Vortex Nucleation by Collapsing Bubbles in Bose-Einstein Condensates

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Nucleation of vortex rings accompanies the collapse of ultrasound bubbles in superfluids. Using the Gross-Pitaevskii equation for a uniform condensate we elucidate the various stages of the collapse of a stationary spherically symmetric bubble and establish conditions necessary for vortex nucleation. The minimum radius of the stationary bubble, whose collapse leads to vortex nucleation, was found to be 28 ± 1 healing lengths. The time after which the nucleation becomes possible is determined as a function of the bubble's radius. We show that vortex nucleation takes place in moving bubbles of even smaller radius if the motion makes them sufficiently oblate.

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In this Letter we establish a new mechanism of vortex nucleation in a uniform condensate. Understanding the production of topological defects is one of the most significant questions in condensed matter systems, with applications to vortex nucleation in the early Universe [1], in superfluid helium [2], in atomic Bose-Einstein condensates (BEC) [3–5] and in liquid-crystal systems [6]. Previously, the nucleation of vortices in a uniform condensate has been connected to critical velocities [7–9], instabilities of the initial states [10] or to an energy transfer among the solitary waves [11]. Experiments in superfluid helium have demonstrated the production of quantized vortices and turbulence [12] by the collapse of cavitated bubbles [13] generated by ultrasound in the megahertz frequency range. The aim of this Letter is to analyze theoretically, for the first time, the physics of this process in the context of the Gross-Pitaevskii (GP) equation. The physics of vortex nucleation by collapsing bubbles is quite different from that previously analyzed, as it does not involve critical velocities, previously assumed to be prerequisite to vortex formation. For example, moving positive [8] and negative [9] ions were shown to generate vortex rings on their surface where the speed of sound was exceeded. In classical viscous flows at large Reynolds numbers a shock is formed at the surface of the object where the criticality occurs, but shocks are disallowed in a condensate since they represent a violation of the Landau criterion and a breakdown of superfluidity. When the velocity exceeds the speed of sound, the condensate evades shocks by shedding vortex rings. Our mechanism is also very different from the formation of vortices during a first-order phase transition of an early Universe [1] where vortices form as a result of a collision of at least three bubbles and only if the Higgs phases inside the bubbles have appropriate values. We show that vortices can be formed by collapse of a single stationary or slowly moving bubble. Vortex nucleation by collapsing bubbles could also be studied in the context of (nonuniform) atomic condensates, for which the GP equation provides a quantitative model, thus providing experimentalists with a new mechanism to produce vortices in BEC systems, alongside rotation [3], the decay of solitons [4] and phase imprinting [5]. Moreover, our work illustrates a new aspect of vortex-sound interaction in BEC, a topic which is receiving increased attention [10,11,14].

We consider the collapse of a stationary spherically symmetrical bubble of radius a. The GP equation that governs the evolution of the condensate is written as

$$-2i\frac{\partial \psi}{\partial t} = \nabla^2\psi + (1 - |\psi|^2 - V(\mathbf{x}, t))\psi, \qquad (1)$$

in dimensionless variables such that the unit of length corresponds to the healing length ξ, the speed of sound $c = 1/\sqrt{2}$, and the density at infinity $\rho_\infty = |\psi_\infty|^2 = 1$. To convert the dimensionless units into values applicable to superfluid helium-4, we take the number density as $\rho = 2.18 \times 10^{28}\ \mathrm{m}^{-3}$, the quantum of circulation as $\kappa = h/m = 9.92 \times 10^{-8}\ \mathrm{m}^2\,\mathrm{s}^{-1}$, and the healing length as $\xi = 0.128$ nm. This gives a time unit $2\pi\xi^2/\kappa \sim 1$ ps. For a sodium condensate with $\xi \approx 0.7\ \mu$m and the Bogoliubov speed of sound 2.8 $\mathrm{mm\,s}^{-1}$, the corresponding time unit is about 0.18 ms. $V(\mathbf{x}, t)$ is the potential of interaction between a boson and the bubble. We shall assume that the bubble acts as an infinite potential barrier to the condensate, so that no bosons can be found inside the bubble ($\psi = 0$) before the collapse. This is achieved by setting V to be large inside the bubble and zero outside. Equivalently, we set $V(\mathbf{x}, t) = 0$ and impose the condition that $\psi = 0$ inside the bubble. The spherical symmetry allows us to reduce the problem to one dimension, so that Eq. (1) for $\psi = \psi(r, t)$ and $V(r, t) = 0$ becomes

