Nonlinear Wave Interactions in Nonlinear Nonintegrable Systems

By Natalia G. Berloff and Louis N. Howard

The main goal of this article is to understand the qualitative appearances of regular arrays of pulses that come up in nonintegrable systems in a variety of contexts, particularly in fluid dynamics. It is shown that even nonintegrable systems have a kind of particle dynamics made up of solitary waves. But the interaction of these solitary waves is not absolutely "clean" as in the case of the KdV and other integrable equations.

1. Introduction

The main advances in the theory of integrable equations are due to the fact that such nonlinear problems have a simple underlying structure and the general solution of an appropriate initial value problem can be obtained by the inverse scattering transform. To describe the dynamics of interacting localized structures the particle approach may also be used, and then we are primarily interested in special solutions, such as solitons, rather than in the general solution of an initial value problem. But then the description of the dynamics in terms of the solitons has some linear properties. Whitham showed for the Korteweg–de Vries (KdV) and some other equations [1] that an infinite superposition of the solitons in a periodic pulse train is also a solution, and this fact can be viewed as another instance of "clean interaction" of solitons.

Address for correspondence: Professor Louis N. Howard, Florida State University, Department of Mathematics, Tallahassee, FL 32306-4510.

Published by Blackwell Publishers, 350 Main Street, Malden, MA 02148, USA, and 108 Cowley Road, Oxford, OX4 1JF, UK.

Some nonintegrable equations may also have a simple underlying structure that allows one to treat them in a way similar to integrable equations. For these equations the inverse scattering transform method fails, since its implementation requires some quite restrictive analytical properties. Then we may abandon the search for a general solution and concentrate on the special solutions instead. In our previous article [2] we introduced the method of constructing the periodic pulse train solution as an infinite superposition of the solitary waves. The form of such periodic wave trains can be obtained in two seemingly different ways. The first presumes the knowledge of an exact solitary wave solution, which often can be expressed in terms of hyperbolic functions, of the given equation. When we substitute an infinite superposition of such solutions into the equation and use a partial fraction decomposition in exponentials (see [2] for details) we obtain some "extra" terms that prevent such superposition to satisfy the equation exactly. It was shown [2] that by taking a linear combination of the solitary wave and such an extra term, and by introducing a scaling parameter on the traveling wave coordinate, we can create a periodic pulse train that satisfies the equation exactly. It was noted that the same periodic pulse train can be obtained by taking the building block of the superposition in the form suggested by a formal Painlevé expansion even if the equation does not possess a Painlevé property.

Here we concentrate on two aspects of pulse interactions. First, we extend this approach to equations that are not Galilean invariant and do not have a solitary wave solution; nevertheless the periodic solution still can be found. Second, we illustrate how this method can be used to analyze the multiparticle solutions.

This article is organized as follows. In Section 2 we discuss some results on the periodic solutions of the generalized Kuramoto−Sivashinsky equation from the point of view of interaction between the solitary waves. Section 3 gives an example of a system for which no solitary solution was found, but the periodic pulse train solution can be constructed for some spacing between pulses. In Section 4 we construct the approximation for the two-particle solution for the generalized Kuramoto−Sivashinsky equation. The comparison of this approach with the method suggested by Kudryashov for finding the cnoidal wave solutions is given in Section 5. Conclusions close the article in Section 6.

2. "Almost clean" interactions of solitary waves

The generalized Kuramoto−Sivashinsky equation,

$$u_t + 2uu_x + u_{xx} + \sigma u_{xxx} + u_{xxxx} = 0, \tag{2.1}$$

describes waves in a vertical and inclined falling film [4], in liquid films that are subjected to interfacial stress from adjacent gas flow, interfacial instability between two cocurrent viscous fluids, unstable drift waves in plasma, and phase evolution for the complex Ginzburg–Landau equation. The Painlevé transform for this equation was found by Kudryashov [5] and, omitting the "constant level" term, is given by

$$u = 30(\ln F)_{xxx} + \frac{15}{2}\sigma(\ln F)_{xx} + \frac{15}{152}(16-\sigma^2)(\ln F)_x. \tag{2.2}$$

Following the discussions given in [2] we take the specific form of the singularity manifold $F = 1 + e^{a(x-ct)}$. The Painlevé expansion takes the form

$$\begin{aligned} u(x,t) = & \frac{30a^3 e^{a(x-ct)}(1-e^{a(x-ct)})}{(1+e^{a(x-ct)})^3} \\ & + \frac{15a^2 e^{a(x-ct)}\sigma}{2(1+e^{a(x-ct)})^2} + \frac{15ae^{a(x-ct)}(16-\sigma^2)}{152(1+e^{a(x-ct)})}. \end{aligned} \tag{2.3}$$

The first and second terms of this expression have the form of solitary waves that decay exponentially fast as $x \to \pm\infty$. The last term is a traveling front and as $x \to +\infty$ approaches a nonzero constant. We can get rid of the last term by setting $\sigma = 4$

$$u_s(x-ct) = 30a^2 \frac{(1+a)e^{a(x-ct)} + (1-a)e^{2a(x-ct)}}{(1+e^{a(x-ct)})^3}. \tag{2.4}$$

