THE BEHAVIOR OF THE FREE BOUNDARY CLOSE TO A FIXED BOUNDARY IN A PARABOLIC PROBLEM

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ABSTRACT. A parabolic obstacle-type problem without sign restriction on the solution is considered. An exact representation of the global solutions (i.e., solutions in the entire half-space $\{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$) is found. Finally, the local properties of the free boundary near a fixed boundary is studied. Under the homogeneous Dirichlet condition on the given boundary the smoothness of the free one is proved.

1. INTRODUCTION

In this paper, a parabolic obstacle-type problem with no restriction on the sign of the solution is considered. More precisely, we study the regularity properties of free boundary in a neighborhood of the fixed boundary of a domain. The exact mathematical formulation follows.

Let function u and an open set $\Omega \subset \mathbb{R}^{n+1}_+$ solve the problem:

$$\begin{gathered} H(u) = \chi_\Omega \quad \text{in} \quad Q_1^+, \\ u = |Du| = 0 \quad \text{in} \quad Q_1^+ \setminus \Omega, \qquad u = 0 \quad \text{on} \quad \Pi \cap Q_1, \end{gathered} \tag{1.1}$$

where $H = \Delta - \partial_t$ is the heat operator, χ_Ω denotes the characteristic function of Ω, Q_1 is the unit cylinder in $\mathbb{R}^{n+1}$, $Q_1^+ = Q_1 \cap \{x_1 > 0\}$, $\Pi = \{(x,t) : x_1 = 0\}$. The first equation in (1.1) is satisfied in the sense of distributions.

The regularity of the free boundary for the problem (1.1) was investigated earlier only in the special case of the parabolic obstacle problem (see [3]). There the additional information $u \geqslant 0$ permits to establish the Lipschitz regularity of the free boundary in a neighborhood of Π as well as the $C^{1,\alpha}$-regularity for the parts of a free boundary lying inside Q_1^+.

The results for an elliptic problem related to (1.1) were obtained in [9]. The parabolic problem without sign restriction on the solution was studied in [5] where the interior case was considered. It was supposed

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there that the solution is defined in the whole cylinder Q_1, without conditions on the fixed boundary Π.

The main strategy used in [5] and in the present paper follows. The first step is the proof of the optimal regularity of the solution u. In [5] this result is local in Q_1. The corresponding estimates up to Π for the solutions of (1.1) were obtained in [1].

The next step is to investigate the global solutions u, defined for all $t \leqslant 0$ and for either $x \in \mathbb{R}^n$ or $x \in \mathbb{R}^n_+ = \mathbb{R}^n \cap \{x_1 \geqslant 0\}$. In the latter case u satisfies the boundary condition $u = 0$ on Π. Also, in both cases, it is supposed that $|u(x,t)| \leqslant M(1+|x|^2+|t|)$. The complete description of homogeneous global solutions defined in $\mathbb{R}^n \times \{t \leqslant 0\}$ was given in [5]. Such global solutions appear naturally as blow-up limits of local solutions at a fixed point of the free boundary. It is an open problem to find such a description for general, nonhomogeneous, global solutions defined in $\mathbb{R}^n \times \{t \leqslant 0\}$. But in our case, where solutions defined in $\mathbb{R}^n_+ \times \{t \leqslant 0\}$ vanish on Π, we give the complete representation of solutions (see Theorem I). The global analysis is based essentially on the monotonicity formulas due to L. Caffarelli and G. Weiss. In the present paper we apply also the local version of Weiss's formula from [3].

As a last step we study the local properties of the free boundary. We essentially use the closeness of the solution to the corresponding blow-up limit at a point z^0 of the free boundary. In the interior case considered in [5] there is an additional requirement on the thickness of the complement of Ω in a neighborhood of z^0. In the case considered in this paper this additional condition is replaced by the boundary condition $u = 0$ on Π. We prove a certain directional monotonicity of the solution which implies the Lipschitz continuity of the free boundary. Namely, we get (see Theorem II) that in a neighborhood of Π the free boundary Γ is a Lipschitz graph $x_1 = f(x', t)$, where the Lipschitz constant of f with respect to x-variables is an absolute one while the Lipschitz constant of f with respect to t depends on the distance to Π. Finally, in Theorem III we prove the higher regularity of the free boundary. It should be noted that Theorem III and remark after that ensure the C^∞-regularity of the free boundary in space and time only at the interior points. The corresponding counterexample, showing that even $C^{1,\alpha}$-regularity with $0 < \alpha < 1$ may fail to occur at the points of contact between the free boundary and the fixed boundary, can be found in [3].

1.1. **Notations and definitions.**

Throughout the paper we will use the following notations:
$z = (x, t)$ are points in $\mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^1$;
if $n \geqslant 2$ then $x = (x_1, x') = (x_1, x_2, \dots, x_n)$;
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$;

$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$;
$\mathbb{R}_- = (-\infty, 0]$;
$\Pi = \{(x,t) \in \mathbb{R}^{n+1} : x_1 = 0\}$;
$e_1, \dots, e_n$ is the standard basis in $\mathbb{R}^n_x$;
e_0 is the standard basis in $\mathbb{R}^1_t$;
χ_Ω denotes the characteristic function of the set Ω ($\Omega \subset \mathbb{R}^{n+1}$);
$v_+ = \max\{v, 0\}$; $\qquad v_- = \max\{-v, 0\}$;
$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$B^+_r(x^0) = B_r(x^0) \cap \mathbb{R}^n_+$;
$B_r = B_r(0)$;
$K_r(z^0) = \{|x_1 - x^0_1| < r\} \times \{|x' - (x^0)'| < r\} \times]t^0 - r^2, t^0[$;
$K^+_r(z^0) = K_r(z^0) \cap \mathbb{R}^{n+1}_+$;
$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0]$;
$Q^+_r(z^0) = Q^+_r(x^0, t^0) = Q_r(x^0, t^0) \cap \mathbb{R}^{n+1}_+$;
$Q_r = Q_r(0,0)$;
$Q^+_r = Q^+_r(0,0)$.

