Vortices in Nonlocal Condensate Models of Superfluid Helium

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Abstract. Nonlocal nonlinear Schrödinger equations are considered as models of superfluid helium. The models contain a nonlocal interaction potential that leads to a phonon-roton-like dispersion relation. It is shown that for any such potential the generalized Gross-Pitaevskii (GP) model has non-physical features, specifically the development of catastrophic singularities and unphysical mass concentrations. The GP equation is remedied by introducing a higher order term in the local density approximation for the correlation energy. The resulting theory is applied in two ways. The family of superfluid vortex rings is derived. The nucleation of vortex rings by a moving ion is considered.

1 Introduction

Superfluid helium at $0^\circ K$ has a large interatomic spacing and is often described in terms of a weakly interacting Bose gas. The imperfect Bose condensate in the Hartree approximation is governed by equations that were derived by Gross and by Ginsburg and Pitaevskii. In terms of the single-particle wavefunction $\psi(\mathbf{x},t)$ for N bosons of mass M, the time-dependent self-consistent field equation is

$$i\hbar\psi_t = -\frac{\hbar^2}{2M}\nabla^2\psi + \psi\int|\psi(\mathbf{x}',t)|^2 V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}', \qquad (1)$$

where $V(|\mathbf{x}-\mathbf{x}'|)$ is the potential of the two-body interactions between bosons. The normalization condition is $\int|\psi|^2\,d\mathbf{x} = N$.

The internal energy per unit volume, $\mathcal{E}$, at point $\mathbf{x}$ and time t is given by

$$\mathcal{E}(\rho) = \frac{\hbar^2}{8M^2\rho}(\nabla\rho)^2 + \frac{1}{2M^2}\int\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}'|)\rho(\mathbf{x}')\,d\mathbf{x}', \qquad (2)$$

and the total energy, W, is

$$W = \int\mathcal{E}(\rho)\,d\mathbf{x} = \int\frac{\hbar^2}{8M^2\rho}(\nabla\rho)^2\,d\mathbf{x} + W_c(\rho). \qquad (3)$$

The first term on the right-hand side of (3) describes the quantum kinetic energy of a Bose gas of nonuniform density; $W_c(\rho)$ is a potential or correlation energy that incorporates the effect of interactions.

For a weakly interacting Bose system (1) is simplified by replacing $V(|\mathbf{x}-\mathbf{x}'|)$ with a δ - function repulsive potential of strength $V_0 = \int V\,d\mathbf{x}'$, which leads to the local GP model. Several aspects of the local GP model are qualitatively or quantitatively unrealistic for superfluid helium. The dispersion relation between the frequency, ω, and wave number, k, of sound waves according to the local GP model is

$$\omega^2 = c^2k^2 + \left(\frac{\hbar}{2M}\right)^2 k^4, \tag{4}$$

where $c = (V_0\rho_\infty)^{\frac{1}{2}}/M$, $\rho_\infty = E_v/V_0$. This shows that the velocity, c, of long wavelength sound waves is proportional to $\rho^{\frac{1}{2}}$ (here we have replaced the bulk density, ρ_∞, by ρ). That this is unrealistic is seen from the experiments on Grüneisen constant $U_G = (\rho\partial c/c\partial\rho)_T \approx 2.8$ at the vapor pressure [7]. Also, the dispersion curve (4) has no roton minimum. At best, (4) describes the phonon branch of the excitation spectrum.

There is significant interest attached to the question of whether the introduction of a realistic two-particle interaction potential, V, that leads to a phonon-roton-like spectra in the GP model, gives a better description of the properties of superfluid helium than the local model [5]. The minimum requirements on such a potential would be the correct position of the roton minimum and the correct bulk normalization (see below).