$$-2i\psi_t = \psi'' + 2\psi'/r + (1 - |\psi|^2)\psi, \qquad (2)$$

where $r^2 = x^2 + y^2 + z^2$. Equation (2) is integrated numerically using fourth-order finite-difference discretization in space and a fourth-order Runge-Kutta method in time. Before collapse the field around the bubble of radius a is stationary, $\psi_t = 0$. The boundary conditions are

$\psi(a, t) = 0$ and $\psi(\infty, t) = 1$. Stationary solutions for various values of the bubble radius a were found by Newton-Raphson iteration. The solutions are $\psi(r) = 0$ if $r \leq a$ and $\psi(r) = R_a(r)$ if $r > a$, where $R_a(r)$ is real. If the radius of the bubble, a, is sufficiently large, then we can set $r = a + \xi$ and to leading order obtain $R''(\xi) + [1 - R(\xi)^2]R(\xi) = 0$, which has the solution that satisfies the boundary conditions $R(\xi) = \tanh(\xi/\sqrt{2})$. Such solutions of the stationary GP equation are used as the initial state $\psi(\mathbf{x}, t = 0)$ in the numerical integration of (2).

The time-dependent evolution of the condensate during and after collapse of the bubble involves several stages. During the first stage dispersive and nonlinear wave trains are generated at the surface of the collapsed bubble. The Fourier components propagate at different velocities generating wave packets moving in opposite directions. This stage of the evolution is characterized by a flux of particles towards the center of the cavity, while oscillations of growing amplitude are being formed on the real and imaginary parts of the wave function and the slope of the steep density front is getting smaller; see Fig. 1. To follow the evolution of the particle flux we calculated the average density inside spheres of different radii centered at the origin. Figure 2 shows these average densities as functions of time for a cavity of initial radius $a = 128$ and for averaging radii of 4, 8, 16, 32, and 64. It is clear from Fig. 2 that there is a definite time $t^* \approx 97$ when the average density reaches its maximum at the same time (and taking the same value) for the two smallest spheres. This moment indicates the start of a qualitatively new stage of the evolution in which there is an outward flux of particles as the condensate that overfilled the cavity begins to expand. The density gradually approaches the uniform state $\rho = 1$.

It is after time t^* that we expect an instability to set in leading to vortex nucleation because outward flow can support outward moving vortex rings. We calculated t^* for various initial radii of the bubble and determined that for $a > 10$ the time of the start of the outflow from the center of the cavity is approximated quite well by the linear function $t^* \sim (1.96 + 0.72)a$.

For sufficiently large a, the minimum distance between the surface on which $\mathrm{Re}\psi = 0$ and the surface on which $\mathrm{Im}\psi = 0$ are of the order of a healing length apart. For instance, for a bubble of initial radius $a = 128$, the nearest surfaces of zero real and imaginary parts of ψ are spheres of radius 16.02 and 17.11 correspondingly. Small perturbations of these zero surfaces can lead to their intersection which forms a closed curve on which $\psi = 0$ and, therefore, represents a closed vortex line. Closed vortex lines emit sound waves as they evolve to form axisymmetric vortex rings.

