Dissipative dynamics of superfluid vortices at non-zero temperatures

Natalia G. Berloff and Anthony J. Youd
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Cambridge, CB3 0WA

We consider the evolution and dissipation of vortex rings in a condensate at non-zero temperatures, in the context of the classical field approximation, based on the defocusing nonlinear Schrödinger equation. The temperature in such a system is fully determined by the total number density and the number density of the condensate. The collisions with non-condensed particles reduce the radius of a vortex ring until it competely disappears. We obtain a universal decay law for a vortex line length and relate it to mutual friction coefficients in the fundamental equation of vortex motion in superfluids.

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The processes of self-organization, formation of large-scale coherent localized structures and interactions of these structures with small-scale fluctuations are at the heart of nonlinear sciences, ranging from classical turbulence, superfluids, ultracold gases and Bose–Einstein Condensates (BECs), to the formation of the early Universe. The key to our understanding of turbulence is to elucidate physics of interactions between large scales (eg. large eddies) and small scales (eg. turbulent fluctuations), and to develop mathematical models that account for the effects of small scales without actually solving for them. According to the Landau description, superfluid ^{4}He consists of the ground state and the excitations — quasiparticles drifting on top of the ground state. In the language of relativistic quantum fields this corresponds to vacuum and matter (eg. gravity waves interacting with a vacuum). Indeed, there is a close relationship between superfluid hydrodynamics and quantum gravity, so that at some level of hierarchy of parameters the interactions of the quantum vacuum and matter can be described by the defocusing nonlinear Schrödinger (NLS) equation [1]. The dynamics of Bose condensates depends on the energy exchange between the condensed and non-condensed parts of the gas. Again, the NLS equation (reformulated as the Gross–Pitaevskii (GP) equation [2]) describes equilibrium and dynamical properties of BEC as well as the formation of BEC from a strongly degenerate gas of weakly interacting bosons [3, 4]. The formation of the large-scale coherent localized ground state (condensate) from a non-equilibrium initial state has been studied in a number of papers addressing different stages of the formation: weak turbulence [5, 6], strong turbulence in the long-wavelength region of energy space [7], and finally, the formation of a genuine condensate [8, 9]. The related question about the effect of finite temperature on the BEC dynamics has also been addressed recently [10].

The problem of a vortex tangle interacting with the normal fluid (thermal cloud) is the key question in superfluid turbulence. The Landau two-fluid theory of superfluidity pre-dated the discovery of quantised vortex lines and therefore omitted significant dynamical effects. This was remedied— in the limit in which the mean spacing between the vortex lines is small compared with any other length scale of interest — by HVBK theory [11, 12]. In this limit, the superfluid vorticity is treated as a continuum, but the discrete nature of the vorticity gives rise to an extra force on the superfluid component, arising from the tension in the vortex lines. This term is absent from the classical Euler equation of motion for an inviscid fluid. The vortex lines also create a force of mutual friction between superfluid and normal fluid in addition to the mutual friction included by Landau in his equations, and represents the effects of collisions of the quasiparticles with the vortex cores. Such forces were introduced into the Landau model in an *ad hoc* way. This Letter is the first attempt to study the effect of these collisions quantitatively: we shall find *the vortex line decay law at non-zero temperature in the context of the defocusing NLS equation.* The NLS equation is a good starting point, as the non-dissipative Landau two-fluid model can be obtained from the equations of conservation of mass and momentum for a one-component barotropic fluid using a general expression for the internal energy functional of the density [13]. Through the Madelung transformation the NLS equation can be written in that form. Analogously, the transport coefficients in the Landau model have been obtained directly from the NLS equation by following the Chapman–Engskog expansion [14]. Note that the separation of scales needed to carry out the derivation of the Landau two-fluid model from the NLS equation does not allow the inclusion of vortices as part of the ground state. It is natural, therefore, to attempt to derive the corresponding effects of the interactions of vortices with the quasiparticles directly from the NLS equation.

