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Vortex nucleation by a moving ion in a Bose condensate

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Abstract

The nonlinear Schrödinger equation is used to analyze the superfluid flow around an ion and to elucidate the vortex nucleation process. Asymptotic expansion for the flow is used to find the critical velocity of the ion for vortex production. 3D numerical calculations demonstrate, that if the axisymmetry of the flow is broken by introducing a solid boundary several healing length from the sphere, vortex loops rather than vortex rings may be formed. These loops may detach from the ion and attach themselves to the wall. In the presence of small 3D random noise, the system still favors the creation of vortex rings. © 2000 Elsevier Science B.V. All rights reserved.

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The deliberately introduced impurity can be a fruitful experimental probe of the structure and behavior of superfluid helium. These impurities are: ^{3}He atoms of radius 4 Å, electrons that by their motion create a bubble of about 16 Å radius, and $^4\mathrm{He}_2^+$ positive ions of radius 8 Å. Vortex nucleation by an ion moving in superfluid helium at low temperature has been studied experimentally and theoretically (see [1] for a review) and has led to a number of interesting results. The superfluid offers no resistance to the ion provided that its speed, v, relative to the ion is less than a certain critical value, v_c. At speeds greater than v_c, the ion continually sheds vortex rings and these create a time-varying drag on the ion [2]. The critical speed v_c may be estimated by modeling the ion as a solid sphere and noting that the maximum relative velocity, $u_{\max}$, between fluid and sphere is greatest on the equator of the sphere (defined relative to the direction Oz of motion as polar axis), and is approximately $3v/2$, assuming that v is small enough for compressibility to be negligible.

Muirhead et al. [3] developed the theory of vortex nucleation using a semiclassical (hydrodynamic) approach; the vortex was taken in the form derived in [4] from the Gross–Pitaevskii (GP) condensate model [5]. They analyzed two scenarios for the vortex shedding that occur for $v > v_c$. In the first, a fully-formed ring detaches, simultaneously and as a whole, from the equator of the sphere. We call this the 'complete ring' scenario to distinguish it from the second or 'vortex loop' scenario, in which a vortex loop grows from the ion's equator, is stretched by the flow, and later detaches from the sphere to become eventually a circular vortex ring also. Muirhead et al. [3], using an energy barrier argument, conclude that loop nucleation is favored over ring nucleation. The principal aim of this Letter is to compare the two scenarios by numerical simulations using the Bose condensate model.

It is well known that, when the Reynolds number is large enough, the viscous boundary layer in contact

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with a moving sphere separates, so creating vorticity in the wake of the sphere and an associated drag. Helmholtz's theorem would forbid this process in a classical fluid that is strictly inviscid. The superfluid can, however, defeat Helmholtz's by the separation (breakdown) of the healing layer on the surface of the ion [6]. In [6] we used the GP model to investigate the complete ring scenario. If $v > v_c$ ($u_{\max} > c$, where c is the local speed of sound) circular vortices are emitted at or near the equator of the ion, defined by the direction of its motion. The flow created by this flow, when combined with the flow over the surface of the ion, at first reduces $u_{\max}$ below c. The self-induced speed of the ring is, however, less than v, so that the ring falls increasingly far behind the ion, and its effect on the ion diminishes. Eventually criticality again occurs and another ring is emitted. Frisch et al. [7] and Winiecki et al. [8] have demonstrated analogous phenomena for the emission of vortex pairs from a cylinder moving with speed $v > v_c \approx \frac{1}{2}c$. The drag on the cylinder created by the vortices in its wake was evaluated in [8].

In terms of the single-particle wavefunction $\psi(\mathbf{x}, t)$ for the N bosons of mass M, the time-dependent self-consistent field equation of the GP model is [5]

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2M}\nabla^2\psi + \psi\bigl(E - V_0|\psi|^2\bigr), \tag{1}$$

where E is the single particle energy and V_0 is the strength of the δ-function interaction potential between the bosons. The wavefunction is required to obey the normalized condition on the total number of the bosons $N = \int |\psi|^2\, dV$. The sphere is an infinite potential barrier to the condensate, so that the boundary condition is $\psi = 0$ at $r = b$, where b, the radius of the sphere, is assumed much greater than the healing length $a = \hbar/(2ME)^{1/2}$, i.e., $\epsilon \equiv a/b \ll 1$. The Madelung transformation $\psi = R\exp(iS)$ converts (1) into a mass continuity equation $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0$, and an integrated momentum equation involving a quantum potential:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + c^2\left(\frac{\rho}{\rho_\infty} - 1\right) - \frac{\hbar^2}{2M^2}\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}} = 0. \tag{2}$$