The smaller the value of a, the larger are the distances between zero surfaces during expansion of the condensate, so it requires a much larger perturbation to bring these surfaces to intersect. In this case, the instability mechanism would be somewhat different and take longer to develop. The radial depletions of the density of the expanding condensate, noticeable in Fig. 1, are unstable to nonspherically symmetric perturbations, similar to the instability of the Kadomtsev-Petviashvili 2D solitons in 3D [10]. Depending on the energy carried by these depletions, they evolve into either vortex solutions or sound waves. From these considerations we expect three possible outcomes after the bubble has collapsed: (i) if the radius of the bubble is smaller than some critical radius a^*, the density depletions generated by the expanding

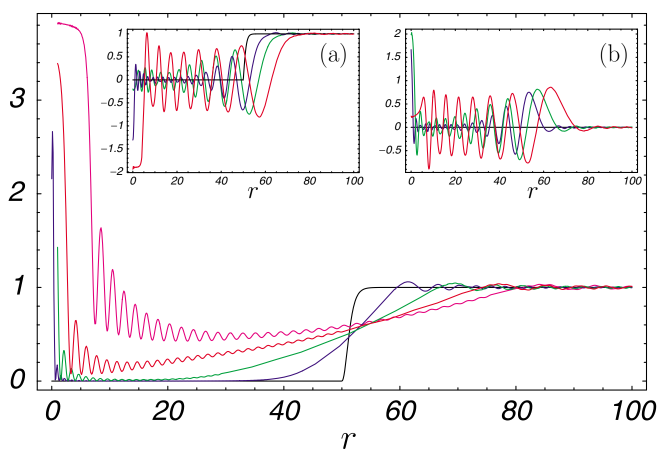

FIG. 1 (color online). The density of the condensate as a function of the distance from the center of the cavity for $t = 0, 10, 20, 30, 40$ after the collapse of the bubble of the radius $a = 50$. The later times correspond to the larger densities at $r = 0$. The insets show the real (a) and imaginary (b) parts of the wave function of the condensate for $t = 0, 10, 20, 30, 40$, where the later times correspond to oscillations of the larger amplitudes.

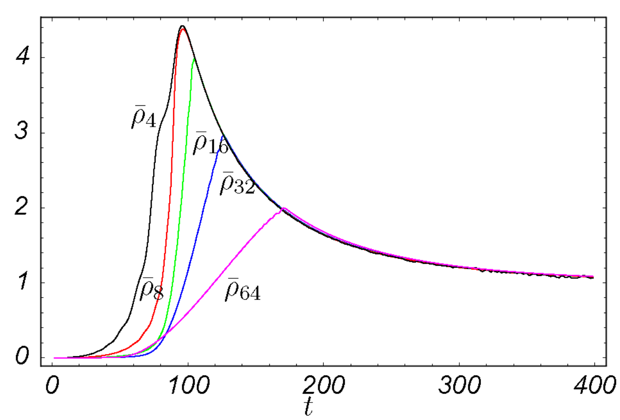

FIG. 2 (color online). The plots of the average density as function of time for various radii of spheres over which the averaging is performed. The initial radius of the cavity in this case is $a = 128$. The radii of averaging spheres are $b = 4, 8, 16, 32, 64$. The average density is calculated as $\bar{\rho}_b = 3\int_0^b \rho(r) r^2 dr/b^3$.

condensate have rather small amplitude that decreases even further as they travel away from the center, quickly becoming sound waves before the instability has time to develop; (ii) after the collapse of a bubble of an intermediate size of radius $a^* < a < \hat{a}$, waves of sufficiently large amplitude are generated and the instability of these waves develops on a time scale inversely proportional to the radius a; (iii) if the radius is sufficiently large, $a > \hat{a}$, the time of the first vortex nucleation is approximately given by the moment of the start of the outward flux of the particles t^* approximated above. As the condensate continues to expand, the instability mechanism described in (ii) is further facilitated by the broken symmetry resulting from the previous nucleation events so that even more vortex rings are nucleated.