We consider the normalised defocusing NLS equation for the complex function ψ [2]:

$$i\partial_t\psi = -\nabla^2\psi + |\psi|^2\psi. \qquad (1)$$

The dynamics conserves the total number of particles $N = \int |\psi|^2 d\mathbf{x}$, and the total energy $E = \int \left(|\nabla\psi|^2 + \frac{1}{2}|\psi|^4\right) d\mathbf{x}$. We consider the uniform discrete system of volume $V = \mathcal{N}^3$, which is a periodic box on a computational grid with 128^3 discrete points.

Our goal is to determine the universal decay law for the vortex line density in the entire range of temperatures from 0 to the critical temperature of condensation, T_λ. Our approach consists of three essential steps. We aimed to: (1) achieve the thermal equilibrium state for the given number of particles and given energy, starting from a non-equilibrium stochastic initial condition for the wavefunction ψ; (2) introduce a vortex ring into this state and follow its decay via interactions with non-condensed quasiparticles; (3) relate the decay rate to the temperature at equilibrium, where we derive the expression for the relative temperature, T/T_λ, as a function of the total number density, $\rho = N/V$, and the number density of the condensate, ρ_0.

We performed large scale numerical simulations of Eq. (1) starting from a strongly non-equilibrium initial condition[7], where the phases of the complex Fourier amplitudes $a_{\mathbf{k}}(t) = \int \psi(\mathbf{x},t)e^{-i\mathbf{p}\cdot\mathbf{x}}\, d\mathbf{x}$ are distributed randomly at $t = 0$. Here the momentum $\mathbf{p}$ takes quantised values $\mathbf{p} = (2\pi/\mathcal{N})\mathbf{n}$ with $\mathbf{n} = (0,0,0), (\pm 1,0,0), \cdots$. The time evolution consists of thermal equilibrium with a quasiparticle cascade from high energies to low energies in the wave number space until the thermadynamical equilibrium is reached with some portion ($\rho_0 \equiv |a_{\mathbf{0}}|^2/V$) of particles occupying the zero momentum state (genuine condensate) and the rest of the non-condensed particles being distributed according to the Rayleigh–Jeans equilibrium distribution [16], modified by the presence of nonlinear interactions with the condensate [9]:

$$|a^{eq}_{\mathbf{p}\neq\mathbf{0}}|^2 = \frac{T}{\omega_B(p)}, \tag{2}$$

where T is the temperature and $\omega_B(p)$ is the Bogoliubov dispersion relation (see below). An ultraviolet cutoff for this distribution appears naturally through the spatial discretization of the NLS equation. The numerical scheme consists of fourth-order finite difference discretization in space and fourth-order Runge–Kutta in time, so it is globally fourth-order accurate. This scheme corresponds to the Hamiltonian system in the discrete variables ψ_{jkn}, such that $i\dot{\psi}_{jkn} = \partial H/\partial\psi^*_{jkn}$, for $j,k,n = 1,...,\mathcal{N}$ and where $H = \sum_{jkn}\left(\psi^*_{jkn}[\frac{1}{12}\Psi_2 - \frac{4}{3}\Psi_1 + \frac{15}{2}\psi_{jkn}] + \frac{1}{2}|\psi_{jkn}|^4\right)$ with $\Psi_2 = \psi_{j+2,k,n} + \psi_{j-2,k,n} + \psi_{j,k+2,n} + \psi_{j,k-2,n} + \psi_{j,k,n+2} + \psi_{j,k,n-2}$ and $\Psi_1 = \psi_{j+1,k,n} + \psi_{j-1,k,n} + \psi_{j,k+1,n} + \psi_{j,k-1,n} + \psi_{j,k,n+1} + \psi_{j,k,n-1}$.