We emphasize that in this paper the top of the cylinder $Q_r(z^0)$ is included in the set $Q_r(z^0)$, while the top of the cylinder $K_r(z^0)$ is not included in the set $K_r(z^0)$. If $Q = \mathbb{R}^{n+1}_+ \cap Q_r(z^0)$, then $\partial' Q$ is the parabolic boundary of Q, i.e., $\partial' Q = \overline{Q} \setminus Q$.
D_i denotes the differential operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$;
$D = (D_1, D') = (D_1, D_2, \dots, D_n)$ denotes the spatial gradient; D_ν stands for the operator of differentiation along the direction $\nu \in \mathbb{R}^{n+1}$, i.e., $|\nu| = 1$ and

$$D_\nu u = \sum_{i=1}^{n} \nu_i D_i u + \nu_0 \partial_t u;$$

$D^2 u = D(Du)$ denotes the Hessian of u;
$H = \Delta - \partial_t$ is the heat operator.

We adopt the convention that the index τ runs from 2 to n. We also adopt the convention regarding summation with respect to repeated indices.

$\|\cdot\|_{p,E}$ denotes the norm in $L_p(E)$, $1 < p \leqslant \infty$;
$W^{2,1}_p(E)$ is the anisotropic Sobolev space with the norm

$$\|u\|_{W^{2,1}_p(E)} = \|\partial_t u\|_{p,E} + \|D(Du)\|_{p,E} + \|u\|_{p,E};$$

We use letters M, N, A, and C (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parentheses: $C(\dots)$.

Let M be a constant, $M \geqslant 1$.

Definition 1. We say a function $u \in W^{2,1}_\infty(Q^+_R)$ belongs to the class $P^+_R(M)$ if u satisfies:

(a) $H[u] = \chi_\Omega$ in Q^+_R, for some open set $\Omega = \Omega(u) \subset Q^+_R$;

(b) $u = |Du| = 0$ in $Q_R^+ \setminus \Omega$;
(c) $u = 0$ on $\Pi \cap Q_R$;
(d) $\operatorname*{ess\,sup}\limits_{Q_R^+}(|\partial_t u| + |D^2 u|) \leqslant M$

and the equation in (a) is understood in the sense of distributions. The elements of $P_R^+(M)$ will be called *local solutions*.

Definition 2. Let $P_\infty^+(M)$ be the class of functions $u \in W_\infty^{2,1}(\mathbb{R}_+^n \times \mathbb{R}_-)$ such that

(a') $H[u] = \chi_\Omega$ in $\mathbb{R}_+^{n+1}$ for some open set $\Omega = \Omega(u) \subset \mathbb{R}_+^n \times \mathbb{R}_-$;
(b') $u = |Du| = 0$ in $\mathbb{R}_+^{n+1} \setminus \Omega(u)$;
(c') $u = 0$ on Π;
(d') $\operatorname*{ess\,sup}\limits_{\mathbb{R}_+^n \times \mathbb{R}_-}(|\partial_t u| + |D^2 u|) \leqslant M$,

where the equation in (a') is understood in the sense of distributions. The elements of $P_\infty^+(M)$ will be called *global solutions*.

In both cases we shall use the following notations:

- $\Lambda(u) = \{(x,t) : u(x,t) = |Du(x,t)| = 0\}$;
- $\Gamma(u) = \{z = (x,t) \in \Lambda(u) : Q_\rho(z) \cap \Omega(u) \neq \emptyset \quad \forall \rho > 0\}$ is the free boundary; that is $\Gamma(u)$ is the part of $\partial\Omega(u) \cap \Lambda(u)$ which "can be seen from below".
- $\Gamma(u) \cap \Pi$ is the set of contact points.

We also introduce the class $P_\infty(M)$. Here we consider $\mathbb{R}^{n+1} \cap \{t \leqslant 0\}$ instead of $\mathbb{R}_+^{n+1} \cap \{t \leqslant 0\}$ and omit the condition $u|_\Pi = 0$. The class $P_R(M)$ with $R < \infty$ is defined similarly .

Definition 3. Let $z^* = (x^*, t^*)$ be a point in $\mathbb{R}^{n+1}$. We say that a function v is *parabolic homogeneous of degree 2 with respect to* z^* if either of the following statements is satisfied:

(i) a function v is defined in $\mathbb{R}^{n+1} \cap \{t \leqslant t^*\}$ and the identity

$$v(\lambda x + x^*, \lambda^2 t + t^*) = \lambda^2 v(x + x^*, t + t^*) \tag{1.2}$$

holds for all $x \in \mathbb{R}^n$, $\lambda > 0$, and for all $t \leqslant 0$.

(ii) $x_1^* \geqslant 0$, a function v is defined in $\{(x,t) \in \mathbb{R}^{n+1} : x_1 \geqslant 0, t \leqslant t^*\}$, and the identity (1.2) holds for all $\lambda > 0$, $t \leqslant 0$, and for all $x \in \mathbb{R}^n$ such that $x_1 + x_1^* \geqslant 0$, $\lambda x_1 + x_1^* \geqslant 0$.

1.2. **Main results.**

Our prime goal in this paper is to prove the following theorems:

Theorem I. *Let $u \in P_\infty^+(M)$, and let $z^0 = (x^0, t^0) \in \Gamma(u)$. Then u is independent of t and the variables $x_3, \dots, x_n$. More precisely, for $(x,t) \in \mathbb{R}_+^{n+1} \cap \{t \leqslant t^0\}$ we have*

$$x_1^0 = 0 \quad \Longrightarrow \quad u(x,t) = \frac{x_1^2}{2} + a x_1 x_2 \tag{1.3}$$

in some suitable rotated coordinate system in $\mathbb{R}^n$ *that leaves* e_1 *fixed, and for some real number* a,

$$x_1^0 > 0 \quad \Longrightarrow \quad u(x,t) = \frac{((x_1 - x_1^0)_+)^2}{2}. \tag{1.4}$$

Theorem II. *Let* $u \in P_2^+(M)$, *let* $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/8}^+$, *and let* $r := x_1^0$.