2 Applicability of the generalized Gross-Pitaevskii model

We transform (1) by introducing the average energy level E_v, so that $\psi = \Psi\exp(-iE_vt/\hbar)$, and rescale it by

$$\mathbf{x} \to \frac{\hbar}{(2ME_v)^{1/2}}\mathbf{x}, \qquad t \to \frac{\hbar}{2E_v}t. \tag{5}$$

The dimensionless form of (1) becomes

$$-2i\frac{\partial\Psi}{\partial t} = \nabla^2\Psi + \Psi\Big(1 - \int |\Psi(\mathbf{x}',t)|^2 V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}'\Big), \tag{6}$$

with the bulk normalization condition $\int V(|\mathbf{x}'|)\,d\mathbf{x}' = 1$. We get the dispersion curve by linearizing about the uniform state. We write $\Psi = 1 + \Psi'$ and consider plane waves of the form $\mathrm{Re}(\Psi') = \exp i(\omega t - kx)$. Then the dispersion relation can be written as

$$\omega^2 = \frac{1}{4}k^4 + 2k\pi\int_0^\infty \sin(kr)V(r)r\,dr. \tag{7}$$

We require $\omega'(k_{rot}) = 0$, $\omega(k_{rot}) = \omega_{rot}$, where (k_{rot}, ω_{rot}) is the position of the roton minimum on the $k\omega-$ dispersion curve, which in dimensional units is found from experiments [9] to be $k_{rot} = 1.926\mathring{A}^{-1}$, and $\omega = 8.62K^\circ k_B/\hbar$. By taking the limit of (7) for $k \to 0$ and using the normalization condition, the sound speed is found to be $1/\sqrt{2}$ as in the local model. By relating this to the known value of the sound speed at low k in He II, 238 m/s, we find that the healing length of the model (6) is fixed as $[L] = 0.47\mathring{A}$, and therefore $k_{rot} = 0.907$ and $\omega_{rot} = 0.158$.

After V has been selected to give a good account of the roton minimum it is typically found that, after the normalization condition has been enforced, the convolution

$$\frac{\delta P}{\delta\rho} = \frac{1}{2M^2}\int \rho(\mathbf{x}')V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}' \tag{8}$$

is not necessary positive. This convolution is the variational derivative of the mechanical pressure, P, of the hydrodynamic formulation of (1) in the semi-classsical limit ($\hbar \to 0$). When the integral on the right-hand side of (8) is negative the pressure P decreases when the mass density increases, which is unphysical.

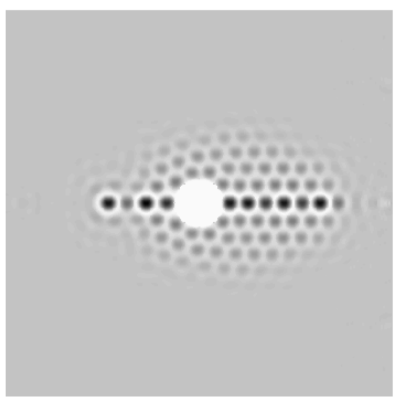
It can be shown [1] that a virial theorem, similar to the one used in establishing the catastrophic singularities of the focusing nonlinear Schrödinger equation, indicates that the solutions of the nonlocal model (1) blow up in a bounded domain.

Finally, to illustrate the development of mass concentrations we solved (6) numerically for the flow around a positive ion moving with the dimensionless velocity $U = \frac{1}{2}U_L$, where U_L is the Landau critical velocity. The ion is modeled as the infinite potential barrier so that $\Psi = 0$ on $r = b$, where b is the radius of the positive ion. The interaction potential was used in the form suggested by Jones [11]

$$V(r) = (\alpha + \beta A^2 r^2 + \gamma A^4 r^4) \exp(-A^2 r^2), \tag{9}$$

where the parameters α, β, γ, and A are chosen to give agreement with the experimentally determined dispersion curve as discussed above. Notice that the local flow velocity is below U_L everywhere. Nevertheless, persistent mass concentrations develop along the axis of symmetry; see Figure 1.

Fig. 1. The density plot in a cross-section of the solution of (6) for the flow around a sphere of radius $10a$ moving to the right with velocity $0.5v_L$. After [1].