The thermodynamic description of the condensation process has been obtained in [9] by adapting the Bogoliubov theory of a weakly interacting Bose gas [17] to the classical system (1). We follow the same basic idea to derive expressions for the energy and non-condensed number density from the discretised energy, H, written in terms of the Fourier amplitudes $a_{\mathbf{p}}$ as

$$H = \sum_{\mathbf{p}} K_2(p)a^*_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2V}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} a^*_{\mathbf{p}_1}a^*_{\mathbf{p}_2}a_{\mathbf{p}_3}a_{\mathbf{p}_4}\delta_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4}, \tag{3}$$

where $\delta_{\mathbf{p}}$ is the Kronecker delta symbol and

$$K_2(p) = \frac{2}{3}\sum_{i=1}^{3}\sin^2(p_i/2)(7-\cos(p_i)). \tag{4}$$

The Bogoliubov transformation $b_{\mathbf{p}} = u_{\mathbf{p}}a_{\mathbf{p}} - v_{\mathbf{p}}a^*_{-\mathbf{p}}$, such that $u_{\mathbf{p}} = 1/\sqrt{1-Q^2_{\mathbf{p}}}$ and $v_{\mathbf{p}} = Q_{\mathbf{p}}/\sqrt{1-Q^2_{\mathbf{p}}}$ with $Q_{\mathbf{p}} = [-K_2 - 2\rho_0 + \omega_B(p)]/\rho_0$ diagonalises the term in (3), which is quadratic in a_0, to $\sum_{\mathbf{p}}{}'\omega_B(p)\, b^*_{\mathbf{p}}b_{\mathbf{p}}$, where $\sum'_{\mathbf{p}}$ excludes the $\mathbf{p} = \mathbf{0}$ mode. Here $\omega_B(p) = \sqrt{K_2^2 + 2\rho_0 K_2}$ is the Bogoliubov-type dispersion relation.

Using the equilibrium distribution of the non-condensed particles (2) (here we neglect the effects of the anisotropy introduced by our finite-differences scheme) the non-condensed number density can then be expressed in terms of the basis used in this diagonalisation as

$$\rho - \rho_0 = \frac{T}{V}\sum_{\mathbf{p}}{}'\frac{K_2(p)+\rho_0}{\omega^2_B(p)}. \tag{5}$$

The discretised energy density H/V in the new basis takes the form

$$\frac{H}{V} = \frac{1}{2}\left[\rho^2 + (\rho-\rho_0)^2\right] + \frac{T}{V}\sum_{\mathbf{p}}{}'1. \tag{6}$$

The Eqs. (5)–(6) are analogous to Eqs. (8)–(9) of [9] but modified for the discrete Hamiltonian, H. Given the energy density, H/V, and the total number density, ρ, one can determine the temperature, T, at equilibrium and the number density of the condensed particles, ρ_0, from Eqs. (5) and (6). The condensate fraction ρ_0/ρ as a function of the energy density H/V is shown in FIG.1. This figure can be compared with FIG.2 of [9] for the spectral representation of the total energy. The analytical formulae (5)–(6) predict the subcritical behaviour of condensation, whereas the numerics does not support this conclusion, as shown in the insert of FIG.1. We use a linear approximation for small ρ_0 to determine the critical maximum energy for condensation as shown in the insert. This energy is then used to determine the critical temperature for condensation T_λ ($= T$ for $\min H/V$ for which $\rho_0 = 0$) from (5)–(6). We found a phenomenological formula that determines T/T_λ as a function of ρ_0 and ρ as

$$\frac{T}{T_\lambda} = 1 - \left(1 - \alpha\sqrt{\rho}\right)\frac{\rho_0}{\rho} - \alpha\sqrt{\rho}\left(\frac{\rho_0}{\rho}\right)^2, \tag{7}$$

where α is the only fitting parameter that we found as $\alpha = 0.227538$. The insert in FIG.1 shows the graph of

FIG. 1: (colour online) Condensate fraction, ρ_0/ρ, as a function of the energy density as obtained from the numerical simulations (points) and from the analytical expressions (5)–(6) (solid line). The inserts show (a) the plot of T/T_λ as a function of ρ_0/ρ, obtained using Eqs. (5)–(6) or Eq. (7) and (b) subcritical condensation predicted by Eqs. (5)–(6) (black line), the linear approximation used to obtain the critical temperature of condensation (gray (red) line), and numerical calculations (blue dots). The total number density is $\rho = 1/2$.