For the traveling wave equation $(u(x,t) = u(x-ct) = u(\xi))$

$$-cu + u^2 + u' + 4u'' + u''' = B \tag{2.5}$$

we considered the superposition of solitary waves (2.3) spaced 2δ apart

$$u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta). \tag{2.6}$$

We used the partial fraction decomposition in exponentials applied to the double product of two solitary waves taken with a shift j. It was shown that such a product can be written as a sum of some solitary waves and traveling fronts. The sum of the latter does not converge individually, but can be combined as

$$\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} a(j)(b(m) - b(m+j)) = \sum_{j=1}^{+\infty} a(j)j, \tag{2.7}$$

and thus is a constant B in (2.5). The superposition (2.6) was shown to satisfy (2.5) exactly for any $\delta > \pi$ for the following relations among the parameters δ, a, c, and B, where $w = e^{-2a\delta}$:

$$c = \frac{2a^2 - 1 + 11a^4 + 240a^4 g_1}{2a^2}, \tag{2.8}$$

$$\delta = (a\delta)(1 + 240 g_2)^{1/4}, \tag{2.9}$$

$$B = 180a^4 \left(-(1+a^2) \sum_{j=1}^{+\infty} \frac{6w^{2j}(1+w^j)}{(1-w^j)^5} j + (1-a^2) \sum_{j=1}^{+\infty} \frac{w^j + 5w^{2j} + 5w^{3j} + w^{4j}}{(1-w^j)^5} j \right), \tag{2.10}$$

where the coefficients g_1 and g_2 depend on the product $a\delta$ only and are given by the infinite series

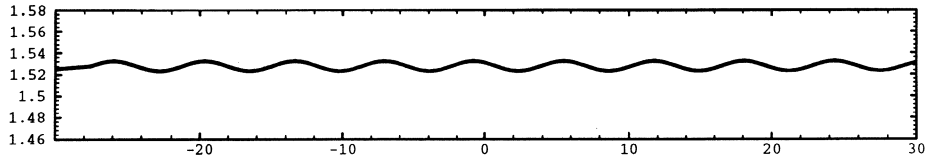
$$g_1 = \sum_{j=1}^{+\infty} \frac{w^{2j}}{(1-\omega^j)^4}, \tag{2.11}$$

$$g_2 = \sum_{j=1}^{+\infty} \frac{w^j + 4w^{2j} + w^{3j}}{(1-w^j)^4}. \tag{2.12}$$

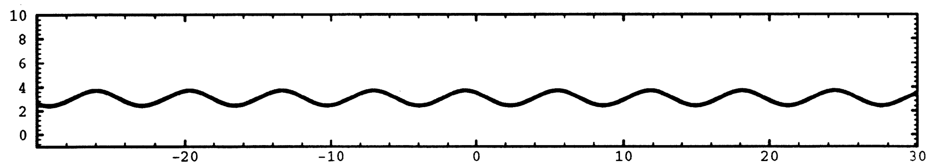
Figure 1 gives the periodic solutions for different values of δ. As δ increases, this periodic pulse train approaches the exact solitary wave solution

$$u(\xi) = 60 \frac{e^{\xi}}{(1+e^{\xi})^3}. \tag{2.13}$$

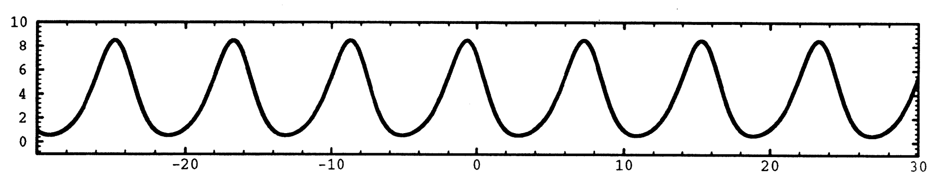
Equation (2.6) may be viewed as an "almost clean" interaction of such solitons. Under nonlinear coupling, solitons adjust their form but do not destroy each other. In general, it reveals the formation of localized persistent structures, their interaction, and the particle dynamics made up of solitary waves. It is not so obvious that the periodic solution for small values of δ is made out of solitary waves (2.13) (see Figure 1a).



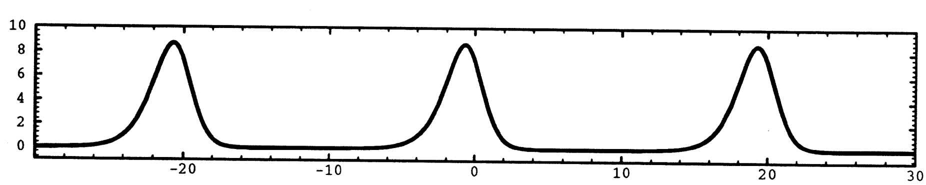
(a) $\delta = 3.141593$



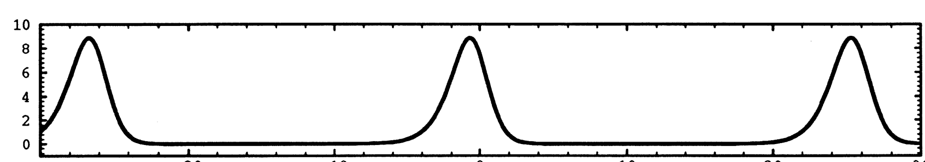
(b) $\delta = 3.151302$



(c) $\delta = 4.078418$



(d) $\delta = 10$



(e) $\delta = 13$

Figure 1. The graphs of the periodic solutions of the generalized Kuramoto–Sivashinsky equation for different values of the spacing 2δ between pulses. The periodic pulse train for $\delta = 13$ is unstable.

3. The Bretherton model equation

In this section we illustrate how the exact periodic solutions can be found for the different types of equations that particularly are not Galilean invariant and for which no solitary solutions are known. To do so we consider the Bretherton model equation that appears in the study of the three-wave interaction process where one-dimensional dispersive waves interact weakly through a cubic term [6]

$$u_{tt} + u_{xx} + u_{xxxx} + u = u^3. \tag{3.1}$$

It has the Painlevé expansion

$$u(x,t) = \pm 2\sqrt{30}\left(-\frac{F_x^2}{F^2} + \frac{F_{xx}}{F}\right) = \pm 2\sqrt{30}\,(\ln F)_{xx}. \tag{3.2}$$

