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Roton creation and vortex nucleation in superfluids

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Abstract

The nonlocal nonlinear Schrödinger equation is used to analyze the superfluid flow around an impurity. The differences between the processes of vortex nucleation and roton creation are elucidated. It is argued that vortices are nucleated when the velocity around the ion exceeds the velocity of sound. © 2000 Elsevier Science B.V. All rights reserved.

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A half a century has passed since the discovery of superfluidity and superfluid vortices, but the mechanisms of vortex nucleation are still not properly understood. The main reason is that there is no truly microscopic picture of superfluid helium available, so the appearance of vortices “from nothing”, or intrinsic nucleation, cannot be derived from first principles.

In the absence of such theory the dynamics of vortices are quite often derived from the Ginzburg–Pitaevskii (GP) model [1,2] which is assumed to be linked to the condensate fraction of the superfluid. This model has been extensively studied particularly for the motion of ions in a dilute Bose condensate [3,4]. The main conclusion of the numerical integration and asymptotic analysis of the GP equation is that the vortices nucleate when the velocity somewhere on the surface of the moving object exceeds the speed of sound. The condensate escapes the formation of a shock wave through the breakdown of the healing layer, resulting in vortex nucleation. Unfortunately, there are many shortcomings of the GP model, so that the model can claim only a qualitative significance for actual superfluid helium. The dispersion relation of the GP model has no roton minimum, which is held responsible for many of the properties of the superfluid. The velocity, c, of long wavelength sound is proportional to $\rho^{1/2}$ where ρ is the density, while experiments evaluating the Grüneisen constant $U_G = (\rho\partial c/c\partial\rho)_T$, show that, in the bulk (i.e., on lengthscales long compared with the healing length, κ/c), the fluid behaves as a barotropic fluid ($p \propto \rho^\gamma$, where p is the pressure) with $\gamma = 2.8$ [5]. Finally, the healing length and correlation length in real helium are known to be quite different.

For some time there has been a belief that, as soon as a realistic two-particle interaction potential, V, that leads to a phonon–roton-like spectra is introduced in the GP model, the properties of superfluid helium will be well represented [6–11]. The mini-

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mum requirements on such a potential would be the correct position of the roton minimum and the correct bulk normalization (see below). Unfortunately, as was shown in Ref. [12], such a model is not applicable since it has nonphysical solutions, having catastrophic mass concentrations. To remedy this, we adopted [13] a density–functional theory approach [14,15] and included short-range correlations into the total energy in the simplest way. This allowed us to correct the nonphysical features of the model, while retaining not only an adequate representation of the Landau dispersion relation, but also simplicity in the analytical and numerical studies. We used this model, which is a nonlocal nonlinear Schrödinger equation (NNLSE), to elucidate the properties of vortex rings. For this model we showed that the vortex core parameter and the healing length can be brought into agreement, so that the energy of large vortex rings agrees with experimental observations [16]. The goal of this Letter is to use the NNLSE model to elucidate vortex nucleation from, and roton emission by, moving ions.

The deliberately introduced impurity has been a valuable experimental probe of the structure and behavior of superfluid helium. These impurities are: ^{3}He atoms of radius 4 Å, electrons that by their motion create a bubble of about 16 Å radius, and $^4\mathrm{He}_2^+$ positive ions of radius 8 Å. Vortex nucleation by an ion moving in a superfluid helium at low temperature has been studied experimentally and theoretically (see Ref. [17] for a review) and has led to a number of interesting results as well as some controversy. The superfluid offers no resistance to the ion provided that the speed, v, of the ion relative to the superfluid is less than a certain critical value, v_c. At speeds greater than v_c, the ion continually sheds vortices and these create a time-varying drag on the ion [16]. The critical speed may be estimated by modeling the ion as a solid sphere and noting that the maximum relative velocity, u_{max}, between fluid and sphere is greatest on the equator of the sphere and is approximately $3v/2$, assuming that v is small enough for the compressibility of the fluid to be ignored.