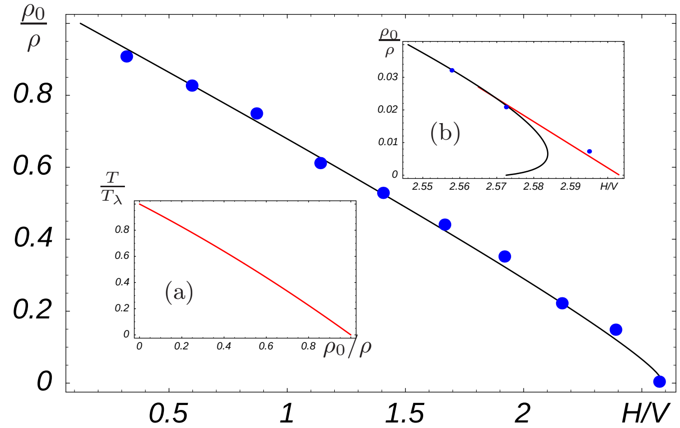
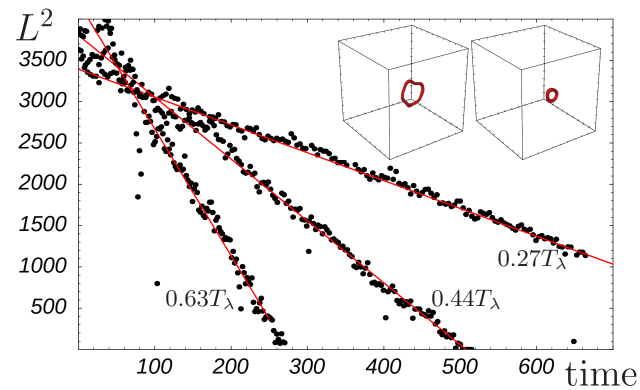


FIG. 2: (colour online) The decay of the square of the vortex line length as a function of time for various T indicated next to the graphs. The fit to the linear function is shown by the gray (red) lines. The inserts show isosurface plots of the vortex line (for filtered fields ψ; see text) for $T = 0.52T_\lambda$ at time=130 (left) and time=1300 (right); between these two times the vortex line length is reduced by a factor of 2. The perturbations to the vortex line due to collisions with non-condensed particles are clearly seen on the left insert. These collisions generate Kelvin waves that also radiate energy to sound. The total number density is $\rho = 1/2$.



T/T_λ as a function of ρ_0/ρ for $\rho = 1/2$. Eq. (7) gives an excellent fit to the values computed from (5)–(6) across all the values of ρ_0 and ρ.

In order to analyze the decay of the vortex line length at non-zero temperatures, we insert a vortex ring into a state of thermal equilibrium and follow its decay due to the interactions with the non-condensed particles. The condensate healing length, which determines the size of the vortex core, is calculated based on the density of the condensate, and in our non-dimensional units is $\xi = 1/\sqrt{\rho_0}$. In healing lengths, the radius of the ring is set to $R_0 = 10$. The new initial state is $\psi_v(t = 0) = \psi_{\rm eq} * \psi_{\rm vortex}$, where $\psi_{\rm eq}$ is the equilibrium state and $\psi_{\rm vortex}$ is a wavefunction of the vortex ring. The vortex line length, L, is calculated as a function of time with high frequencies being filtered out from the field ψ, according to $\tilde{a}_{\mathbf{p}} = a_{\mathbf{p}} * \max(\sqrt{1-p^2/p_c^2}, 0)$, where the cut-off wavenumber is chosen as $p_c = 10(2\pi/\mathcal{N})$. The first important conclusion of our numerical simulations is that at all temperatures, the square of the vortex line length decays linearly with time,

$$\frac{dL^2}{dt} = -\gamma(\rho, T/T_\lambda), \tag{8}$$

where γ does not depend on t. FIG. 2 shows this dependence for various temperatures. The actual isosurfaces of the decaying vortex line are shown in the inserts.

