Evolution of vortex rings after the collapse of ultrasound bubbles in superfluids

Natalia G. Berloff[1] and Carlo F. Barenghi[2]

[1] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA*
[2] *School of Mathematics and Statistics, University of Newcastle, Newcastle upon Tyne NE1 7RU*

Collapse of ultrasound bubbles in superfluids leads to the nucleation of vortex rings. Using the Gross-Pitaevskii equation for a uniform condensate we elucidate the various stages of the vortex formation and establish conditions necessary for the vortex nucleation after the collapse of an oblate moving bubble. In the case of a stationary spherically symmetrical bubble we calculate the vortex line length as a function of time after collapse.

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The experimental realisation of Bose-Einstein condensates (BEC) in dilute alkali and hydrogen gases [1] and in a gas of metastable helium [2] has stimulated a great interest in the dynamics of BEC. In the case of a pure condensate, both equilibrium and dynamical properties of the system can be described by the Gross-Pitaevskii (GP) equation [3]. The GP model has been remarkably successful in predicting the condensate shape in an external potential, the dynamics of the expanding condensate cloud, the motion of quantised vortices; it is a popular qualitative model of superfluid helium.

There are several ways how the vortices can be created in condensates: by the process of strongly non-equilibrium condensate formation in a weakly interacting Bose gas [4]; by moving an object with supercritical velocities [5]; by an energy transfer among the solitary waves [6]. Experiments in superfluid helium have demonstrated the production of quantised vortices [7] by the collapse of cavitated bubbles [8] generated by ultrasound in the megahertz frequency range. Recently [9] we have shown that this process can be modelled by the GP equation, therefore, vortex nucleation by collapsing bubbles could also be studied in the context of (non-uniform) atomic condensates (BEC), for which the GP equation provides a quantitative model, thus providing experimentalists with a new mechanism to produce vortices in BEC systems, alongside rotation [10], the decay of solitons [11] and phase imprinting [12].

The goal of this paper is to elucidate the various stages of the vortex formation and establish conditions necessary for the vortex nucleation after the collapse of an oblate moving bubble.

First we consider the collapse of a stationary spherically symmetrical bubble of radius a. The GP equation that governs the evolution of the condensate is written as

$$-2\mathrm{i}\frac{\partial\psi}{\partial t} = \nabla^2\psi + (1 - |\psi|^2 - V(\mathbf{x},t))\psi, \tag{1}$$

in dimensionless variables such that the unit of length corresponds to the healing length ξ, the speed of sound is $c = 1/\sqrt{2}$, and the density at infinity is $\rho_\infty = |\psi_\infty|^2 = 1$. To convert the dimensionless units into values applicable to sodium condensate, we take the healing length as $\xi = 0.7\mu$m and the Bogoliubov speed of sound as 2.8 mm s^{-1} as in NIST experiments [13]. This gives a time unit as 0.18ms. $V(\mathbf{x}, t)$ is the potential of interaction between a boson and a bubble. We will assume that the bubble acts as an infinite potential barrier to the condensate, so that no bosons can be found inside the bubble ($\psi = 0$) before the collapse. Numerically this is achieved by setting V to be large inside the bubble and zero outside.

The stationary solutions for a spherically symmetrical bubble are found by the Newton-Raphson iterations. The solutions are $\psi(r) = 0$ if $r = \sqrt{x^2+y^2+z^2} \le a$ and $\psi(r) = R_a(r)$ if $r > a$, where $R_a(r)$ is real, with the graphs of $R_a(r-a)$ for $a = 1, 2, 10, 30$ given on Figure 1. If the radius of the bubble, a, is sufficiently large, then we can set $r = a + \xi$ and to the leading order get $R''(\xi) + [1 - R(\xi)^2]R(\xi) = 0$ which has the solution, satisfying the boundary conditions, $R(\xi) = \tanh(\xi/\sqrt{2})$. The total energy of the system [14],