Here $\rho = M\psi\psi^*$ is the mass density, and $\mathbf{j} = (\hbar/2i)(\psi^*\nabla\psi - \psi\nabla\psi^*) = \rho\mathbf{u}$; $\mathbf{u} = \nabla\phi$, where $\phi = (\hbar/M)S$, is the fluid velocity; $c^2 = E/M$ is the sound velocity; $\rho_\infty = ME/V_0 = M\psi_\infty^2$ is the mass density at great distances. It is clear from (2) that the fluid is compressible. Indeed, 'in the bulk' (the region far from the sphere) where gradients of the density, ρ, are negligible, (2) shows that the pressure, p, is proportional to ρ^2. Compressibility is therefore important at criticality, since the velocity of the sphere is then comparable with c.

In [6], we solved (1) in the ion reference frame, so that the ion is permanently situated at O. The condensate is in steady motion at infinity in the negative z-direction with speed $V \equiv v/c$. Grant and Roberts [9] developed an asymptotic expansion of the solution for $U \ll 1$, so that the flow is approximately incompressible. Their results can be extended to show that the dimensionless flow velocity, u/c, is $3V/2 + 551\epsilon^2V^3/1760$. For $V = \mathcal{O}(1)$, compressibility becomes important. The solution consists of two parts, a boundary layer (inner solution) close to the surface of the sphere and the mainstream flow (outer solution) in the far field. In the mainstream, quantum effects are negligible at leading order in ϵ, and the condensate becomes a compressible inviscid fluid. By matching smoothly the boundary layer to the mainstream solution we show [6] that the maximum flow velocity, u/c, on the equator of the sphere is

$$\begin{aligned} U_\theta = {} & 3V/2 + 0.313V^3 + 0.3924V^5 + 0.648V^7 \\ & + 1.24V^9 + 2.63V^{11} + 5.96V^{13} + \cdots \\ & + \epsilon\bigl(2.12V + 1.58V^3 + 2.89V^5 + \cdots\bigr). \end{aligned} \tag{3}$$

This expression reaches the local speed of sound when $V_c \approx 0.53$ for $\epsilon \to 0$.

The ring scenario was studied in [6] by numerically integrating axisymmetric solutions of (1), i.e., solutions independent of χ, where (r, θ, χ) are spherical coordinates and $\theta = 0$ in the direction of motion of the sphere. We also demonstrated the style of healing layer separation, which occurs near, or slightly downstream of, the ion equator.

In this Letter, we examine the possibility that vortex shedding is preferentially a non-axisymmetric process, as for example is envisaged by the vortex loop scenario. For this purpose, a fully 3D program was constructed, which finds solutions of (1) that depend on χ, as well as on r and θ. Eq. (1) was solved in an integration box of dimensions $80a$ in each of the Cartesian dimensions (for the details of numerical integration see [6]).