There exist a positive constant $\delta = \delta(n, M)$ *and a Lipschitz continuous nonnegative function* f *defined on* $\{|x' - (x^0)'| < r\} \times]t^0 - r^2, t^0]$ *such that if* $r < \delta/2$ *then*

$$u \geqslant 0 \qquad \text{in} \quad K_{3r/2}^+(z^0)$$

and

$$\Omega(u) \cap K_r(z^0) = \{(x,t) \in K_r(z^0) : x_1 > f(x', t)\}.$$

Here the Lipschitz constant of f *with respect to the* x*-variables depends on* n *and* M *only, while the Lipschitz constant of* f *with respect to* t *equals* $C(n, M)r^{-1}$.

Theorem III. *Let* f *be the same function as in Theorem II. Then in a neighborhood of every point* (x', t) *satisfying* $f(x', t) \in K_r(z^0)$, *the function* f *belongs to the class* $C^{1,\alpha}$ *for some* $0 < \alpha < 1$.

Remark. In fact, applying the Kinderlehrer-Nirenberg technique (see [8]), one can show that $\partial\Omega(u) \cap K_r(z^0)$ is not only $C^{1,\alpha}$ but also space-time C^∞-regular.

2. Useful facts.

For the reader's convenience and for future references, we recall and explain some facts.

2.1. Nondegeneracy.

Fact 2.1. *Let* $u \in P_R^+(M)$ *with* $0 < R \leqslant +\infty$. *Then there exists a constant* $C(n) > 0$ *such that*

(i) *for all* $z^0 \in \overline{\{u > 0\}}$, *and for all* ρ *satisfying* $Q_\rho(z^0) \subset Q_R$ *we have*

$$\sup_{Q_\rho^+(z^0)} u \geqslant u(z^0) + C(n)\rho^2;$$

(ii) *for all* $z^0 \in \Lambda(u)$, *and for* $Q_\rho(z^0)$ *as above we have either*

$$\sup_{Q_\rho^+(z^0)} u \geqslant C(n)\rho^2 \tag{2.1}$$

or $u \equiv 0$ *in* $Q_{\rho/2}^+(z^0)$;

(iii) *for all* $z^0 \in \Gamma(u)$, *and for* $Q_\rho(z^0)$ *as above, inequality (2.1) holds true;*

(iv) *if* $u(x,t) < 0$ *for all* $(x,t) \in Q_\rho(z^0) \subset Q_R^+$ *then we have*

$$u(z^0) \leqslant -C(n)\rho^2.$$

Proof. The proofs of cases (i) and (iii) with $C(n) = \frac{1}{2n+1}$ are just the same as the proof of Fact 2 in [2].

The proof of Case (ii) for $z^0 \notin \Pi$ follows from [5] (see the proof of Lemma 5.1). For $z^0 \in \Pi$ we have to use additionally the Hopf lemma.

It remains only to verify case (iv). It is clear that $H[u] = 1$ in $Q_\rho(z^0)$. Further, we consider the function

$$w(x,t) = u(x,t) - u(x^0,t^0) - \frac{1}{2n+1}(|x-x^0|^2 - (t-t^0)).$$

Then w is caloric in $Q_\rho(z^0)$, and $w(x^0,t^0) = 0$. Moreover, the maximum principle yields

$$\begin{aligned} 0 \leqslant \sup_{Q_\rho(z^0)} w &= \sup_{\partial' Q_\rho(z^0)} w \\ &\leqslant \sup_{\partial' Q_\rho(z^0)} u - u(x^0,t^0) - \inf_{\partial' Q_\rho(z^0)} \frac{(|x-x^0|^2 - (t-t^0))}{2n+1} \\ &\leqslant -u(x^0,t^0) - \frac{\rho^2}{2n+1}. \end{aligned}$$

Choosing $C(n) = \frac{1}{2n+1}$ we are done. □

Fact 2.2.

(i) *Suppose that $u \in P_R^+(M)$. If $z^0 = (x^0,t^0) \in \partial\Omega(u) \setminus \Gamma(u)$, then there exists $r_0 = r_0(z^0) > 0$ such that $u \equiv 0$ in $Q_{r_0}^+(z^0)$. If, in addition, $t^0 < 0$ then $u < 0$ in $Q_{r_0}^+(x^0, t^0 + r_0^2)$.*

(ii) *Suppose that $v_k \in P_r(M)$ and $z^k \in \Gamma(v_k)$ for $k = 1, 2, \ldots$. If $v_k \to v$ in $C_{loc}(Q_r)$ and $z^k \to z^0$, as $k \to \infty$, then $v \in P_r(M)$ and $z^0 \in \Gamma(v)$.*

Proof. The proofs of both statements follows from Fact 2.1 and [5] (see subsections 5.1, 7.1 and 5.2, respectively). □

2.2. **Monotonicity formulas.** So-called monotonicity formulas will play an essential role in this paper and will appear in almost every section.

We use two kinds of monotonicity formulas: the first due to G.Weiss [10] and the second due to L.Caffarelli (see [6], [7] and [2], Lemma 2.1).

To formulate the first monotonicity formula, we define Weiss's functional as follows:

$$W(r,x^0,t^0,v) := \frac{1}{r^4} \int\limits_{t^0-4r^2}^{t^0-r^2} \int\limits_{\mathbb{R}^n} \left(|Dv|^2 + 2v + \frac{v^2}{t-t^0} \right) G(x-x^0, t^0-t)\,dxdt,$$

where (x^0,t^0) is a point in $\mathbb{R}^{n+1}_+$, r is a positive constant, the heat kernel $G(x,t)$ is defined as

$$G(x,t) = \frac{\exp(-|x|^2/4t)}{(4\pi t)^{n/2}} \text{ for } t > 0 \text{ and } G(x,t) = 0 \text{ for } t \leqslant 0, \tag{2.2}$$

and v is a continuous function defined on $\mathcal{Q} := \mathbb{R}^n \times [t^0 - 4\mathcal{R}^2, t^0]$, $\mathcal{R} \geqslant r$. We also suppose that $D_i v \in L_{2,loc}(\mathcal{Q})$ and Dv have at most polynomial growth with respect to x, as $|x| \to \infty$.