We let

$$F(x,t) = 1 + e^{a\xi}$$

with $\xi = x - ct$, so that

$$u_s(x,t) = u_s(x-ct) = u_s(\xi) = A\frac{e^{a\xi}}{(1+e^{a\xi})^2}, \tag{3.3}$$

where $A = \pm 2\sqrt{30}\,a^2$. The traveling wave form of (3.1) is given by

$$(c^2+1)u_{\xi\xi} + u_{\xi\xi\xi\xi} + u - u^3 = 0. \tag{3.4}$$

Note that this equation lacks the constant of integration and, therefore, to compensate the constant that will appear as a result of the partial fraction decomposition we consider a periodic pulse train of the form

$$u(\xi) = \sum_{m=-\infty}^{+\infty} u_s(\xi - 2m\delta) + B. \tag{3.5}$$

Then $u(\xi)$ satisfies (3.4) if

$$\sum_{m=-\infty}^{+\infty} \{\mu u''_{s,m}(\xi) + u''''_{s,m}(\xi) + u_{s,m}\} - \left(\sum_{m=-\infty}^{+\infty} u_{s,m} + B\right)^3 + B = 0, \tag{3.6}$$

where we used the notation $\mu = c^2 + 1$ and $u_{s,m}(\xi) = u_s(\xi - 2m\delta)$. Equation (3.6) can be written as

$$\sum_{m=-\infty}^{+\infty} \left\{ \mu u''_{s,m} + u''''_{s,m} + (1 - 3B^2) u_{s,m} - u^3_{s,m} - 3Bu^2_{s,m} \right\}$$
$$- 3 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} u^2_{s,m} u_{s,m+j} - 3 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} u_{s,m} u^2_{s,m+j}$$
$$- 6 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \sum_{i=1}^{+\infty} u_{s,m} u_{s,m+j} u_{s,m+j+i}$$
$$- 6B \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} u_{s,m} u_{s,m+j} - B^3 + B = 0. \tag{3.7}$$

From the partial fraction decomposition in the variables $z = e^{-a\xi}$ and $w = e^{-2a\delta}$ we have the following expressions:

$$\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} u_{s,m} u_{s,m+j} = A^2 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{z^2 w^{2m+j}}{(z + w^m)^2 (z + w^{m+j})^2}$$
$$= A^2 p_1 \sum_{m=-\infty}^{+\infty} \frac{z w^m}{(z + w^m)^2} + A^2 q_1$$
$$= A p_1 \sum_{m=-\infty}^{+\infty} u_{s,m} + A^2 q_1, \tag{3.8}$$

$$\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} u^2_{s,m} u_{s,m+j} + \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} u_{s,m} u^2_{s,m+j}$$
$$= A^3 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{z^3 w^{3m+j}}{(z + w^m)^4 (z + w^{m+j})^2}$$
$$+ A^3 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{z^3 w^{3m+2j}}{(z + w^m)^2 (z + w^{m+j})^4}$$
$$= A^3 p_2 \sum_{m=-\infty}^{+\infty} \frac{z w^{3m} + z^3 w^m}{(z + w^m)^4} + A^3 p_3 \sum_{m=-\infty}^{+\infty} \frac{z^2 w^{2m}}{(z + w^m)^4} + A^3 q_2, \tag{3.9}$$

$$\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \sum_{i=1}^{+\infty} u_{s,m} u_{s,m+j} u_{s,m+j+i}$$
$$= A^3 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \sum_{i=1}^{+\infty} \frac{z^3 w^{3m+2j+i}}{(z + w^m)^2 (z + w^{m+j})^2 (z + w^{m+j+i})^2}$$
$$= A^3 p_4 \sum_{m=-\infty}^{+\infty} \frac{z w^m}{(z + w^m)^2} + A^3 q_3 = A^2 p_4 \sum_{m=-\infty}^{+\infty} u_{s,m} + A^3 q_3, \tag{3.10}$$

where

$$p_1 = -\sum_{j=1}^{+\infty} \frac{2w^j}{(w^j-1)^2},$$

$$q_1 = -\sum_{j=1}^{+\infty} \frac{w^{2j}+w^j}{(w^j-1)^3} j,$$

$$p_2 = \sum_{j=1}^{+\infty} \frac{w^j+6w^{2j}+w^{3j}}{(w^j-1)^4},$$

$$p_3 = \sum_{j=1}^{+\infty} \frac{16w^{2j}}{(w^j-1)^4},$$

$$q_2 = 4\sum_{j=1}^{+\infty} \frac{w^{2j}+w^{3j}}{(w^j-1)^5} j,$$

$$p_4 = \sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty} \frac{w^{i+j}+w^{2i+j}+w^{i+2j}-6w^{2i+2j}+w^{3i+2j}+w^{2i+3j}+w^{3i+3j}}{(w^i-1)^2(w^j-1)^2(w^{i+j}-1)^2},$$

$$q_3 = 2\sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty} \frac{w^{2i+4j}-w^{i+2j}}{(w^j-1)^3(w^{i+j}-1)^3} j + 2\sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty} \frac{w^{4i+2j}-w^{2i+j}}{(w^i-1)^3(w^{j+i}-1)^3} i.$$

To obtain the last expression we used the formula (similar to (2.7)):

$$\sum_{m=-\infty}^{+\infty}\sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty} (a_1 b(m)) + (-a_1(j)-a_2(j))b(m+j) + a_2(j)b(m+j+i)$$
$$= \sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty} a_1(j)j - \sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty} a_2(j)i. \tag{3.11}$$

Now (3.7) has a summation in m only or is a constant and we can set each term of this sum equal to zero, collect coefficients of similar exponentials, and solve this equation for an arbitrary m. This gives a system of three algebraic equations in δ, B, μ, a. Then μ is found as

$$\mu = -\frac{1+a^4-3B^2-12\sqrt{30}\,a^2p_1-360a^4p_2-720a^4p_4}{a^2}. \tag{3.12}$$

For B and a we have the equations

$$(1-3B^2) - \sqrt{30}\,B(1+12p_1)a^2 - (4+240p_2+60p_3+720p_4)a^4 = 0 \tag{3.13}$$

and

$$B - B^3 - 360a^4q_1 - 720\sqrt{30}\,a^6q_2 - 1440\sqrt{30}\,a^6q_3 = 0. \tag{3.14}$$

The first equation is quadratic in a^2, so it can be solved for a^2 and used in the second equation, which is cubic in B. Since the coefficients $p_1, p_2, p_3, p_4, q_1, q_2, q_3$ depend on the product $a\delta$ only (recall that $w = e^{-2a\delta}$) we can fix $a\delta$, solve the quadratic equation (3.13) for a^2, find B from (3.14), and recover δ. Note that by definition $\mu \geq 1$; therefore, the solution exists only for restricted values of δ, which in our case are $3.5697 < \delta < 4.4143$. Figure 2 gives the dependence of different parameters on the spacing between pulses.