To confirm the scenario outlined above we performed fully three-dimensional calculations for cavities of various radii in a computational cube with sides of 300 healing lengths [15]. We determined that there is a critical radius of the bubble for which vortex rings nucleate $a^* \approx 28 \pm 1$. The borderline radius between regime (ii) and (iii) was found as $\hat{a} \approx 55$. Figure 3 shows the density isoplots and the density plots of the portion of the cross section at $z = 0$ at various times after the collapse of a bubble of radius $a = 50$. Notice that the breakdown of the spherical symmetry of the field facilitates the production of even more vortex rings as the condensate expands. At the moment of its birth, each vortex ring has zero radius, as the surfaces of zero real and imaginary parts of ψ touch each other. Such "point defects" gradually evolve into vortex rings of increasingly larger radius, as clearly seen in Fig. 3. The process by which solitary waves evolve into states of a higher energy was elucidated in [11]. A finite-amplitude sound wave that moves behind a vortex ring transfers its energy to it, allowing the vortex ring to grow in size. The radius of the vortex ring stabilizes only when it has travelled sufficiently far from the center of the collapsed bubble, where the flow is almost uniform. The larger the radius of the bubble, the more finite-amplitude sound waves are generated at shorter distances, and the larger the radii of the final rings become.

So far we have considered the collapse of a stationary bubble, where the vortex nucleation is connected to the instabilities developed in the spherically symmetric flow. There are situations when the nucleation is facilitated by an initial lack of symmetry of the flow as in the case of a moving bubble or a bubble in a nonuniform (trapped) condensate. The surrounding condensate exerts a net inward pressure across the surface, which is balanced by the pressure inside the bubble. It was shown in [9], by asymptotic analysis of the GP equation coupled with the equation of motion for the wave function of an electron, that a moving bubble becomes oblate in the direction of its motion. This flattening is created by the difference in

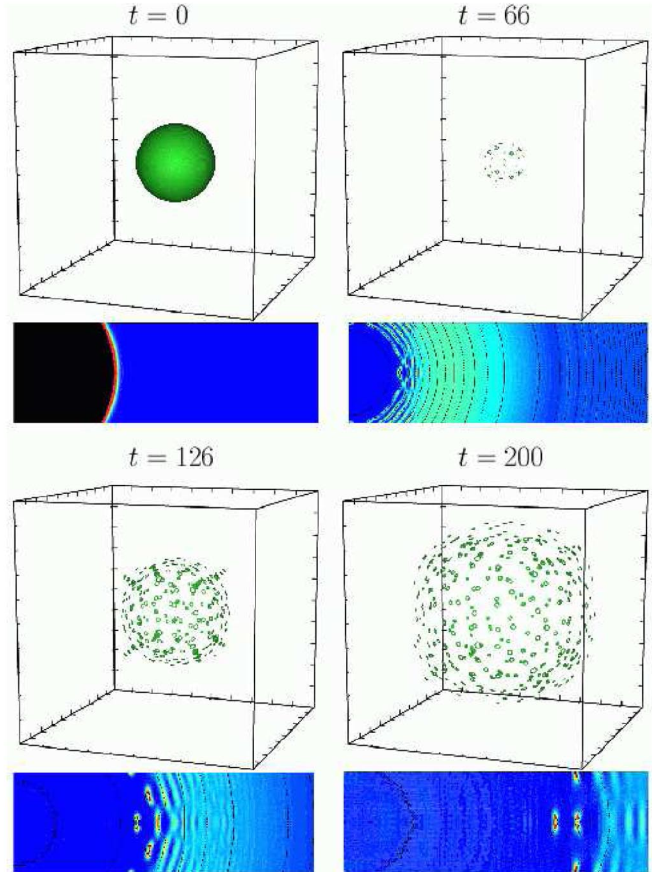

FIG. 3 (color online). Time snapshots of the density isoplots $\rho = 0.2$ after the collapse of the bubble of radius $a = 50$ centered at the origin. The density plots of the cross section of the solution at $z = 0$, $x \in [0150]$, $y \in [-25, 25]$ are given at the bottom of each isoplot to indicate the details of the vortex formation. Black solid and dashed lines show zeros of real and imaginary parts of ψ correspondingly, therefore, their intersection shows the position of topological zeros. Both low and high density regions are shown in darker shades to emphasize intermediate density regions. The side of the computational box is 300 healing lengths and the distance between ticks on the side corresponds to 30 healing lengths. The vortex rings nucleate at about $t \sim 50$ (compared with $t^* \sim 38$).