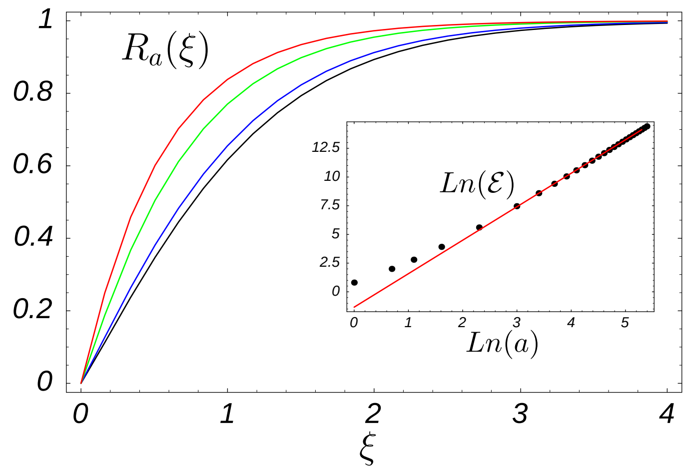
$$\mathcal{E} = \frac{1}{2}\int |\nabla\psi|^2\, dV + \tfrac{1}{4}\int (1 - |\psi|^2)^2\, dV \tag{2}$$

depends on the radius of the bubble, and therefore, on the form of R_a:

$$\mathcal{E} = \frac{\pi a^3}{3} + 2\pi\int_a^\infty [R_a'(r)^2 + \frac{1}{2}(1 - R_a(r)^2)^2] r^2\, dr. \tag{3}$$

The insert on Figure 1 shows the loglog plot of the energy vs radius of the bubble together with the linear fit. For $a > 20$ the energy depends on the radius as $\mathcal{E} \sim 1.65\, a^{2.9}$. From the energy conservation it is clear that after the bubble collapses and the condensate fills the cavity the necessary (but not sufficient) condition for vortex nucleation is that the energy has to be

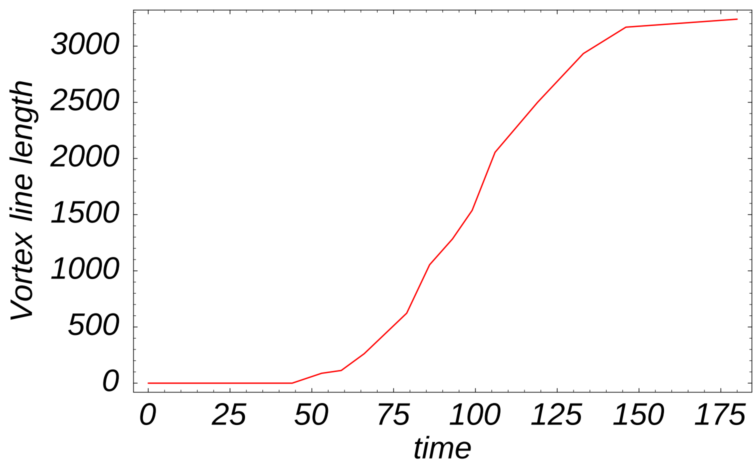
Fig. 1. (colour online) The plot of the amplitude of the solution around the stationary bubbles of radii $a = 1$ (red),2 (green), 10 (blue) and 30 (black) (the smaller a corresponds to a steeper amplitude). The loglog plot of the energies of the solutions with various a are shown on the inset together with the linear fit.



greater than that of one vortex ring. The minimal energy of the vortex solution was found in [14] to be about $\mathcal{E} \sim 55 \pm 1$ which corresponds to the minimum radius of $a = 2.2$ with $\mathcal{E} = 55.7$. As the condensate fills the cavity, most of the energy will be emitted via the sound waves, so the energy of the bubble has to be sufficiently greater than the energy of a single vortex ring to allow for such an emission.

The time-dependent evolution of the condensate after the bubble collapses involves several stages. The overall picture is complicated by a complex interplay between dispersive and nonlinear effects. Dispersive effects become important on the wavelengths of order of the healing length with the group velocity approximately given by $\partial(\sqrt{k^2/2 + k^4/4})/\partial k$ for the perturbation propagating along the uniform state $\psi = 1$ towards infinity and with the group velocity approximated by $\partial(k^2 - 1)/2\partial k$ for the perturbation moving along the uniform state $\psi = 0$ towards the centre of the cavity. The wavetrain generated by the nonlinearity moves slower with the larger wavelengths than the dispersive wavetrain. During the first stage of the evolution dispersive and nonlinear wavetrains are generated at the surface of the collapsed bubble. The Fourier components propagate at different velocities generating wave packets moving in opposite directions. This stage of the evolution is characterised by a flux of particles towards the centre