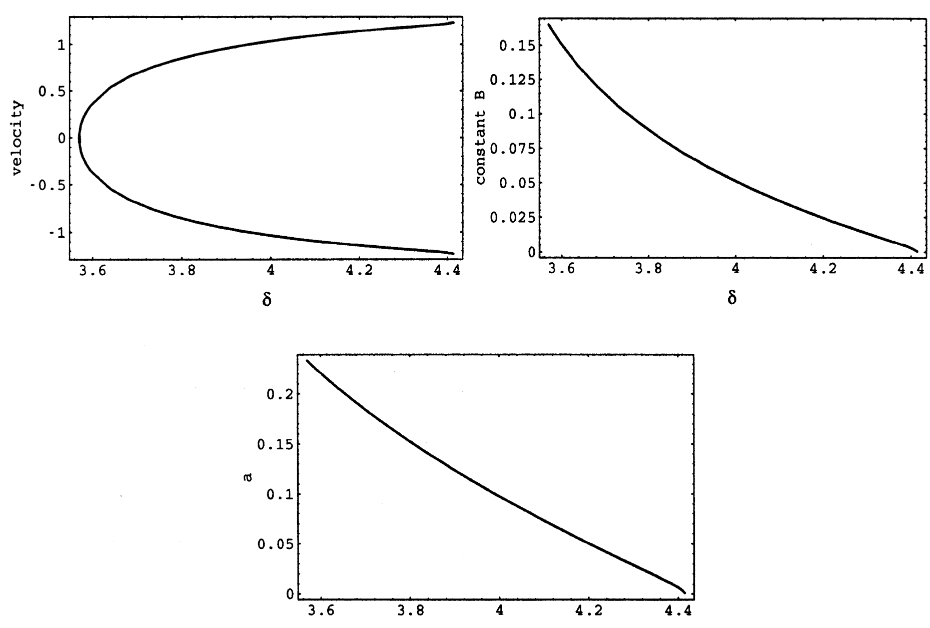


Figure 2. The dependence of the velocity of propagation c, the constant B, and the amplitude of the pulse on the half period δ of the exact periodic solution of the Bretherton equation.

3. The two-particle periodic pulse train

An extensive study of multiparticle solutions of the generalized Kuramoto–Sivashinsky equations for relatively small values of the dispersion parameter was performed in [7]. In [8] the dynamics of the trains of well-separated pulses was investigated using singular perturbation methods and with numerical simulations. We used the Fourier–Galerkin spectral method with periodic boundary conditions to analyze the steady traveling wave solutions of the GKSE. The numerical results demonstrate that the solutions tend to form stable lattices of pulses that are steady in some particular frame. Periodic solutions containing a single pulse occur if $\pi < \delta < 12.75$. For $\delta > 12.75$ the single particle solution becomes unstable and is thus not realized for greater δ. For $\delta > 2\pi$ two-particle solutions can emerge. Some are characterized by equal spacing and are indeed just the single-particle solutions with spacing twice as small. Others have unequal spacing between pulses. In general, for $\delta > N\pi$, solutions ranging from one to N pulses are all possible, but as δ increases, first the one-particle solution loses its stability (at $\delta = 12.75$), then the two-particle solution becomes unstable (at $\delta = 25.5$), and so on.

The families of multiparticle solutions found in our case are similar to results achieved in [7] in the small dispersion case. Figure 3 represents the two-particle solutions with equally (a) and unequally (b) spaced pulses.

As one compares the form of these two periodic solutions it may look as if they are made out of the same solitary waves. To verify this hypothesis we consider the periodic pulse train that is a superposition of solitary waves with two spacings 2δ and ξ_0 between pulses (two periodic single-particle

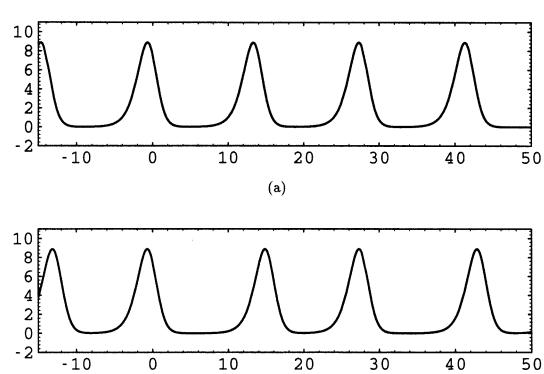


Figure 3. Two-particle periodic solutions of the GKSE with (a) equally spaced pulses for $\delta = 14$, where spacing between pulses is 14; (b) unequally spaced pulses for $\delta = 14$, where the smallest spacing between pulses is 13.28.

solutions added with a shift ξ_0). So, the solution of

$$-cu + u^2 + u' + 4u'' + u''' = B \tag{4.1}$$

or

$$\mathscr{L}u + u^2 = B$$

with a linear operator

$$\mathscr{L} = -c + \frac{d}{d\xi} + 4\frac{d^2}{d\xi^2} + \frac{d^3}{d\xi^3}$$

is sought in the form

$$u(\xi) = \sum_{m=-\infty}^{+\infty} \left(u_s(\xi - 2m\delta) + u_s(\xi - 2m\delta - \xi_0) \right), \tag{4.2}$$

where

$$u_s(\xi) = \frac{\beta_1 e^{a\xi} + \beta_2 e^{2a\xi}}{(1 + e^{a\xi})^3}. \tag{4.3}$$