It is easy to check that for any $\lambda \in]0, \mathcal{R}/r]$ the functional W has the following scaling property:

$$W(\lambda r, x^0, t^0, v) = W(\lambda, 0, 0, v_r), \tag{2.3}$$

where

$$v_r(x,t) = \frac{v(rx + x^0, r^2 t + t^0)}{r^2} \tag{2.4}$$

is the parabolic scaling of v around $z^0 = (x^0, t^0)$.

Suppose now that a function $u \in P^+_\infty(M)$ and $z^0 = (x^0, t^0) \in \partial\Omega(u) \cap \Lambda(u)$. Then we extend u as zero across the plane Π into the set $\mathbb{R}^{n+1} \cap \{x_1 < 0, t \leqslant 0\}$; we will preserve the notation u for this extension. From Lemma 1.2 [3] it follows that

$$\begin{aligned}\frac{dW(r, x^0, t^0, u)}{dr} &= \frac{1}{r}\int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{|\mathcal{L}u_r|^2}{-t} G(x,-t)\,dxdt \\ &+ \frac{x_1^0}{r^2}\int_{-4}^{-1}\int_{x_1 = \frac{-x_1^0}{r}} |D_1 u_r|^2 G(x,-t)\,dx'dt \geqslant 0,\end{aligned} \tag{2.5}$$

where u_r is as in (2.4), and

$$\mathcal{L}u_r(x,t) := x \cdot Du_r(x,t) + 2t\partial_t u_r(x,t) - 2u_r(x,t).$$

Relation (2.5) guaranties that the functional W is monotone nondecreasing with respect to r. In particular, the equality $\frac{dW}{dr} = 0$ for all $r > 0$ is equivalent to

$$\begin{aligned}\mathcal{L}u_r(x,t) = 0 \quad &\text{for} \quad (x,t) \in \mathbb{R}^n \times]-4,-1[, \\ D_1 u_r = 0 \quad &\text{on} \quad \{x_1 = \frac{-x_1^0}{r}\}.\end{aligned} \tag{2.6}$$

It is evident that the first equality in (2.6) gives the degree 2 parabolic homogeneity of the function u with respect to z^0.

For our purposes it is also essential to introduce a local version of the Weiss functional. In particular, this permits us to derive the homogeneity of the blow-up and blow-down limits. Similarly to [3] we define the local Weiss functional as follows:

$$W_b(r, x^0, t^0, v) := \frac{1}{r^4}\int_{t^0-4r^2}^{t^0-r^2}\int_{B_b(x^0)} \left(|Dv|^2 + 2v + \frac{v^2}{t - t^0}\right) G(x - x^0, t^0 - t)\,dxdt,$$

where b and r are positive constants, $z^0 = (x^0, t^0)$ is a point in $\mathbb{R}^{n+1}$, the function G is the same as in (2.2), and v is a continuous function defined on $\mathcal{Q}_b(z^0) := B_b(z^0)\times]t^0-4\mathcal{R}^2, t^0[$, $\mathcal{R} \geqslant r$ and satisfying $|Dv| \in L_2(\mathcal{Q}_b(z^0))$.

Let us note that for the local Weiss functional and for any $\lambda \in]0, \mathcal{R}/r]$ Lemma 1.1 [3] guarantees the scaling property

$$W_b(\lambda r, x^0, t^0, v) = W_{b/r}(\lambda, 0, 0, v_r), \tag{2.7}$$

with v_r defined by (2.4).

To apply the local Weiss functional W_b to $u \in P_b^+(M)$ we always assume that u is extended as zero across the plane Π into the set $Q_b \cap \{x_1 < 0\}$. We preserve the notation u for this extension.

In [3] (see Lemma 1.2 and remark after that) the following estimate was proved: if $u \in P_2^+(M)$ and $(x^0, t^0) \in \Lambda(u) \cap Q_1$, then for arbitrary ρ and α satisfying $\frac{1}{2} \geqslant \rho > \alpha > 0$ we have

$$W_1(\rho, x^0, t^0, u) - W_1(\alpha, x^0, t^0, u) \geqslant -C_0(n, M)(\rho - \alpha). \tag{2.8}$$

From here it follows that the function $W_1(r, x^0, t^0, u)$ has a limit as $r \to 0^+$ (see Corollary 1.3 [3]). The corresponding limit

$$\omega(x^0, t^0, u) := \lim_{r\to 0^+} W_1(r, x^0, t^0, u) \tag{2.9}$$

will be called the *balanced energy* of the function u at the point $(x^0, t^0) \in \Lambda(u) \cap \partial\Omega(u)$.

Remark. It is easy to see that $\omega(x^0, t^0, u) = \lim\limits_{r\to 0^+} W_b(r, x^0, t^0, u)$ for an arbitrary fixed $b > 0$. Moreover, for global solutions from $P_\infty^+(M)$ the convergence of the Weiss functional W to the balanced energy as $r \to 0^+$ is monotone.

For the second monotonicity formula we denote

$$I(r, v, z^0) = \int\limits_{t^0-r^2}^{t^0} \int\limits_{\mathbb{R}^n} |Dv(x,t)|^2 G(x - x^0, t^0 - t)\, dx dt,$$

where $r \in]0, R]$, $z^0 = (x^0, t^0) \in \mathbb{R}^{n+1}$, the function v is defined in $E = \mathbb{R}^n \times [t^0 - R^2, t^0]$, and $G(x,t)$ is defined in (2.2).

Suppose now that h_1 and h_2 are nonnegative sub-caloric functions in E with a at most polynomial growth at infinity, satisfying

$$h_1(z^0) = h_2(z^0) = 0 \qquad \text{and} \qquad h_1 \cdot h_2 \equiv 0.$$

Then the functional

$$\Phi(r) = \Phi(r, h_1, h_2, z^0) = \frac{1}{r^4} I(r, h_1, z^0) I(r, h_2, z^0) \tag{2.10}$$

is monotone nondecreasing in r. More precisely, if the supports of $h_1(\cdot, t^0 - r^2)$ and $h_2(\cdot, t^0 - r^2)$ are not complementary halfspaces in $\mathbb{R}^n$ containing x^0 on their boundaries, then either $\Phi'(r) > 0$, or $\Phi(\rho) \equiv 0$ for $\rho \in (0, r]$.