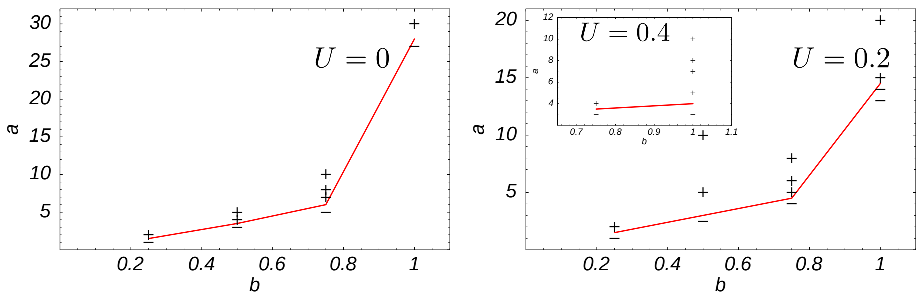
Fig. 2. The vortex line length as a function of time after collapse of the stationary spherically symmetric bubble of radius $a = 50$.



of the cavity, while the oscillations of growing amplitude are being formed on the real and imaginary parts of the wave function and the slope of the steep density front is getting smaller. When the condensate that overfilled the cavity began to expand it becomes possible to nucleate vortices. The radial depletions of the density of the expanding condensate are unstable to non-spherically symmetric perturbations, similar to the instability of the Kadomtsev-Petviashvili 2D solitons in 3D [15]. This instability leads to vortex nucleation. It is important to notice that the nucleated rings grow in size as they move away from the centre of the collapsed bubble, which means that they move to a higher energy state. The process in which solitary waves evolve into states of a higher energy was elucidated in [6]. A finite-amplitude sound wave (rarefaction pulse) that moves behind a vortex ring transfers its energy to it, allowing the vortex ring to grow in size. The radius of the vortex ring stabilises only when it has travelled sufficiently far from the centre of the collapsed bubble, where the flow is almost uniform. Figure 2 plots the total vortex line length as a function of time passed after the bubble of radius $a = 50$ collapsed.

In [9] we demonstrated that the vortex nucleation is facilitated by an initial lack of the symmetry in the flow as in the case of a moving bubble or a bubble in the non-uniform (trapped) condensate. The surrounding condensate exerts a net inward pressure across the surface, which is balanced by the pressure inside the bubble. Moving bubble becomes oblate, with the precise form depending on the velocity, surface tension and pressure inside

Fig. 3. (Colour online) Summary of the numerical integrations of the GP equation (1) performed for the various bubble parameters. Initially the bubble moves with the velocity U and takes on an oblate spheroidal form $x^2 + b(y^2 + z^2) = a^2$. After the collapse the vortex rings either nucleate (the corresponding point on the ab-plot is marked with +sign) or not ($-$). The red (grey) line shows the critical values of the parameters for vortex nucleation.



the bubble. The non-uniformity of the flow in the collapsing oblate bubble leads to vortex ring nucleation for bubble sizes much smaller than in the case of a stationary spherically symmetric bubble. Next, we consider the effect that different parameters, such as the size, oblateness and velocity have on the vortex nucleation. Figure 3 summarises our findings. We considered the bubble initially given by an oblate spheroid $x^2 + b(y^2 + z^2) = a^2$ that moves with velocity $U = 0, 0.2$ and 0.4. As Figure 3 illustrates the critical radius of nucleation decreases with velocity and increases with b.

In summary, we studied collapse of stationary and moving bubbles in the context of the GP equation. Such bubbles can be created in condensates by superimposing several laser beams inside a condensate. Our findings are also related to the experiments in superfluid helium in which the cavitated bubbles are generated by ultrasound in the megahertz frequency range. We calculated the vortex line length as a function of time after collapse and showed that the vortex line length stabilises after a rapid initial growth. We have also established conditions necessary for the vortex nucleation after the collapse of oblate moving bubbles in terms of the bubble velocity, size and oblateness.

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