To simplify the notation we denote

$$U_s(\xi, m) = u_s(\xi - 2m\delta) + u_s(\xi - 2m\delta - \xi_0).$$

$u(\xi)$ satisfies (4.1) if we can find constants c, δ, ξ_0, and B such that

$$\sum_{m=-\infty}^{+\infty} \mathscr{L}\{U_s(\xi, m)\} + \left(\sum_{m=-\infty}^{+\infty} U_s(\xi, m) \right)^2 = B, \tag{4.4}$$

which can be written as

$$\sum_{m=-\infty}^{+\infty} \left(\mathscr{L}\{U_s(\xi, m)\} + U_s(\xi, m)^2 \right) + 2 \sum_{m=-\infty}^{+\infty} U_s(\xi, m) \sum_{j=1}^{+\infty} U_s(\xi, m + j) = B. \tag{4.5}$$

Since we would like to get the summation in m only, so that we can set each term in the sum equal to zero independent of m, we need to separate the summation in j from m in the last term of the previous expression.

Using the substitutions $z = e^{-a\xi}$, $w = e^{-2\delta a}$, and $p = e^{-a\xi_0}$ this term can be written as

$$2 \sum_{m=-\infty}^{+\infty} U_s(\xi, m) \sum_{j=1}^{+\infty} U_s(\xi, m+j)$$

$$= 2A^2 \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \left(\frac{\beta_1 w^m z^2}{(z+w^m)^3} + \frac{\beta_2 w^{2m} z}{(z+w^m)^3} + \frac{\beta_1 p w^m z^2}{(z+pw^m)^3} + \frac{\beta_2 p^2 w^{2m} z}{(z+pw^m)^3} \right)$$

$$\times \left(\frac{\beta_1 w^{m+j} z^2}{(z+w^{m+j})^3} + \frac{\beta_2 w^{2m+2j} z}{(z+w^{m+j})^3} + \frac{\beta_1 p w^{m+j} z^2}{(z+pw^{m+j})^3} + \frac{\beta_2 p^2 w^{2m+2j} z}{(z+pw^{m+j})^3} \right). \tag{4.6}$$

Using the partial fraction decomposition the term under the summation signs can be decomposed into linear combination of

$$\frac{z^2 w^m}{(z+w^m)^3}, \frac{z^2 w^{m+j}}{(z+w^{m+j})^3}, \frac{z w^{2m}}{(z+w^m)^3},$$

$$\frac{z w^{2m+2j}}{(z+w^{m+j})^3}, \frac{p z^2 w^m}{(z+pw^m)^3}, \frac{z^2 p w^{m+j}}{(z+pw^{m+j})^3}, \frac{z p^2 w^{2m}}{(z+pw^m)^3}, \frac{z p^2 w^{2m+2j}}{(z+pw^{m+j})^3},$$

$$\frac{w^m}{z+w^m}, \frac{w^{m+j}}{z+w^{m+j}}, \frac{pw^m}{z+pw^m}, \frac{pw^{m+j}}{z+pw^{m+j}}.$$

The sequences

$$\frac{w^m}{z+w^m}, \frac{w^{m+j}}{z+w^{m+j}}, \frac{pw^m}{z+pw^m}, \frac{pw^{m+j}}{z+pw^{m+j}},$$

tend to 0 for $m \to +\infty$, but for $m \to -\infty$ they do not (individually) since the terms of the m-sequences tend to 1. Furthermore, $w^m/(z+w^m)$ and $w^{m+j}/(z+w^{m+j})$ (as well as $pw^m/(z+pw^m)$ and $pw^{m+j}/(z+pw^{m+j})$) do not have equal magnitude, but opposite sign coefficients, so they cannot be put into the form

$$\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} a(j)(b(m)-b(m+j)) = \sum_{j=1}^{+\infty} a(j)j$$

and thus be a constant term. Instead, we must be able to add series of the form

$$\sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \xi(j,p)\frac{w^m}{z+w^m} + \phi(j,p)\frac{w^{m+j}}{z+w^{m+j}}$$
$$- \phi(j,p)\frac{pw^m}{z+pw^m} - \xi(j,p)\frac{pw^{m+j}}{z+pw^{m+j}}, \qquad (4.7)$$

where the coefficients $\xi(j,p)$ and $\phi(j,p)$ also depend on β_1 and β_2. These coefficients have quite complicated structures, but the symmetry between them can easily be seen by looking at their Taylor expansion in p:

$$\xi(j,p) = \beta_1\beta_2 w^j p + \left(-3\beta_1^2 + 10\beta_1\beta_2 - 3\beta_2^2\right)w^{2j}p^2$$
$$+ 9\left(-2\beta_1^2 + 5\beta_1\beta_2 - 2\beta_2^2\right)w^{3j}p^3 + O(p^4), \qquad (4.8)$$

$$\phi(j,p) = \frac{\beta_1\beta_2}{w^j}p + \frac{-3\beta_1^2 + 10\beta_1\beta_2 - 3\beta_2^2}{w^{2j}}p^2$$
$$+ 9\frac{-2\beta_1^2 + 5\beta_1\beta_2 - 2\beta_2^2}{w^{3j}}p^3 + O(p^4). \qquad (4.9)$$

These two expansions consist of terms that are opposite in the sign of exponents of w. Therefore, denoting by μ_k the coefficients of w^{kj} or w^{-kj} (they are equal) in these two expansions we can rewrite (4.7) as

$$\sum_{k=1}^{+\infty} \mu_k \sum_{m=-\infty}^{+\infty} \sum_{j=1}^{+\infty} \left(w^{kj}\frac{w^m}{z+w^m} + w^{-kj}\frac{w^{m+j}}{z+w^{m+j}} \right.$$
$$\left. - w^{-kj}\frac{pw^m}{z+pw^m} - w^{kj}\frac{pw^{m+j}}{z+pw^{m+j}} \right). \qquad (4.10)$$