2.3. **Blow-up and blow-down.** For a function $u \in P^+_R(M)$ with $0 < R \leqslant +\infty$ and for a point $z^0 = (x^0, t^0) \in \partial\Omega(u) \cap \Lambda(u)$ we consider the parabolic scaling u_r defined by (2.4).

By a standard compactness arguments we may pass to the limit along a subsequence $r_k \to 0$, obtaining as a result a global solution u_0 from $P_\infty(M)$ or $P^+_\infty(M)$ respectively for the cases $x^0_1 > 0$ and $x^0_1 = 0$. In both cases the point $(0, 0) \in \partial\Omega(u_0)$. Moreover, if $z^0 \in \Gamma(u)$ then due to item (ii) in Fact 2.2 we have $(0, 0) \in \Gamma(u_0)$. Otherwise from item (i) in Fact 2.2 we get $u_0 \equiv 0$ for $t \leqslant 0$. Usually, such a process is referred to as passage to blow-up limit. Any global solution u_0 thus obtained is called a *blow-up* of the function u at the point z^0.

Similarly, for a global solution $u \in P^+_\infty(M)$ and $z^0 = (x^0, t^0) \in \Gamma(u)$ we can consider the scaled functions u_r around z^0 and let $r \to +\infty$. Then u_r converge (for a subsequence) to a function u_∞ uniformly on compact subsets of $(\mathbb{R}^n_+ \cup \Pi) \times \mathbb{R}_-$. It is easy to see that $u_\infty \in P^+_\infty(M)$ and $(0, 0) \in \Gamma(u_\infty)$. The limit function u_∞ is called *blow-down* of the global solution u at the point z^0.

In general, different subsequences r_k may result different blow-up and blow-down limits at the same point.

Fact 2.3. *The functions u_0 and u_∞ are degree 2 parabolic homogeneous with respect to the origin.*

Proof. We prove Fact 2.3 for the blow-up of u at the point z^0. The case of u_∞ is treated in the same way.

It suffices to observe that for any $\lambda > 0$ we have

$$\begin{aligned} W(\lambda, 0, 0, u_0) &= \lim_{r\to 0+} W_{1/r}(\lambda, 0, 0, u_r) = \lim_{r\to 0+} W_1(\lambda r, x^0, t^0, u) \\ &= \text{const} = \omega(x^0, t^0, u). \end{aligned}$$

The second equality in the above relation follows from scaling property (2.7), while the regularity properties of u provide the first equality. □

Fact 2.4. *For $u \in P^+_2(M)$ and any point $z^0 = (x^0, t^0) \in \partial\Omega(u) \cap Q^+_2$ the following statements hold:*

(i) *the balanced energy $\omega(x^0, t^0, u)$ may equal one of the following values 0, $A := \frac{15}{4}$ or $2A$. Correspondingly, z^0 is called a zero energy point, a low energy point or a high energy point.*

(ii) *If $z^0 \in \partial\Omega(u) \setminus \Gamma(u)$ then $\omega(x^0, t^0, u) = 0$, while $\Gamma(u)$ contains low and high energy points only.*

(iii) *If z^0 is a zero energy point then $u \equiv 0$ in $Q_r(z^0)$ for some $r > 0$. If, in addition $t^0 < 0$, then $u < 0$ in $Q_r(x^0, t^0 + r^2/2)$.*

(iv) *If z^0 is a low energy point then the sets $\Lambda(u)$ and $\{u > 0\}$ have non-empty interiors in $Q_\rho(z^0)$ for all $\rho > 0$.*

(v) *If the set $\Lambda(u)$ does not have non-empty interior in $Q_\rho(z^0)$ for some $\rho > 0$ then z^0 is a high energy point.*

Proof. See Section 7, [5]. □

Fact 2.5. *Let* $u \in P_2^+(M)$ *and* $z^0 = (x^0, t^0) \in \Gamma(u)$. *If for some* $\rho > 0$ *the cylinder* $Q_\rho(x^0, t^0 + \rho^2/2)$ *contains only low energy points, then*

$$\lim_{\Omega(u) \ni z \to z^0} \partial_t u(z) = 0.$$

Proof. See the proof of 7.7 [5]. □

3. Global solutions

Lemma 3.1. *Let* $u \in P_\infty^+(M)$ *be a degree 2 parabolic homogeneous function with respect to the origin, and let* $(0,0) \in \Gamma(u)$. *Then for some* $a \in \mathbb{R}$ *we have*

$$u(x,t) = \frac{x_1^2}{2} + a x_1 x_2 \quad in \quad \mathbb{R}^n_+ \times \mathbb{R}_-. \tag{3.1}$$

Proof. Let e be a unit spatial direction orthogonal to e_1. We claim that the function $v := D_e u$ does not change its sign. To prove this, first we extend v by zero across the plane Π to the entire space $\mathbb{R}^n \times \mathbb{R}_-$ and keep the notation v for the extension. From homogeneity of u with respect to the origin it follows that

$$\Phi(\lambda, v_+, v_-, 0, 0) = C(e), \tag{3.2}$$

where Φ is as in (2.10). However, in our case the equality (3.2) is possible only if $C(e) = 0$, see subsection 2.2. This implies $v \geqslant 0$ or $v \leqslant 0$ for all points of $\mathbb{R}^n_+ \times \mathbb{R}_-$. So, we have proved that $D_e u$ preserves its sign. Since this is true for all spatial directions e orthogonal to e_1, it follows that $u(x,t)$ is two-space dimensional, i.e., in suitable spatial coordinate

$$u(x,t) = u(x_1, x_2, t). \tag{3.3}$$

For definiteness, we assume in the rest of the proof that

$$D_2 u \geqslant 0 \tag{3.4}$$

(otherwise we replace e_2 by $-e_2$).

Further, we consider the case where the interior of $\Lambda(u)$ is empty. For $(x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_-$ we define the function w by the formula

$$w(x,t) = u(x,t) - \frac{x_1^2}{2}.$$

It is easy to see that w is caloric in $\mathbb{R}^n_+ \times \mathbb{R}_-$ and has at most quadratic growth with respect to x and at most linear growth with respect to t. Then we extend w by the odd reflection to the whole space $\mathbb{R}^{n+1} \cap \{t \leqslant 0\}$ and preserve the notation w for the extended function. By the Liouville theorem (see Lemma 2.1 in [1]), the function w, and, consequently, the function u, is a polynomial of degree 2. Taking into account the homogeneity of u, the equality (3.3) and the condition $u|_\Pi = 0$ we get the exact representation

$$u(x,t) = \frac{x_1^2}{2} + ax_1x_2, \qquad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_-$$

for some constant a.