At first it was believed, on the basis of the experimental data (see Table 8.2(a) of Ref. [17]) that the critical state arises when $u_{max} \approx v_L$, the Landau critical velocity, so that $v_c \approx 2v_L/3$, a result in rough agreement with observation [16]. A great surprise came from the discovery by Bowley et al. [18] that even a tiny concentration of ^{3}He is enough to decrease the critical velocity of nucleation dramatically. After the helium was cleansed of ^{3}He, the critical velocity of nucleation by negative ions increased by approximately 10 m/s (from 49.9 m/s to 59.5 m/s, at a pressure of 12 bars). Their explanation of this remarkable observation was based on the fact that ^{3}He atoms tend to congregate on a free surface, such as that of the electron bubble, and to reduce surface tension. An electron bubble is already made oblate by its motion [19], and the reduction in surface tension makes it more oblate, so increasing u_{max} on the equator and reducing v_c. The picture of nucleation that emerges after the Hendry et al. [20] experiments was that vortex nucleation and roton emission are independent processes, and that the latter is linked to v_L but the former is not. At pressures greater than 11 bars, pairs of rotons are emitted when $v > v_L$, and these decelerate the ion. When the applied electric field is increased sufficiently the ion may reach a critical velocity v_c for vortex nucleation before being decelerated by roton emission. From this picture it seems clear that $v_c > v_L$ and, since u_{max} is greater than v by at least a factor of $3/2$ (but by a larger factor when compressibility is taken into account), the vortex nucleates when u_{max} exceeds some critical speed u_c that greatly exceeds v_L. The main contention of this Letter is that u_c is the speed of sound c. We use our NNLSE theory to support this conclusion.

The NNLSE equation that we suggested and studied in Ref. [13] is written for the single-particle wavefunction $\psi(\mathbf{x},t)$ for N bosons of mass M as

$$i\hbar\psi_t = -\frac{\hbar^2}{2M}\nabla^2\psi + \psi\int|\psi(\mathbf{x}',t)|^2 V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}' + W\psi|\psi|^{2(1+\gamma)}, \tag{1}$$

where V is the interaction potential, and W and γ are phenomenological constants. If $W=0$ and $V(|\mathbf{x}-\mathbf{x}'|) = V_0\delta(|\mathbf{x}-\mathbf{x}'|)$, Eq. (1) reduces to the GP model. Several authors have considered models in which $W=0$ but V is not a $\delta-$ function [6–11,21]. It is then easily arranged that (1) gives a good representation of the Landau dispersion relation, and especially that the roton minimum is well described and the velocity of sound, c, is correct. (The last condition is not fulfilled by the nonlocal model considered in Ref. [11]). It has been shown [12] that, for any potential that satisfies these conditions, the general model (1) with $W=0$ has nonphysical features, such as loss of hyperbolicity, leading to the creation of nondissipative mass concentrations. Higher order terms must be retained in the expression for the correlation energy, and this motivates the introduction of the last term in (1).

If E_v is the average energy level of a boson, we write $\Psi = \exp(iE_v t/\hbar)\psi$, so that (1) becomes

$$i\hbar\Psi_t = -\frac{\hbar^2}{2M}\nabla^2\Psi + \Psi\left(\int|\Psi(\mathbf{x}',t)|^2 V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}' + W|\Psi|^{2(1+\gamma)} - E_v\right).$$

Casting this equation into dimensionless form by the transformation $\mathbf{x}\to a\mathbf{x}$, $t\to\hbar t/2E_v$, where $a=\hbar(2ME_v)^{-1/2}$ is the healing length, we obtain

$$-2i\frac{\partial\Psi}{\partial t} = \nabla^2\Psi + \Psi\left[1-\int|\Psi(\mathbf{x}')|^2 V(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}' - \chi|\Psi|^{2(1+\gamma)}\right], \tag{2}$$

where $\chi = W\rho_\infty^{1+\gamma}/E_v$ is a nondimensional parameter and ρ_∞ is the density at infinity, where the fluid is undisturbed. The dispersion relation is

$$\omega^2 = \tfrac{1}{4}k^4 + 2\pi k\int_0^\infty \sin kr V(r)\,r\,dr + \tfrac{1}{2}(1+\gamma)\chi k^2.$$

The normalization condition on the interaction potential is $\int V(|\mathbf{x}'|)\,d\mathbf{x}' = 1-\chi$, and this gives the slope at the origin (the dimensionless speed of sound) as $\sqrt{(1+\gamma\chi)/2}$. By relating this to the known speed of sound (238 m/s), we find that the unit length and unit time of our model are $a = 0.471\sqrt{1+\gamma\chi}$ Å and $a/c\sqrt{2} = 1.4\times10^{-13}(1+\gamma\chi)$ s.