Each term of this series in k can be shown to give us a constant and a new periodic pulse train that consists of the terms $e^{ax}/(1+e^{ax})(1+pe^{ax})$.

$$
\begin{aligned}
&\sum_{m=-\infty}^{+\infty}\sum_{j=1}^{+\infty}\left(\frac{w^{m+kj}}{z+w^m}+\frac{w^{m-(k-1)j}}{z+w^{m+j}}-\frac{pw^{m-kj}}{z+pw^m}-\frac{pw^{m+(k+1)j}}{z+w^{m+j}}\right)\\
&=\sum_{m=0}^{+\infty}\sum_{j=1}^{+\infty}\frac{w^{m+kj}}{z+w^m}-\sum_{m=0}^{+\infty}\sum_{j=1}^{+\infty}\frac{pw^{m-kj}}{z+pw^m}+\sum_{m=0}^{+\infty}\sum_{j=1}^{+\infty}\frac{w^{m-(k-1)j}}{z+w^{m+j}}\\
&\quad-\sum_{m=0}^{+\infty}\sum_{j=1}^{+\infty}\frac{pw^{m+(k+1)j}}{z+pw^{m+j}}\\
&\quad+\sum_{m<0}\sum_{j=1}^{+\infty}\left(\frac{w^{m+kj}}{z+w^m}-w^{kj}\right)-\sum_{m<0}\sum_{j=1}^{+\infty}\left(\frac{pw^{m-kj}}{z+pw^m}-w^{-kj}\right)\\
&\quad+\sum_{m<0}\sum_{j=1}^{+\infty}\left(\frac{w^{m-(k-1)j}}{z+w^{m+j}}-w^{-kj}\right)-\sum_{m<0}\sum_{j=1}^{+\infty}\left(\frac{pw^{m+(k+1)j}}{z+pw^{m+j}}-w^{kj}\right)\\
&=\sum_{j=1}^{+\infty}w^{kj}\sum_{m=0}^{+\infty}\frac{w^m}{z+w^m}-\sum_{j=1}^{+\infty}w^{kj}\sum_{m=j}^{+\infty}\frac{pw^m}{z+pw^m}-\sum_{j=1}^{+\infty}w^{-kj}\sum_{m=0}^{+\infty}\frac{pw^m}{z+pw^m}\\
&\quad+\sum_{j=1}^{+\infty}w^{-kj}\sum_{m=j}^{+\infty}\frac{w^m}{z+w^m}+\sum_{j=1}^{+\infty}w^{kj}\sum_{m<0}\left(\frac{w^m}{z+w^m}-1\right)\\
&\quad-\sum_{j=1}^{+\infty}w^{kj}\sum_{m<j}\left(\frac{pw^m}{z+pw^m}-1\right)\\
&\quad+\sum_{j=1}^{+\infty}w^{-kj}\sum_{m<j}\left(\frac{w^m}{z+w^m}-1\right)-\sum_{j=1}^{+\infty}w^{-kj}\sum_{m<0}\left(\frac{pw^m}{z+pw^m}-1\right)\\
&=\sum_{j=1}^{+\infty}w^{kj}\sum_{m=0}^{+\infty}\frac{w^m}{z+w^m}-\frac{pw^m}{z+pw^m}+\sum_{j=1}^{+\infty}w^{-kj}\sum_{m=0}^{+\infty}\frac{w^m}{z+w^m}-\frac{pw^m}{z+pw^m}\\
&\quad+\sum_{j=1}^{+\infty}w^{kj}\sum_{m=0}^{j-1}\frac{pw^m}{z+pw^m}-\sum_{j=1}^{+\infty}w^{-kj}\sum_{m=0}^{j-1}\frac{w^m}{z+w^m}\\
&\quad+\sum_{j=1}^{+\infty}w^{kj}\sum_{m<0}\frac{w^m}{z+w^m}-\frac{pw^m}{z+pw^m}+\sum_{j=1}^{+\infty}w^{-kj}\sum_{m<0}\frac{w^m}{z+w^m}-\frac{pw^m}{z+pw^m}\\
&\quad-\sum_{j=1}^{+\infty}w^{kj}\sum_{m=0}^{j-1}W\left(\frac{pw^m}{z+pw^m}-1\right)+\sum_{j=1}^{+\infty}w^{-kj}\sum_{m=0}^{j-1}\left(\frac{w^m}{z+w^m}-1\right)\\
&=\sum_{j=1}^{+\infty}(w^{kj}+w^{-kj})\sum_{m=-\infty}^{+\infty}\left(\frac{w^m}{z+w^m}-\frac{pw^m}{z+pw^m}\right)+\sum_{j=1}^{+\infty}(w^{kj}+w^{-kj})j.
\end{aligned}
\tag{4.11}
$$

So (4.7) becomes

$$\sum_{k=1}^{+\infty} \mu_k \sum_{j=1}^{+\infty} (w^{kj} + w^{-kj}) \sum_{m=-\infty}^{+\infty} \left(\frac{w^m}{z+w^m} - \frac{pw^m}{z+pw^m} \right) + \sum_{j=1}^{+\infty} (w^{kj} + w^{-kj}) j$$

$$= \sum_{j=1}^{+\infty} (\xi(j,p) + \phi(j,p)) \sum_{m=-\infty}^{+\infty} \left(\frac{(1-p) z w^m}{(z+w^m)(z+pw^m)} \right)$$

$$+ \sum_{j=1}^{+\infty} (\xi(j,p) + \phi(j,p)) j. \tag{4.12}$$

Now we let $B = \sum_{j=1}^{+\infty}(\xi(j,p) + \phi(j,p))j$. What is left in (4.5) has the summation in m only and we can set each term of this sum equal to zero; that leads to the equation

$$-cu_s(\xi) + u_s(\xi)^2 + u_s'(\xi) + 4u_s''(\xi) + u_s'''(\xi)$$

$$+ 2\left(\frac{A_1 e^{a\xi} + A_2 e^{2a\xi}}{(1+e^{a\xi})^3} + \frac{Q_1 p e^{a\xi} + Q_2 p^2 e^{2a\xi}}{(1+e^{a\xi}p)^3} \right.$$

$$\left. + G(1-p) \frac{e^{a\xi}}{(1+e^{a\xi})(1+pe^{a\xi})} \right) = 0. \tag{4.13}$$