Now we claim that the interior of $\Lambda(u)$ is always empty. Suppose, towards a contradiction, that we may fix a cylinder $Q_{2r}(z^0)$ in the interior of $\Lambda(u)$. Without loss of generality we may suppose that $t^0 < 0$. In this part our arguments are similar to that of the proof of Theorem B in [9]. Due to (3.4) we must have $u \leqslant 0$ in

$$\mathcal{G}_{2r}(z^0) := \{(x_1, x_2 - s, t) : (x_1, x_2, t) \in Q_{2r}(z^0), s \geqslant 0\}.$$

From here one infers that for the smaller set $\mathcal{G}_r(z^0)$ the following holds

$$\partial\Omega(u) \cap \mathcal{G}_r(z^0) = \emptyset. \tag{3.5}$$

Indeed, if there exists $z^* \in \partial\Omega(u) \cap \mathcal{G}_r(z^0)$, then the maximum principle applied to the sub-caloric function u in $Q_r(z^*)$ gives that $u \equiv 0$ in $Q_r(z^*)$. Then, by homogeneity of u, it vanishes also in $B_{\lambda r}(\lambda x^*) \times \{\lambda^2 t^*\}$ for any $\lambda > 0$. Hence $z^* \notin \partial\Omega(u)$.

Combining (3.5) and the fact $Q_r(z^0) \subset \Lambda(u)$, we conclude that $\mathcal{G}_r(z^0) \subset \Lambda(u)$. Hence we can translate u in the x_2-direction by considering the functions $u_m(x,t) = u(x_1, x_2 - m, t)$ in $\mathbb{R}^n_+ \times \mathbb{R}_-$. Since u is homogeneous with respect to the origin, each element of $\{u_m\}$ satisfies

$$u_m(\lambda x, \lambda^2 t) = \lambda^2 u(x_1, x_2 - \lambda^{-1} m, t) = \lambda^2 u_{m/\lambda}(x_1, x_2, t). \tag{3.6}$$

In addition, for all $x_2 \leqslant x_2^0$ we have

$$|u(x,t)| \leqslant M(|x_1 - x_1^0|^2 + |t - t^0|),$$

and, hence

$$|u_m(x,t)| \leqslant C(1 + x_1^2 + |t|). \tag{3.7}$$

Due to (3.4) and (3.7), the sequence $\{u_m\}$ is non-increasing and bounded for any fixed x_1 and t. Therefore, by compactness $\{u_m\}$ converges to a limit function $\widetilde{u}$, which is a global solution independent of x_2. It should be mentioned also that $\widetilde{u}$ is parabolic homogeneous of degree 2 function with respect to the origin, as provided by (3.6). In other words, $\widetilde{u}$ is a one-space dimensional homogeneous global solution with $Q_r(z^0) \subset \Lambda(\widetilde{u})$, satisfying $\widetilde{u}(0,t) = 0$ for $t \leqslant 0$.

If $\widetilde{u}$ does not vanish identically in $\mathbb{R}^1_+ \times \mathbb{R}_-$ then, in view of the rest of the proof of Lemma 6.3 [5], we get the representation $\widetilde{u}(x,t) = (x_1)^2/2$. The latter contradicts to the fact $Q_r(z^0) \subset \Lambda(\widetilde{u})$.

If $\widetilde{u} \equiv 0$ in $\mathbb{R}^1_+ \times \mathbb{R}_-$ then due to the monotone convergence of u_m we may conclude that $u \geqslant 0$ in $\mathbb{R}^1_+ \times \mathbb{R}_-$. Now we can apply Theorem 2 [2] to the function u. This gives the exact representation $u(x,t) = (x_1)^2/2$, which contradicts to our assumption about the interior of $\Lambda(u)$. $\square$

Lemma 3.2. *Let $u \in P^+_\infty(M)$ and $z^0 = (x^0, t^0) \in \Gamma(u)$. Then the function u is a parabolic homogeneous function of degree 2 with respect to z^0, and*

$$W(r, x^0, t^0, u) = \frac{15}{4} =: A \quad \textit{for any} \quad r > 0. \tag{3.8}$$

Proof. We only need to prove (3.8). Then the first statement of our lemma follows immediately (see subsection 2.2).

Due to the scaling property (2.3) and the monotonicity of the Weiss functional we have

$$\begin{aligned} W(1,0,0,u_0) \leqslant W(1,0,0,u_r) &= W(r,x^0,t^0,u) \\ &\leqslant W(1,0,0,u_\infty). \end{aligned} \tag{3.9}$$

Here u_0 and u_∞ are blow-up and blow-down limits of the function u at the point z^0 respectively. We recall that according to Fact 2.3 the functions u_0 and u_∞ are degree 2 parabolic homogeneous with respect to the origin.

If $x^0_1 > 0$ then $u_0 \in P_\infty(M)$. Hence we can apply successively Fact 2.3, Lemma 6.3 [5] and Lemma 6.2 [5] to the function u_0, which yields the bound

$$W(1,0,0,u_0) \geqslant A. \tag{3.10}$$

Indeed, otherwise we will have that the function u_0 belongs to the class $P^+_\infty(M)$ and $(0,0) \in \Gamma(u_0)$. In this case Lemma 3.1 guarantees the exact representation $u_0(x,t) = \frac{x_1^2}{2} + ax_1x_2$ where a is a constant. Direct computations (see, for example, Lemma 6.2 [5]) now give

$$W(1,0,0,u_0) = A. \tag{3.11}$$

Similarly, the application of Lemma 3.1 to the function $u_\infty \in P^+_\infty(M)$ with $(0,0) \in \Gamma(u_\infty)$ and direct computations provide the identity

$$W(1,0,0,u_\infty) = A. \tag{3.12}$$

Finally, combining together the inequalities (3.9)-(3.12) we get the desired result. □

Proof of Theorem I. Let us suppose first that $x^0_1 = 0$. In this case the successive application of Lemma 3.2 and Lemma 3.1 to the function u gives the desired representation.