Such an unphysical behavior indicates that assumptions made in the derivation of the equation must be unjustified. This difficulty could be overcome by introducing into the weakly nonlinear theory dissipation or higher order nonlinear terms. But we would like to preserve the Hamiltonian character of the GP model. Instead, we will take the density - functional approach [8], which tries to introduce an accurate microscopic picture of liquid helium.

3 Nonlocal nonlinear Schrödinger equation

The correlation energy of the Skyrme interactions in nuclei [14] is given by

$$W_c(\rho) = \frac{1}{M^2}\int\left[\frac{W_0}{2}\rho^2 + \frac{W_1}{2+\gamma}\rho^{2+\gamma} + W_2(\nabla\rho)^2\right]d\mathbf{x}, \tag{10}$$

where W_0, W_1, W_2 and γ are phenomenological constants. The first two terms give a local density approximation, and the gradient term corresponds to finite range interactions. In a somewhat similar way to [8], we add the necessary nonlocality of interactions directly into the first term of (10) by introducing a two-body interaction potential, $V(|\mathbf{x}-\mathbf{x}'|)$, so that (10) becomes

$$W_c(\rho) = \frac{1}{M^2}\int\left[\frac{1}{2}\int\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}'|)\rho(\mathbf{x}')\,d\mathbf{x}' + \frac{W_1}{2+\gamma}\rho^{2+\gamma}\right]d\mathbf{x}. \tag{11}$$

This incorporates and generalizes the W_2 interaction term in (10), which has therefore been abandoned.

On adopting (11), we find that the nonlinear Schrödinger equation replacing (1) is

$$i\hbar\psi_t = -\frac{\hbar^2}{2M}\nabla^2\psi + \psi\int|\psi(\mathbf{x}',t)|^2V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}' + W_1\psi|\psi|^{2(1+\gamma)}, \tag{12}$$

and equation (6) is replaced by

$$-2i\frac{\partial\Psi}{\partial t} = \nabla^2\Psi + \Psi\left[1 - \int|\Psi(\mathbf{x}',t)|^2V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}' - \chi|\Psi|^{2(1+\gamma)}\right]. \tag{13}$$

The bulk normalization condition becomes $\int V(|\mathbf{x}'|)\,d\mathbf{x}' = 1-\chi$. The dispersion relation of (13) is modified in comparison with (4) by adding the term $\frac{1}{2}(1+\gamma)\chi k^2$ to its right-hand side. The bulk normalization condition gives the slope at the origin (the dimensionless speed of sound) as $\sqrt{(1+\gamma\chi)/2}$ and the unit of length (healing length) as $[L] = 0.47\sqrt{1+\gamma\chi}$ Å. A fit to the Landau dispersion curve can be obtained as in §2.

There are two logical choices of the parameter γ. First, we can view the term $W_1\rho^{2+\gamma}$ in (10) as the second term in the nonlinear expansion of the correlation energy in powers of ρ, and that yields $\gamma = 1$. The second possible choice is to take $\gamma = 2.8$, which gives $c \propto \rho^{2.8}$ in agreement with the experimentally determined Grüneisen constant $U_G \approx 2.8$.

Next, we shall use (13) to study the family of the vortex rings [2]. For a vortex ring of large radius R the results for the straight-line vortex can be used to give [12] the energy and momentum of such ring as

$$\mathcal{E} = \frac{1}{2}\kappa^2\rho_\infty R\left[\ln\left(\frac{8R}{L}\right) - 2 + c\right],$$

and

$$p = \kappa\rho_\infty\pi R^2.$$

After differentiating $\mathcal{E}$ and p with respect to R and substituting into the Hamilton's equation $v = \partial\mathcal{E}/\partial p$ we get the expression for the velocity of the large vortex ring as