pressure between the poles and equator associated with the greater condensate velocity at the latter than at the former. How oblate the bubble becomes during its motion will depend on the velocity of the bubble, its surface tension and the pressure inside the bubble. The nonuniformity of the flow created by an oblate moving collapsed bubble leads to vortex ring nucleation for critical bubble sizes much smaller than in the case of a stationary spherically symmetric bubble. Figure 4 shows snapshots in time of the density plots of the cross section of the collapsed bubble that, prior to $t = 0$, was moving with a constant velocity $U = 0.2$ and acquired an oblate form given by $x^2 + \frac{1}{2}(y^2 + z^2) = 100$. As the result of the bubble's collapse four vortex rings of different radii were

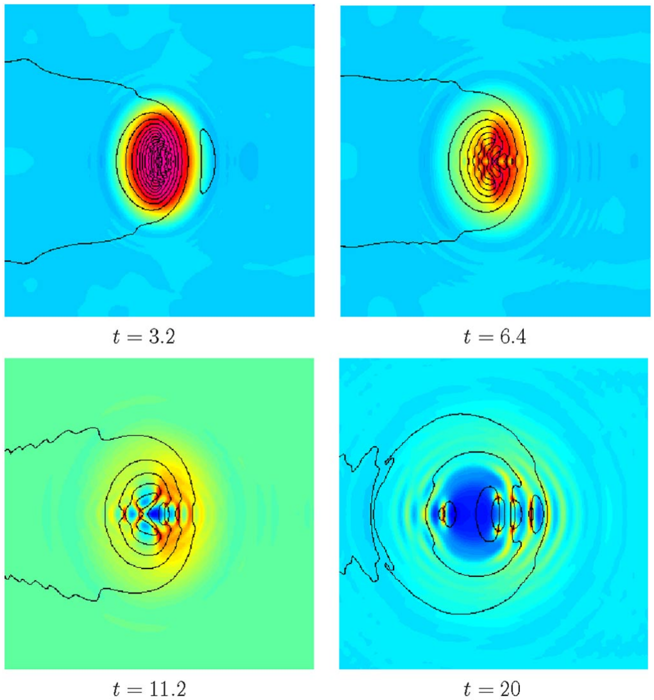

FIG. 4 (color online). The snapshots of the contour plots of the density cross section of a condensate obtained by numerically integrating the GP model (1) for a moving bubble (see text). Black solid lines show zeros of real and imaginary parts of ψ, therefore, their intersection shows the position of topological zeros. Both low and high density regions are shown in darker shades to emphasize intermediate density regions. Only a portion of an actual computational box is shown.

created. Three of them are moving in the same direction as the bubble before the collapse and one vortex ring is moving in the opposite direction. In the model used, the oblateness of a bubble and the velocity of its propagation, U, are independent parameters. A useful quantity that can be determined from further numerical simulations is the critical radius of the bubble as a function of the oblateness and U. This will be further analyzed in our future research.

In summary, we have suggested and studied a new mechanism of vortex nucleation as a result of the collapse of stationary and moving bubbles in the context of the GP equation. This is related to the experiments in superfluid helium in which the cavitated bubbles are generated by ultrasound in the megahertz frequency range. Our findings suggest that a sufficiently large or deformed stationary bubble in a trapped condensate will produce vortex rings as a result of its collapse. This will also be a subject of our future studies.

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