We now turn to the case $x^0_1 > 0$. By Lemma 3.2 the function u is parabolic homogeneous of degree 2 with respect to z^0. This together with the assumption $z^0 \in \Gamma(u)$ guarantees that $u(x,t) \equiv 0$ in the infinite set $\{(x,t) : 0 < x_1 \leqslant x^0_1, t \leqslant t^0\}$. From here it follows that the function $\widetilde{u}(x,t) = u(x + x^0, t + t^0)$ belongs to the class $P^+_\infty(M)$ and $(0,0) \in \Gamma(\widetilde{u})$. It is obvious that $\widetilde{u}$ is parabolic homogeneous of degree 2 with respect to the origin. Therefore, applying Lemma 3.1 to $\widetilde{u}$, we get that in some suitable rotated coordinate system that leaves e_1 fixed $\widetilde{u}$ has the following representation

$$\widetilde{u}(x,t) = \frac{x_1^2}{2} + ax_1x_2 \qquad \text{as} \quad x_1 \geqslant 0,\ t \leqslant 0.$$

Consequently, for $t \leqslant t^0$ we have

$$u(x,t) = \begin{cases} \dfrac{(x_1 - x_1^0)^2}{2} + a(x_1 - x_1^0)(x_2 - x_2^0), \text{ if } \quad x_1 \geqslant x_1^0, \\ 0, \text{ otherwise.} \end{cases} \tag{3.13}$$

Since Du is a continuous function we can conclude that in (3.13) the parameter $a = 0$. This finishes the proof. ☐

4. Characterization of the free boundary points near Π

The information about global solutions that we obtained in Theorem I can be applied to study the behavior of the free boundary near the fixed boundary Π. First we present a preliminary result concerning contact points $z^0 \in \Pi \cap \Gamma(u)$. We show that the free boundary touches the fixed one in a parabolically tangential manner.

Lemma 4.1. *For any $\varepsilon > 0$ there exists $\rho = \rho_\varepsilon > 0$ such that if $u \in P_1^+(M)$ and $z^0 = (x^0, t^0) \in \Pi \cap \Gamma(u) \cap Q_{1/2}$ then*

$$\Gamma(u) \cap \mathbb{K}_\varepsilon(z^0) \cap \{0 < x_1 < \rho_\varepsilon\} = \emptyset. \tag{4.1}$$

Here

$$\mathbb{K}_\varepsilon(z^0) := \left\{(x,t) : x_1 > \varepsilon\sqrt{|x' - (x^0)'|^2 + |t - t^0|}\right\}.$$

Proof. Suppose, towards a contradiction, that the statement of Lemma 4.1 is not true for some $\varepsilon > 0$. Then there exist functions $u_j \in P_1^+(M)$, points $\widehat{z^j} = (\widehat{x^j}, \widehat{t^j}) \in \Pi \cap \Gamma(u_j) \cap Q_{1/2}$, and points $z^j = (x^j, t^j) \in \Gamma(u_j) \cap \mathbb{K}_\varepsilon(\widehat{z^j})$ such that $r_j := x_1^j \to 0$ as $j \to \infty$.

We consider two cases: $t^j > \widehat{t^j}$, for a subsequence, and $t^j \leqslant \widehat{t^j}$. In the first case let us consider for $t \leqslant 0$ the functions

$$v_j(x,t) = \frac{u(x^j - r_je_1 + r_jx, t^j + r_j^2t)}{r_j^2}.$$

We observe that $v_j\big|_{x_1=0} = 0$, $(e_1, 0) \in \Gamma(v_j)$, and

$$\widetilde{z^j} := \left(0, \frac{(\widehat{x^j})' - (x^j)'}{r_j}, -\frac{t^j - \widehat{t^j}}{r_j^2}\right) \in \Gamma(v_j) \cap \Pi.$$

It follows from inclusions $z^j \in \mathbb{K}_\varepsilon(\widehat{z^j})$ that

$$\frac{|(\widehat{x^j})' - (x^j)'|}{r_j} \leqslant \varepsilon^{-1}, \qquad \frac{|t^j - \widehat{t^j}|}{r_j^2} \leqslant \varepsilon^{-2}.$$

Thus there exist subsequences $\{v_j\}$ and $\widetilde{z^j} \in \Gamma(v_j) \cap \Pi$ which converge to a global solution $v \in P_\infty^+(M)$ and to $\widetilde{z} = (0, (\widetilde{x})', \widetilde{t}\,) \in \Gamma(v) \cap \Pi$

with $\widetilde{t} \leqslant 0$, respectively. It is clear that $(e_1, 0) \in \Gamma(v)$. The latter contradicts to Theorem I.

In the case $t^j \leqslant \widehat{t^j}$ we consider for $t \leqslant 0$ the functions

$$v_j(x,t) = \frac{u(\widehat{x^j} + r_j x, \widehat{t^j} + r_j^2 t)}{r_j^2},$$

and observe that $v_j\big|_{x_1=0} = 0$, $(0,0) \in \Gamma(v_j)$, and

$$z_*^j := \left(\frac{x^j - \widehat{x^j}}{r_j}, -\frac{\widehat{t^j} - t^j}{r_j^2} \right) \in \Gamma(v_j).$$

Hence we may conclude that for a subsequence $v_j \to v \in P_\infty^+(M)$, while $z_*^j \to z_* = (1, (z_*)', t_*) \in \Gamma(v)$ with $t_* \leqslant 0$. Since $(0,0) \in \Gamma(v)$, we again get a contradiction to Theorem I. $\square$

Corollary 4.2. *There is a universal constant* $r_0 = r_0(n, M)$ *and a modulus of continuity* σ $(\sigma(0^+) = 0)$ *such that if* $u \in P_1^+(M)$ *and* $z^0 = (x^0, t^0) \in \Gamma(u) \cap \Pi \cap Q_{1/2}$, *then*

$$\Gamma(u) \cap Q_{r_0}(x^0, t^0 + r_0^2/2) \cap Q_{1/2} \subset \{(x,t) : \\ x_1 \leqslant \sigma(\sqrt{|x - x^0|^2 + |t - t^0|}) \cdot \sqrt{|x - x^0|^2 + |t - t^0|}\Big\}.$$