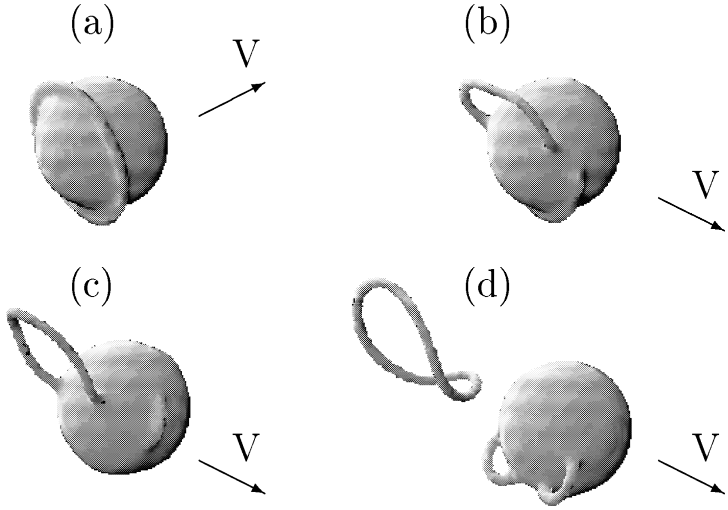

Fig. 1. The results of numerical integration of (1) for the sphere of radius $10a$ with $V = 0.566$. The pictures show the isosurface $\rho = 0.2\rho_\infty$ at (a) $t = 85a/c$, (b) $t = 128a/c$, (c) $t = 160a/c$, and (d) $t = 234a/c$.

We first examined whether a vortex loop grows spontaneously from a small asymmetric perturbation. Our perturbation consisted of random noise added to the motion of the sphere, so that its dimensionless steady velocity $\mathbf{V} = V\hat{\mathbf{z}}$ was replaced by $\mathbf{V} = V\hat{\mathbf{z}} + \mathbf{h}(t)$ where $\mathbf{h}$ is a random perturbation. (This is meant to simulate in an approximate way fluctuations in the electric field dragging the ion through the fluid.) When $\mathbf{h}$ was changed every time step ($\Delta t \approx 0.005a/c$), no tendency to produce vortex loops was observed, even where $|\mathbf{h}|$ was as large as 10% of V. When $\mathbf{h}$ was changed randomly every 2000 time steps, the ion motion $V\hat{\mathbf{z}} + \mathbf{h}$ is oblique to Oz and constant during each of these periods (duration $10.7a/c$), and circular vortices tended to nucleate on or near the equator defined by that direction. Again, there was no perceptible tendency for loops to develop. Loops might start to form but they would evanesce or quickly transform to a complete ring while still in contact with the ion. To accentuate the appearance of the vortex loop we implemented the following strategy: initially we fixed $\mathbf{h} = \frac{1}{5}V_c\hat{\mathbf{y}}$ and allowed the system to evolve up to the nucleation of the first ring (for about 16000 time steps); see Fig. 1(a). After that, $\mathbf{h}$ was reset to $\mathbf{h} = -\frac{1}{5}V_c\hat{\mathbf{y}}$ during the next 24000 time steps. As the sphere changes its direction the flow at part of the nascent vortex ring becomes subcritical. This nonuniformity allows the formation of a vortex loop; see Fig. 1(b). For as long as the sphere maintains its motion, the radius of the vortex loop grows, the feet of the loop move closer to each other and towards the back side of the sphere (Fig. 1(c)) until they reconnect to form a ring; see Fig. 1(d). Such an asymmetric vortex ring balances the velocity field on the surface of the sphere nonuniformly; even though the sphere continues moving steadily in the same direction this nonuniformity influences the formation of the next vortex loop and so on. Note that for the larger values of V the picture of nucleation becomes increasingly more complicated. As the velocity field around the sphere can become supercritical before the complete loop detaches, some other loops and rings may be formed that reconnect among themselves and with the previously formed loops or rings.

Second, we changed the geometry by introducing a plane wall (an infinite potential barrier) at a distance of $10a$ from the ion. Fig. 2 shows the formation and evolution of the vortex loop on the sphere moving subcritically with $V = 0.51$ in a presence of a solid plane at 10 healing lengths from the sphere. A vortex loop appears on the side of the sphere closer to the boundary; see Fig. 2(a), as the velocity reaches criticality there. The loop spreads laterally, interacts with the boundary (Fig. 2(b)), the feet of the vortex line on the surface of the sphere come closer to each other (Fig. 2(c)), and they detach to form a loop on the boundary (Fig. 2(d)).