$$v = \frac{\kappa}{4\pi R}\left[\ln\left(\frac{8R}{L}\right) - 1 + c\right].$$

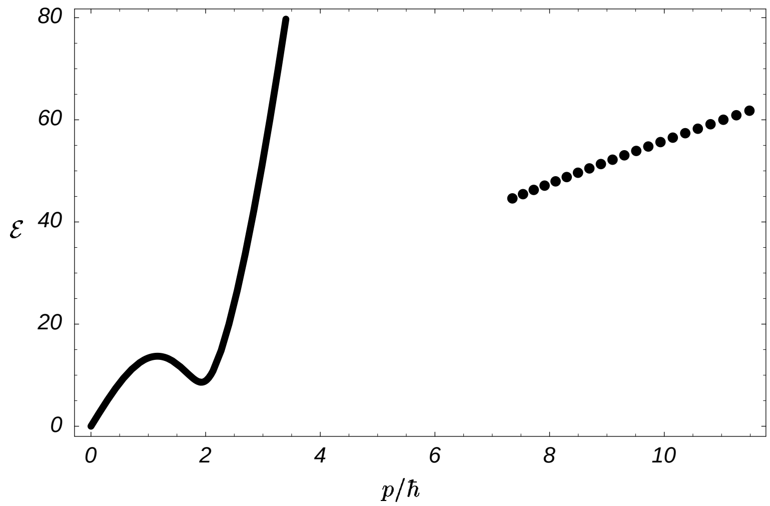
Glaberson and Donnelly [6] used the experimental results of Rayfield and Reif [13] on the relation between the energy and velocity of large vortex rings to estimate the vortex core parameter L. These estimates were based on the hollow core vortex model with $c = 0$ and produced $L \approx 0.81$Å. Jones [11] did similar calculations for the nonlocal model (6) and found $c = -0.13$, so that $L \approx 0.71$Å for the optimal choice of the parameter A. For the local GP model with $c = 0.381$ the vortex core parameter is $L \approx 1.19$Å. These values of L are much larger that the healing length found from the sound speed, which is 0.47 Å for any of the above models. Jones [11] posed the question of whether a self-consistent theory is possible, i.e., one where the vortex core parameter and the healing length are brought into harmony. The answer is "Yes." Our model (13) with $V(r) = V(r) = (\alpha + \beta A^2 r^2 + \gamma A^4 r^4)\exp(-A^2 r^2) + \delta\exp(-B^2 r^2)$ is able to bring about agreement. For $\gamma = 1$, $\chi = 3.5$, $A = 1.6$, $B = 1$, and $\delta = 1$ we numerically integrated (13) to find $c = 0.1825$, so that $L \approx 1$Å, which is the healing length of our model. This gives the energy of a vortex ring traveling at 27 cm/sec as 10 ev, which agrees with the experiments of Rayfield and Reif [13]. This choice of parameters is not very practical and, in the calculations below, we shall use instead the interaction potential (9) with $\chi = 0.2, \gamma = 1$, and $A = 0.9$, which also represent the roton minimum satisfactorily.

A sequence of vortex rings of small radius has been derived numerically [2]. When the velocity of the vortex ring reaches the Landau critical velocity the ring becomes unstable and evanesces into sound waves. For any ring traveling with speed greater than the Landau critical velocity, the amplitude of the far-field solution will not decay exponentially at infinity, which makes the existence of such a ring impossible. Figure 2 plots the sequence of the vortex rings on the $p\mathcal{E}$ plane together with the dispersion curve of (13). Note that, as parameters χ, γ, and A are varied, the position of a corresponding family of vortex rings on $p\mathcal{E}$-plane changes dramatically relative to dispersion curve. Actually it is even possible to bring the termination point of this family close to the roton minimum.

4 Vortex nucleation and roton emission

In this section we shall use the model (13) to elucidate vortex nucleation from, and roton emission by, moving ions [3]. The picture of nucleation that emerges as a result of experiments performed over the years by the McClintock group in Lancaster University (see, for instance [10]) shows that vortex nucleation and roton emission are independent processes, and that the latter is linked to v_L but the former is not. We numerically integrated (13) for the axisymmetric flow around the positive ion of radius $b = 10$ moving uniformly with velocity U. Our numerical calculations (for the details of the numerics see [1]) indicate that, provided U does not exceed the dimensionless Landau critical velocity U_L, the ion experiences no drag and the flow is steady in the frame

Fig. 2. The dispersion relation $p - \mathcal{E}$ and a family of the vortex rings as solutions of (13). Each dot represents the position of a vortex ring on $p\mathcal{E}$-plane.