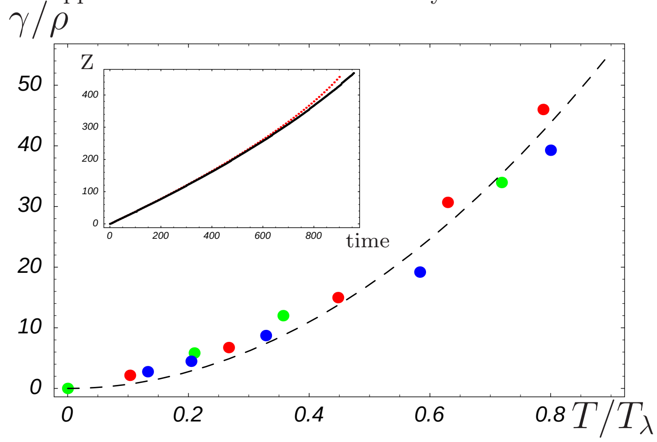
This result agrees with predictions of the HVBK theory for superfluid helium [11] according to which the fundamental equation of the motion of a vortex line, $\mathbf{v_L}$, is given by (see also page 90, Eq. (3.17) of [18])

$$\mathbf{v_L} = \mathbf{v_{sl}} + \alpha \mathbf{s}' \times (\mathbf{v_n} - \mathbf{v_{sl}}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v_n} - \mathbf{v_{sl}})], \tag{9}$$

where $\mathbf{v_{sl}}$ is the local superfluid velocity that consists of the ambient superfluid flow velocity and the self-induced vortex velocity $\mathbf{u_i}$, $\mathbf{v_n}$ is the normal fluid velocity, $\mathbf{s}$ is a position vector of a point on the vortex and $\mathbf{s}'$ is the unit tangent at that point. Mutual friction parameters α and α' are *ad hoc* coefficients in the HVBK theory that are functions of ρ_n, ρ, and T only. Eq. (9) is a general and universal equation used to follow the evolution of three-dimensional vortex motion in an arbitrary flow. When formulated for a single vortex ring Eq. (9) reads $dR/dt = -\alpha u_i$, where $u_i = \kappa[\log(8R/\xi) - \delta + 1]/(4\pi R)$ and δ is the vortex core parameter. For the GP vortices $\delta \approx 0.38$ [2]. In dimensionless units used in our paper $u_i = [\log(8R) - \delta + 1]/R$. After integration of the equation for $\dot{R}$ we get $\alpha t = (R_0^2 - R^2)/[2(\log(8\widehat{R}) + \delta - 1)]$, where $\widehat{R}$ is the mean radius of the ring. When this is compared with (8) we get the following relationship between γ and α: $\gamma = 8\pi^2(\log(8\widehat{R}) + \delta - 1)\alpha$. From our numerics we obtained a general result valid across all ranges of temperatures and total densities: $\gamma \approx K\rho(T/T_\lambda)^2$, where $K \approx 68$, see FIG.3. Note that for a GP condensate $T/T_\lambda \approx \rho_n/\rho$ to the first order (see insert (a) of FIG.1), so alternatively, we can write $\gamma \approx K_1\rho_n(T/T_\lambda)$ Thus, we found that the mutual friction coefficient in condensate superfluids is given by $\alpha \approx K_2\rho_n(T/T_\lambda)$.