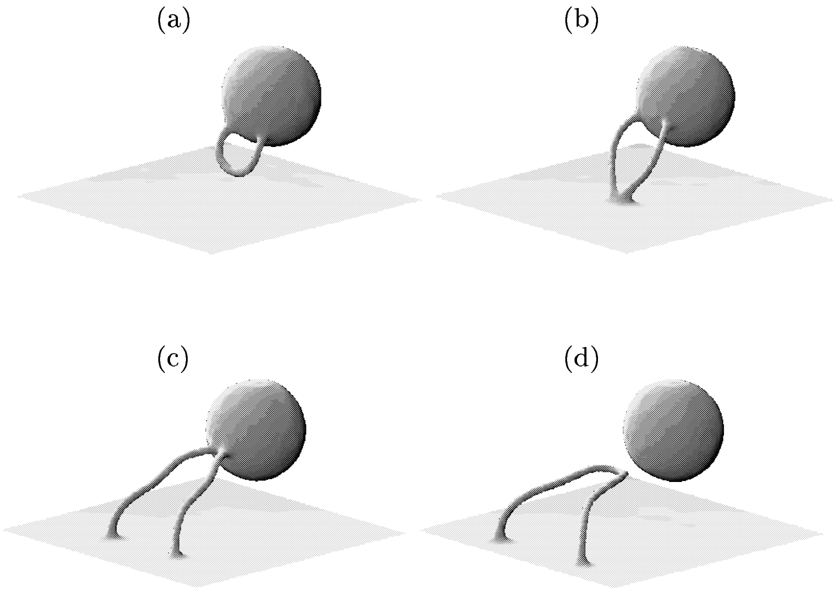

Fig. 2. The results of numerical integration of (1) with $V = 0.51$ for the isosurface $\rho = 0.2\rho_\infty$ at (a) $t = 132a/c$, (b) $t = 168a/c$, (c) $t = 216a/c$, and (d) $t = 240a/c$.

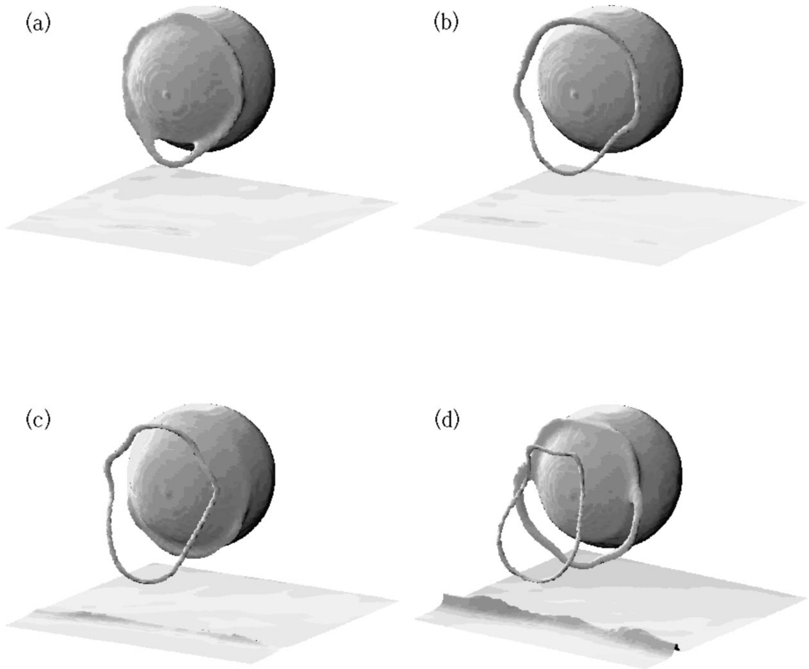

Fig. 3. The results of numerical integration of (1) with $V = 0.59$ for the isosurface $\rho = 0.004\rho_\infty$ at (a) $t = 64a/c$, (b) $t = 81a/c$, (c) $t = 96a/c$, and (d) $t = 113a/c$.

For larger distances between the ion and the boundary it is also possible that a loop on the sphere develops into a ring without touching and reconnecting with the boundary. This scenario is different if the sphere is moving with supercritical velocity. Again the vortex loop is first formed close to the boundary (Fig. 3(a)); almost simultaneously however a complimentary loop appears on the opposite side of the sphere (Fig. 3(b)). Both rings rapidly spread laterally until they reconnect to form a vortex ring; see Figs. 3(c) and (d). These figures also show how the healing layer on the wall is thickened by the presence of the ring. This process, also present in Fig. 2, aids the capture of a vortex loop by the wall.

The simulations described in this Letter have not revealed any marked tendency for vortex loops to nucleate in preference to vortex rings at criticality, although in special circumstances loops do form first and develop rapidly into rings.

Acknowledgements

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