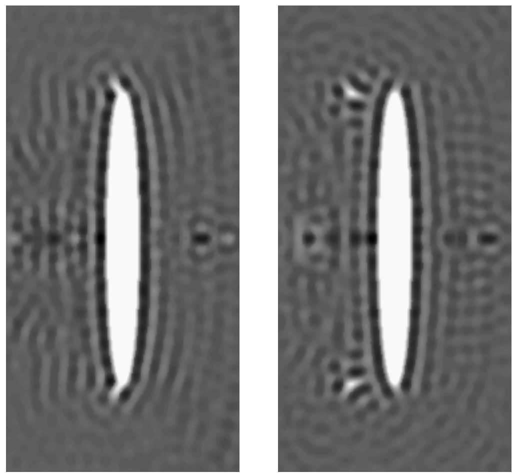


of reference moving with the ion. Notice that the velocity on the equator of the ion may exceed U_L (for incompressible flow the velocity on the equator would be $3U/2$ but is even larger when compressibility is allowed for), but this leads neither to vortex nucleation nor roton emission.

When $U > U_L$ a modulated wave envelope is formed involving wave numbers from the neighborhood of the Landau point, where $\omega'(k_L) = \omega(k_L)/k_L$, $\omega(k) = \bar{\omega}(k) - \mathbf{U} \cdot \mathbf{k}$, and $\bar{\omega}(k)$ refers to stagnant helium. These waves radiate energy to infinity, resulting in drag on the ion. We have not so far been able to observe vortex nucleation for $U > U_L$, but we obtained insight into vortex nucleation with the help of an artificial example which tended to confirm the hypothesis [10] that roton emission and vortex nucleation are different processes.

Our artificial example is motivated by the fact that the critical velocity v_c for vortex nucleation by an electron bubble is (according to the local GP model) reduced by its shape which, when moving, is oblate [4]. The presence of ^{3}He would enhance this effect through the concomitant reduction in surface tension. We can make v_c even smaller by artificially increasing the flattening, to such an extent that v_c becomes less than v_L, so that nucleation can be studied with the model (13) without the complications of roton emission. We therefore consider an ion with an oblate spheroidal surface moving in the direction of its short (symmetry) axis with a velocity less than v_L. The ratio of lengths of axes is 5. Nucleation of vortices occurs when $v = v_c \approx 0.148 \pm 0.007c$ (when the speed of sound is reached on the equator); see Figure 3. To compare this with the corresponding result for the Bose condensate, we performed similar calculations using the local GP model. The critical velocity of nucleation in this case was found to be $0.205 \pm 0.007c$. Such a significant drop ($\sim$ 30%) in the critical velocities between local and nonlocal model can be partially explained by the greater compressibility of the fluid, according to the nonlocal model.

Fig. 3. The density plot in a cross-section of the solution of (13) for the flow around an oblate spheroid (see text) moving to the right with velocity $0.156c$ at $t = 100$ (left) and $t = 300$ (right). The white circles show the core of a vortex ring nucleated from the spheroid and gradually falling astern of it. After [3].



5 Conclusions

In summary, we considered a nonlocal nonlinear Schrödinger equation (13) as a model of superfluidity. The model has a finite range interaction potential that leads to a dispersion curve with a roton minimum and can accommodate a more realistic relationship between the speed of sound, the density and the pressure. The parameters of the model can be chosen to bring the healing length into agreement with the vortex core parameter. According to our model, there is no drag on a positive ion moving with $v < v_L$. As the velocity of the ion exceeds the Landau critical velocity v_L, it starts to experience drag and it creates modulated waves with wave numbers corresponding to the roton minimum. Our model failed to describe vortex nucleation in such circumstances. Nevertheless it could, through an artificial example, provide strong indications that roton emission and vortex nucleation are different processes, the former being connected to the Landau critical velocity, and the latter to the speed of sound.

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