The existence of the transverse force on superfluid vortices which is parametrised by α' has been a subject of

FIG. 3: (Color online) Values of γ/ρ as a function of temperature T/T_λ for various values of the total number density ρ depicted in various shades of gray (in various colours): $\rho = 1/2$ (dark (red)), $\rho = 1/4$ (light (green)) and $\rho = 3/4$ (medium (blue)). The plot of the quadratic fit $\gamma/\rho = 68(T/T_\lambda)^2$ is given by the dashed line. The relative temperature is calculated using Eq. (7). The insert shows the distance travelled by a vortex ring as a function of time for $T/T_\lambda = 0.27$ (red dots – distances calculated using $dz/dt = u_i$, black line using numerics). Curves depart when the vortex ring becomes small in radius and the analytical formula is no longer accurate approximation of the vortex velocity.



much debate in mid-1990s, when calculations of the classical Magnus force applied to superfluid vortices have been offered and argued about [19]. The criticism is based on the observation that the classical hydrodynamic equations are inapplicable in the vortex core. Whether or not the details of the non-classical vortex dynamics are crucial to the existence of the transverse force is still an open question. The estimate of α' can be obtained from our numerical procedure as following. Eq. (9) written for a distance travelled by a single vortex ring takes form (see Eq. (3.53) on page 107 of [18]) $dz/dt = (1-\alpha')u_i$. We compared the distances travelled by a vortex ring at various temperatures obtained numerically with the distances travelled by a vortex ring in the absence of the transverse force according to the analytical formula $dz/dt = u_i$, where $u_i = u_i(R(t))$ and $R(t)$ varies with time according to (8). The insert of FIG.3 shows these distances for $T/T_\lambda = 0.27$. Our calculations fail to detect any significant presence of the transverse force for any temperature considered: the deviation from the analytical curve is insignificant within the accuracy of (8). We plan to perform a more thorough analytical and numerical study of transverse force from a single phonon acting on a single vortex in context of the GP model in future.

In summary, we considered the effect of temperature on the decay of vortex line length via interactions with non-condensed particles in the context of the defocusing NLS equation. We related the obtained decay law to the mutual friction coefficients in the HVBK theory. It has been suggested that the emission of sound by vortex reconnections and vortex motion is the only active dissipation mechanism responsible for the decay of superfluid turbulence. The decay of superfluid turbulence via Kelvin wave radiation and vortex reconnections was studied in the framework of the GP equation [20] at near zero temperature, via collision of two vortex rings, and confirmed that in the Kelvin wave cascade, where energy is transferred to much shorter wavelengths with a cut-off below a critical wavelength, the vortex line density can be described by the famous Vinen equation [21] $d(L/V)/dt = -\chi(L/V)^2$. It has also been shown [22] that the presence of localized finite amplitude sound waves greatly enhances the dissipation of the vortex tangle, essentially changing the decay law to exponential decay. This Letter complements the existing Kelvin wave cascade scenario by considering an opposite limit when there are no reconnections, and the decay mechanism depends only on the energy exchange with non-condensed particles. This mechanism exceeds the energy transfer via the Kelvin wave cascade.

Finally, following a referee suggestion, we would like to emphasise that the effects of finite temperature on a condensate and on the dynamics of coherent structures in BECs (such as matter-wave solitons and vortices) have been extensively studied recently (see e.g. [23]). These studies couple the GP equation (which now describes only the condensate part) to an equation for the thermal cloud (a semiclassical kinetic equation, the Bogoliubov-de Gennes equation, etc.). Our approach is fundamentally different as we use fully classical and self-consistent treatment of the interactions in the context of a single NLS equation with the wavefunction ψ describing both condensed and non-condensed parts of a superfluid. The shortcomings of our approach when applied to a real bosonic system are in inability of the NLS equation to describe any effects coming from highly energetic but scarcely occupied modes and in the replacement of the Bose distribution of the population of excited modes by Eq. (2). To which extent these shortcomings change the *quantitative* characteristics of interactions in a real bosonic system remains the subject of further studies.

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