For the interaction potential we choose the form suggested by Jones [21] as $V(|\mathbf{x}-\mathbf{x}'|) = V(r) = (\alpha + \beta A^2r^2 + \delta A^4r^4)\exp(-A^2r^2)$, where the parameters α,β and δ are chosen so that the normalization condition is satisfied and the dispersion relation has its roton minimum close to that experimentally observed at the vapor pressure $k_{\rm rot} = 1.926$ Å^{-1}, $\hbar\omega_{\rm rot}/k_{\rm B} = 8.62$ K° [22], which in our nondimensional units is at $k_{\rm rot} = 0.9077\sqrt{1+\gamma\chi}$, $\omega_{\rm rot} = 0.158(1+\gamma\chi)$, the Landau velocity being $v_{\rm L} = 0.27c$. We choose the remaining parameters to be $A=0.9$, $\chi=0.2$ and $\gamma=1$. We focus on this model, but note that the alternative choice, $\gamma=2.8$, makes the velocity, c, of long wavelength sound waves proportional to $\rho^{2.8}$, in agreement with the experimentally determined Grüneisen constant $U_{\rm G} = (\rho\partial c/\partial\rho c)_T \approx 2.8$.

Fig. 1 compares the experimentally determined dispersion curve with that employed by our model. The insets give the density in the core of the straight line vortex and in the healing layer at a solid boundary, both for our $\gamma=1$ model and for the GP model. The energy per unit length of the line vortex is $\mathscr{E} = (\kappa\rho_\infty/4\pi)[\ln(L_{\rm c}/a)+L_0]$, where $L_{\rm c}$ is a cut-off distance and L_0 is the vortex core parameter. The GP model gives $L_0\approx0.38$; our model significantly reduces this to $L_0\approx-0.09$. The negative vortex core energy comes from a marked depletion of the vortex core in the nonlocal model. The core of the vortex in our NNLSE model is similar to the one deduced from Monte Carlo simulations [23].

We numerically integrated (2) for the axisymmetric flow around the positive ion of dimensionless radius $b=10$ moving uniformly; its dimensionless velocity is $\mathbf{U} = U\mathbf{1}_z$, where $\mathbf{1}_z$ is the unit vector in the z-direction. The ion is modeled as the infinite potential barrier so that $\psi=0$ on $r=b$. To keep the ion in the center of the computational box, we transform z to $z-Ut$ in (2). Our numerical calculations (for the details of the numerics see Ref. [12]) indicate that, provided U does not exceed the dimensionless Landau critical velocity $U_{\rm L}$, the ion experiences no drag and the flow is steady in the frame of reference moving with the ion. Fig. 2 shows the density variation around an ion moving with the velocity $U=0.78U_{\rm L}$. Notice that the velocity on the equator of the ion exceeds $U_{\rm L}$ (for the incompressible flow the velocity on the equator would be $3U/2$ if the flow were incompressible but is even larger when compressibility is allowed for), but this leads neither to vortex nucleation nor roton emission.

When $U>U_{\rm L}$ a modulated wave envelope is formed with wave number from the neighborhood of the Landau point, where $\omega'(k_{\rm L}) = \omega(k_{\rm L})/k_{\rm L}$, $\omega(\mathbf{k}) = \overline{\omega}(\mathbf{k}) - \mathbf{U}\cdot\mathbf{k}$, and $\overline{\omega}(\mathbf{k})$ refers to stagnant helium.