The coefficients A_1, A_2, Q_1, Q_2, G were obtained from the partial fraction decomposition in (4.6) and have the form

$$A_1 = \beta_1^2(g_3 + l_1) + \beta_2^2(g_1 + m_1) + \beta_1 \beta_2(g_3 - 3g_1 + m_3),$$

$$Q_1 = \beta_1^2(g_3 + l_1) + \beta_2^2(g_1 + m_1 + m_3 - l_3) + \beta_1 \beta_2(g_3 - 3g_1 + l_3),$$

$$A_2 = \beta_1^2(g_1 + m_1 + m_3 - l_3) + \beta_2^2(g_3 + l_1) + \beta_1 \beta_2(g_3 - 3g_1 + l_3),$$

$$Q_2 = \beta_1^2(g_1 + m_1) + \beta_2^2(g_3 + l_1) + \beta_1 \beta_2(g_3 - 3g + m_3),$$

$$G = (\beta_1^2 + \beta_2^2) r_1 + \beta_1 \beta_2 r_2,$$

where

$$r_1 = \sum_{j=1}^{+\infty} \frac{3p^2 w^{2j}(p+w^j)}{(w^j-p)^5} - \frac{3p^2 w^{2j}(1+pw^j)}{(pw^j-1)^5},$$

$$r_2 = \sum_{j=1}^{+\infty} \frac{pw^j+5p^2w^{2j}+5p^3w^{3j}+p^4w^{4j}}{(pw^j-1)^5} - \frac{p^4w^j+5p^3w^{2j}+5p^2w^{3j}+pw^{4j}}{(w^j-p)^5},$$

$$l_1 = \sum_{j=1}^{+\infty} \frac{3p^2(1+p^4)w^{2j}-12p^3(1+p^2)w^{3j}}{(w^j-p)^4(pw^j-1)^4} + \frac{36p^4w^{4j}-12p^3(1+p^2)w^{5j}+3p^2(1+p^4)w^{6j}}{(w^j-p)^4(pw^j-1)^4},$$

$$l_3 = \sum_{j=1}^{+\infty} \frac{-p^3(1+2p^2)w^j+p^2(-3+12p^2-3p^4)w^{2j}+(-2p+6p^5-p^7)w^{3j}}{(w^j-p)^4(pw^j-1)^4} + \frac{(16p^2-36p^4+8p^6)w^{4j}+(-2p+6p^5-p^7)w^{5j}}{(w^j-p)^4(pw^j-1)^4} + \frac{(-3p^6-3p^2+12p^4)w^{6j}-(p^3+2p^5)w^{7j}}{(w^j-p)^4(pw^j-1)^4},$$

$$m_1 = \sum_{j=1}^{+\infty} \frac{p^3w^j+(2p^2-4p^4+2p^6)w^{2j}+(-8p^3-2p^5+p^7)w^{3j}}{(w^j-p)^4(pw^j-1)^4} + \frac{(24p^4-8p^6)w^{4j}+(-8p^3-2p^5+p^7)w^{5j}+(2p^2-4p^4+2p^6)w^{6j}+p^3w^{7j}}{(w^j-p)^4(pw^j-1)^4},$$

$$m_3 = \sum_{j=1}^{+\infty} \frac{(-2p^3-p^5)w^{7j}+p^2(-3+12p^2-3p^4)w^{6j}+(-p+6p^3-2p^7)w^{5j}}{(w^j-p)^4(pw^j-1)^4} + \frac{(8p^2-36p^4+16p^6)w^{4j}+(-p+6p^3-2p^7)w^{3j}}{(w^j-p)^4(pw^j-1)^4} + \frac{p^2(-3+12p^2-3p^4)w^{2j}-(2p^3+p^5)w^j}{(w^j-p)^4(pw^j-1)^4},$$

$$g_1 = \sum_{j=1}^{+\infty} \frac{w^j+4w^{2j}+w^{3j}}{(w^j-1)^4},$$

$$g_3 = \sum_{j=1}^{+\infty} \frac{6w^{2j}}{(w^j-1)^4}.$$

By equating the coefficients of equal powers of exponentials in Equation (4.13) we get the expression for the velocity c as

$$c = 120a^2(-g_1+g_3+l_1-m_1) + \frac{-1-5a^2+3p+15a^2p-3p^2-135a^2p^2+p^3+5a^2p^3}{(p-1)^3}, \quad (4.14)$$

and the compatibility conditions on a and p as

$$l_3 = \frac{(p+p^2)}{(p-1)^3} + m_3, \quad (4.15)$$

$$r_1 = \frac{p(1+p)(a^2p^2-p^2+10a^2p+2p+a^2-1)}{2(1+a^2)(1-p)^5} + \frac{a^2-1}{2(1+a^2)}r_2, \quad (4.16)$$

$$\begin{aligned} l_1 = {} & \frac{-4a^2g_1}{a^2-1} + \frac{1-a^4-4p+60a^2p-56a^4p}{60(a-1)a^2(1+a)(p-1)^4}, \\ & + \frac{6p^2-486a^4p^2-4p^3-60a^2p^3-176a^4p^3+p^4-a^4p^4}{60(a-1)a^2(1+a)(p-1)^4} \\ & + m_3 - \frac{3+a^2}{a^2-1}m_1. \end{aligned} \quad (4.17)$$

By fixing the product $a\delta$ we can find p from (4.15) and then find a that satisfies both (4.16) and (4.17) to good accuracy.

