{7.6}$$

which takes the form of the exact function in the superposition $u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta)$,

$$u_s(\xi) = \frac{140a^2}{13} \frac{(1-13a^2)e^{a\xi} + 2(1+26a^2)e^{2a\xi} + (1-13a^2)e^{3a\xi}}{(1+e^{a\xi})^4} \tag{7.7}$$

if we let $F = 1 + e^{a\xi}$.

The second comment that we make is on the representation of the periodic solutions in terms of cnoidal waves. By finding the exact periodic solution of the KdV equation as a superposition of the solitary waves Whitham [3, 4] showed that this result leads to a known representation of elliptic functions in terms of an infinite sum of sech^2. The building blocks of the periodic solution that we were able to find also can be written in terms of sech^2 and their derivatives, therefore, in terms of elliptic functions and their derivatives.

Equation (7.3) can be written as $u_s(\xi) = 30a^2(\mathrm{sech}^2(a\xi/2) + (\mathrm{sech}^2(a\xi/2))')$, so the solution of (7.1) can be represented as

$$u = C_1 + C_2 cn^2(b\xi, m) + C_2 \frac{d}{d\xi} cn^2(b\xi, m). \tag{7.8}$$

Equation (7.6) can be written as $u_s(\xi) = (140a^2/13)(\mathrm{sech}^2(ax/2) - 13(\mathrm{sech}^2(ax/2))'')$, so the solution of the Kawahara equation can be represented by

$$u = C_1 + C_2 cn^2(b\xi, m) + C_3 \frac{d^2}{d\xi^2} cn^2(b\xi, m). \tag{7.9}$$

The solutions of this form were obtained in [9, 12].

8. Conclusion

An algorithm allowing one to find the exact solutions of some nonlinear nonintegrable PDEs by means of the Painlevé expansion even in the case when the leading-order term is noninteger has been introduced. The use of the Painlevé expansion truncated before the "constant term" level leads to a *multilinear* equation as a constraint on the arbitrary function in this expansion. This *multilinear* equation can be solved for some special cases and specific form of the function. The sum of equally spaced solitons is shown to be an approximation of the exact periodic wave train. The partial fraction decomposition allows one to find the relation of the period of the solution and the speed of the propagation. The sum of the equally spaced solitons exactly satisfies the given equations with the correction term. The exact periodic solution can be obtained introducing this correction term into the sum and a correction on the traveling wave variable of the exact solitary wave solution.

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