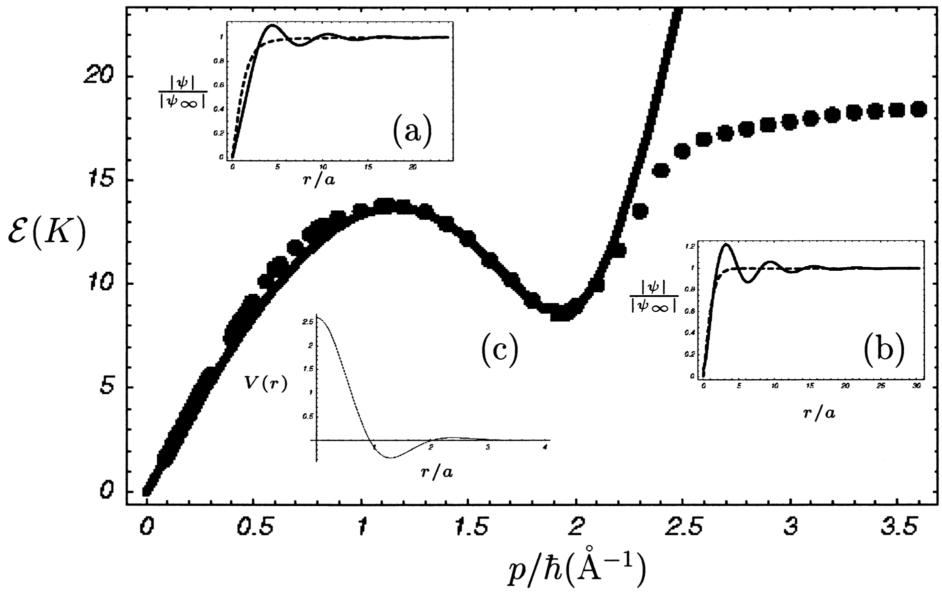

Fig. 1. The dispersion relation $p-\mathscr{E}$. The solid line corresponds to the nonlocal potential $V(|\mathbf{x}-\mathbf{x}'|)$ with $A=0.9$, $\chi=0.2$, and $\gamma=1$. The dots are based on experiment [15]. The insets show (a) the amplitude $|\psi|/\psi_\infty$ of the straight line vortex for the NNLSE (solid line) and the GP model (dashed line); (b) the amplitude $|\psi|/\psi_\infty$ of the healing layer at a solid boundary (an infinite potential barrier) placed at $r=0$ for the NNLSE (solid line) and the GP model (dashed line); (c) the potential $V(|\mathbf{x}-\mathbf{x}'|)$ plotted as a function of $r=|\mathbf{x}-\mathbf{x}'|$, in the nondimensional units defined in the main text.

These waves radiate energy to infinity, resulting in drag on the ion. Fig. 3 shows the wave pattern for an ion moving from left to right with velocity $0.56c$. This does not closely resemble the wave pattern of

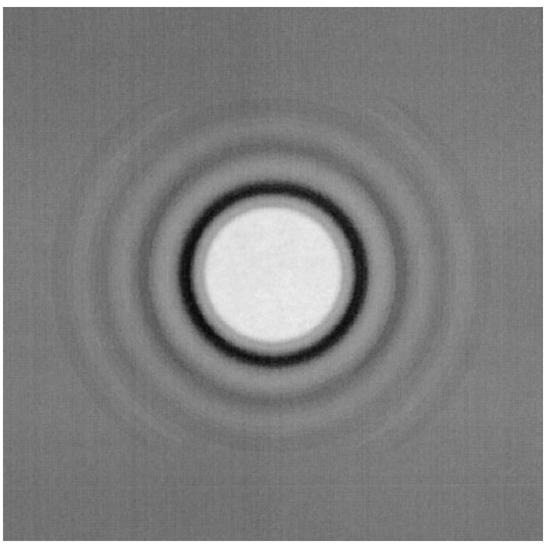

Fig. 2. The density plot in a cross-section of the solution of (2) for the flow around a sphere of radius $10a$ moving to the right with velocity $0.78v_{\mathrm{L}}$.

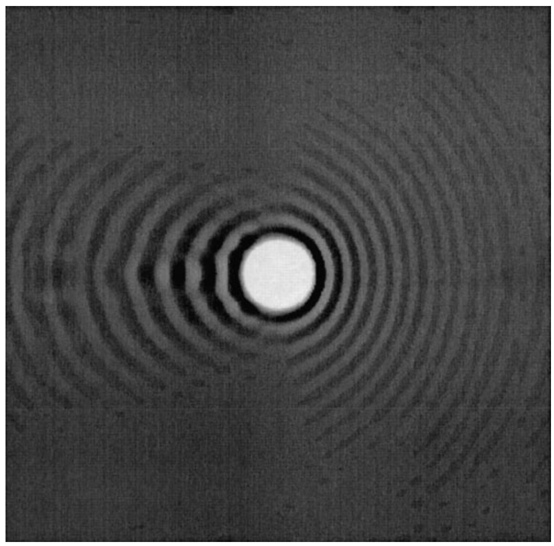

Fig. 3. The density plot in a cross-section of the solution of (2) for the flow around a sphere of radius $10a$ moving to the right with velocity $0.56c$.