From Equation (4.15) we recover p, which corresponds to equally spaced pulses. Then, by taking irregular spacing from numerical results, we can use (4.16), (4.17) to find the approximation for a and use (4.13) to find the error of approximation of the irregular two-particle solution by (4.2). It turns out that the error of approximation (4.2) for the smallest value of δ for which two-particle solutions was observed is about 0.1% of the maximum height of the pulse, and this error quickly decays as δ increases. The appearance of the term $G(1-p)(e^{ax}/(1+e^{ax})(1+pe^{ax}))$ indicates the interaction between two adjacent pulses that slightly modifies the form of each pulse.

5. Cnoidal waves

The periodic solutions in terms of elliptic functions for some integrable equations were obtained by Toda in case of "Toda lattice." Later Whitham suggested constructing the periodic solutions as sums of equally spaced solitons for the Korteweg–de Vries, the modified Korteweg–de Vries, and the Boussinesq equations. For this construction he gave elementary arguments, using identities in sech functions [1, 9]. These are the same solutions since the infinite sums of hyperbolic functions can be represented by the

Jacobi elliptic functions. The Painlevé expansion (or some generalization of the Painlevé expansion for equations that do not formally possess a Painlevé property) can often be written in the derivatives of the logarithmic function. Our choice for the singular manifold function $F = 1 + e^{a\xi}$ naturally leads to the representation of such an expansion in terms of hyperbolic functions. So we can represent the periodic pulse trains as the infinite superposition of the hyperbolic functions and their derivatives and, therefore, as Jacobi elliptic functions and their derivatives.

For example, the generalized Kuramoto–Sivashinsky equation with $\sigma = 4$ (2.1) has a Painlevé expansion with $F = 1 + e^{a\xi}$

$$u_s(\xi) = 30a^2 \frac{(1+a)e^{a\xi} + (1-a)e^{2a\xi}}{(1+e^{a\xi})^3}, \tag{5.1}$$

which can be written as

$$u_s(\xi) = 30a^2\left(\operatorname{sech}^2\frac{a\xi}{2} + \left(\operatorname{sech}^2\frac{a\xi}{2}\right)'\right). \tag{5.2}$$

Then the infinite superposition of such functions can be represented as the cnoidal function and its derivative.

The Bretherton model equation has the Painlevé expansion with $F = 1 + e^{a\xi}$ of the form

$$u_s(\xi) = \pm 2\sqrt{30}\,a^2 \frac{e^{a\xi}}{(1+e^{a\xi})^2}, \tag{5.3}$$

which can be written as

$$u_s(\xi) = \pm\sqrt{30}\,a^2 \operatorname{sech}^2\frac{a\xi}{2}, \tag{5.4}$$

and the infinite superposition of this function is a cnoidal wave.

Kudryashov [10] developed the method of getting the cnoidal wave solution by directly applying the Painlevé expansion to the original equation, thus reducing it to the anharmonic oscillator. Obviously our approach is closely related to his method but involves more elementary arguments and exhibits the interaction between the solitary waves directly.

We may represent our periodic pulse train solutions in the form of elliptic functions and their derivatives using the identity given in [9]:

$$a^2 \sum_{m=1}^{+\infty} \operatorname{sech}^2(a\xi - 2m\delta) = dn^2\xi - C \tag{5.5}$$

with $C = 1 - E'/K'$, $a = \pi/2K'$, and $2\delta = \pi K/K'$.

6. Conclusion

The remarkable result that in some problems the infinite superposition of the soliton solutions is a periodic solution of the governing equation has been known for some time now. This fact can be considered as another indication of a clean interaction between solitons. Although a linear sum of aperiodically spaced solitary wave solutions is not an exact solution, for some systems the superposition of slightly modified waves is a solution. In nonintegrable systems the solitary waves combined into a periodic solution may interact more strongly than in integrable equations and produce some additional terms as the result of such an interaction. The partial fraction decomposition in exponentials is a convenient tool for analyzing such terms. The Painlevé expansion gives the correct form of these interaction terms. This leads to a conclusion that the interaction of the solitary waves in the periodic pulse train depends on the singularity structure of the system.

References

1. G. B. WHITHAM, Comments on periodic waves and solitons, *IMA J. Appl. Math.* 32:353–366 (1984).
2. N. G. BERLOFF and L. N. HOWARD, Solitary and periodic solutions of nonlinear nonintegrable equations, *Stud. Appl. Math.* 99(1):1–24 (1997).
3. J. WEISS, M. TABOR, and G. CARNEVALLE, The Painlevé property for partial differential equations, *J. Math. Phys.* 24(3):522–526 (1983).
4. D. J. BENNEY, Long waves on liquid films, *J. Math. & Phys.* 45:150–155 (1966).
5. N. KUDRYASHOV, Exact solutions of the generalized Kuramoto–Sivashinsky equation, *Phys. Lett. A* 147(5,6):287–291 (1990).
6. F. P. BRETHERTON, Resonant interactions between waves: The case of discrete oscillations, *J. Fluid Mech.* 20:457–479 (1964).
7. C. ELPHICK, G. R. IERLEY, O. REGEV, and E. A. SPIEGEL, Interacting localized structures with galilean invariance, *Phys. Rev. A* 44(2):1110–1122 (1991).
8. N. J. BALMFORTH, G. R. IERLEY, and R. WORTHING, Pulse dynamics in an unstable medium, Institute for Fusion Studies, University of Texas, Austin, 1995.
9. G. B. WHITHAM, *On Shocks and Solitary Waves*, Scripps Institution of Oceanography Reference Series 91–24, 1991.
10. N. KUDRYASHOV, On types of nonlinear nonintegrable equations with exact solutions, *Phys. Lett. A* 155(4,5):269–275 (1991).

FLORIDA STATE UNIVERSITY

(Received February 20, 1997)