Pomeau and Rica [11], who regarded theirs as exhibiting Cherenkov radiation.

We have not so far been able to observe vortex nucleation for $U > U_L$, but we obtained insight into vortex nucleation with the help of an artificial example which tended to confirm the hypothesis [20] that roton emission and vortex nucleation are different processes.

Our artificial example is motivated by the fact that the critical velocity v_c for vortex nucleation by an electron bubble is (according to the GP model) reduced by its shape which, when moving, is oblate [19]. Also, the presence of ^{3}He would enhance this effect through the concomitant reduction in surface tension (see above). We can make v_c even smaller by artificially increasing the flattening, to such an extent that v_c becomes less than v_L and can therefore be studied with the NNLSE model without the complications of roton emission. We therefore consider an ion with an oblate spheroidal surface moving in the direction of its short (symmetry) axis with a velocity less than v_L. The ratio of lengths of axes is 5. Nucleation of vortices occurs when $v = v_c \approx 0.148 \pm 0.007c$ (when the speed of sound is reached on the equator); see Fig. 4. To compare this with the corresponding result for the Bose condensate, we performed similar calculations using the GP model. The critical velocity of nucleation in this case was found to be $0.205 \pm 0.007c$. Such a significant drop ($\sim 30\%$) in the critical velocities between local and nonlocal model can be partially explained by the greater compressibility of the fluid, according to the nonlocal model.

This encourages us to make the following speculative observation. The GP model predicts that the critical velocity of nucleation for the positive ion of radius 10 Å is $0.53c$ [3,4]. The drop of approximately 30% predicted by the nonlocal model reduces this to $0.37c$. The critical velocity of nucleation for the negative ion is a further 20% smaller, or $0.29c$, which is within 15% of experimentally observed critical velocity. By this we imply that the nucleation of vortices occurs when the velocity somewhere on the surface of the moving ion exceeds the speed of sound.

In summary, we considered a nonlocal nonlinear Schrödinger equation (NNLSE) as a model of superfluidity. The model has a finite range interaction potential that leads to a dispersion curve with a roton minimum and can accommodate a more realistic relationship between the speed of sound, the density and the pressure. The parameters of the model can be chosen to bring the healing length into agreement with the vortex core parameter. According to our NNLSE model, there is no drag on a positive ion moving with $v < v_L$. As the velocity of the ion exceeds the Landau critical velocity v_L, it starts to experience drag and it creates modulated waves with wave numbers corresponding to the roton minimum. Our model failed to describe vortex nucleation in such circumstances. Nevertheless it could, through an artificial example, provide strong indications that roton emission and vortex nucleation are different processes, the former being connected to the Landau critical velocity, and the latter to the speed of sound.

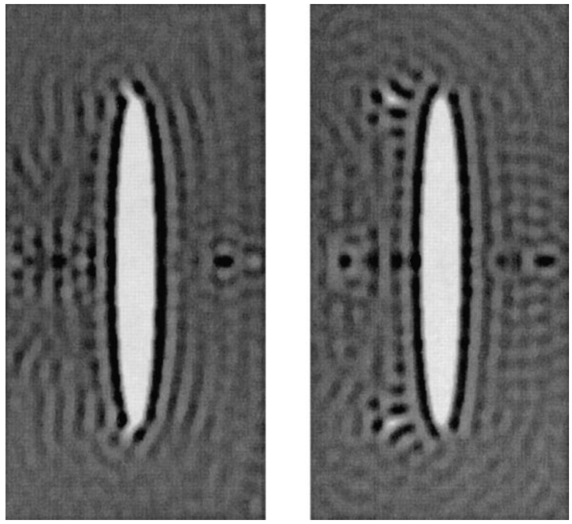

Fig. 4. The density plot in a cross-section of the solution of (2) for the flow around an oblate spheroid (see text) moving to the right with velocity $0.156c$ at $t = 100$ (left) and $t = 300$ (right). The white circles show the core of a vortex ring nucleated from the spheroid and gradually falling astern of it.

Acknowledgements

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