Proof. It suffices to consider the modulus of continuity $\sigma(\rho)$ given by the inverse of the function $\varepsilon \to \rho_\varepsilon$ provided by Lemma 4.1 and to put $r_0 = \rho_{\varepsilon=1}$. $\square$

Lemma 4.3. *There exists* $\delta_0 = \delta_0(n, M) > 0$ *such that if* $u \in P_2^+(M)$ *and* $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_1$ *with* $x_1^0 \leqslant \delta_0$, *then for the balanced energy at the point* z^0 *(see (2.9)) we have*

$$\omega(x^0, t^0, u) = \frac{15}{4} = A. \tag{4.2}$$

Proof. From (2.9) and (2.7) it follows that $\omega(x^0, t^0, u) = W(1, 0, 0, u_0)$, where u_0 is an arbitrary blow-up limit of the solution u at the point z^0. As we have mentioned in subsection 2.3, there are only two possibilities: $u_0 \in P_\infty^+(M)$ or $u_0 \in P_\infty(M)$. In the first case $z^0 \in \Pi$ and Lemma 3.2 immediately provides (4.2). Whereas for the second case Fact 2.2 and Fact 2.4 imply either $W(1, 0, 0, u_0) = A$ (and we are done), or $W(1, 0, 0, u_0) = 2A$.

Now we claim that $\omega(x^0, t^0, u) \neq 2A$, if δ_0 is small enough. Indeed, suppose, towards a contradiction, that there exist sequences $u_k \in P_2^+(M)$ and $z^k = (x^k, t^k) \in \Gamma(u_k) \cap Q_1$ such that $u_k \to u$, $z^k \to z^* = (x^*, t^*) \in Q_1 \cap \Pi$, $r_k := x_1^k \to 0^+$ as $k \to \infty$ and $\omega(x^k, t^k, u_k) = 2A$.

For $(x,t) \in Q_{1/r_k}^+$ we consider the functions

$$v_k(x,t) = \frac{u_k(r_k x + x^k - r_k e_1, r_k^2 t + t^k)}{r_k^2}.$$

We observe that the sequence $\{v_k\}$ converges, for a subsequence, to a global solution $v \in P^+_\infty(M)$. It is evident that $(e_1, 0) = (1, 0, \dots, 0) \in \Gamma(v)$ (see Fact 2.2). By Lemma 3.2 we have

$$W(\rho, e_1, 0, v) = A. \tag{4.3}$$

On the other hand, for arbitrary $\rho > 0$ elementary computations combined with estimate (2.8) and scaling property (2.7) give

$$\begin{aligned} 2A = \omega(x^k, t^k, u_k) &\leqslant W_1(\rho r_k, x^k, t^k, u_k) + C_0 \rho r_k \\ &= W_{1/r_k}(\rho, e_1, 0, v_k) + C_0 \rho r_k \\ &= W(\rho, e_1, 0, v) + \vartheta_k(\rho) + C_0 \rho r_k. \end{aligned} \tag{4.4}$$

Here

$$\vartheta_k(\rho) := \frac{1}{\rho^4} \int\limits_{-4\rho^2}^{-\rho^2} \int\limits_{B_{1/r_k}(e_1)} \left(|Dv_k|^2 - |Dv|^2 + 2(v_k - v) + \frac{v_k^2 - v^2}{t} \right) \times$$

$$\times\, G(x - e_1, -t) dx dt$$

$$- \frac{1}{\rho^4} \int\limits_{-4\rho^2}^{-\rho^2} \int\limits_{\mathbb{R}^n \setminus B_{1/r_k}(e_1)} \left(|Dv|^2 + 2v + \frac{v^2}{t} \right) G(x - e_1, -t) dx dt.$$

It is evident that for fixed ρ the sequence $\{\vartheta_k(\rho)\}$ converges to zero as $k \to \infty$. Thus, (4.4) contradicts (4.3) for large k. The proof is complete. □

5. Regularity properties of solutions

Lemma 5.1. *For any $\varepsilon > 0$ there exists $\delta_1 = \delta_1(n, \varepsilon, M) > 0$ such that if $u \in P^+_2(M)$ and $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q^+_{1/8} \cap \{x_1 < \delta_1\}$ then for $r := x^0_1$ and $\psi(x) = \frac{((x_1 - x^0_1)_+)^2}{2}$ we have*

$$\sup_{Q^+_{8r}(z^0)} |u(x,t) - \psi(x)| \leqslant \varepsilon r^2,$$

$$\sup_{Q^+_{8r}(z^0)} |Du(x,t) - D\psi(x)| \leqslant \varepsilon r.$$

Proof. The statement is proved along the same lines as Lemma 2.3 in [3] or Lemma 5.2 in [9]. □

Remark. It should be mentioned that the additional assumption $u \geqslant 0$ in Lemma 2.3 [3] allows to prove a more general statement in comparison with Lemma 5.1 in this paper.

Lemma 5.2. *Let $u \in P^+_2(M)$, let $\varepsilon > 0$ be a sufficiently small number, and let $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q^+_{1/8} \cap \{x_1 < \delta_1\}$, where $\delta_1 = \delta_1(n, \varepsilon, M)$ is the constant occurring in Lemma 5.1.*

Then the following statements hold:

(i) *there exists a positive number* $N = N(n)$ *such that for* $r := x_1^0$ *and* $\Sigma_r := Q_{7r}(z^0) \cap \{0 < x_1 < r(1 - N\sqrt{\varepsilon})\}$ *we have*

$$u(x,t) \leqslant 0 \quad in \quad \Sigma_r. \tag{5.1}$$

(ii) *If there exists a point* $\widehat{z} = (\widehat{x}, \widehat{t}) \in \Sigma_r$ *such that* $u(\widehat{z}) = 0$, *then*

$$u \equiv 0 \quad in \quad \Sigma_r \cap \{t \leqslant \widehat{t}\}. \tag{5.2}$$

(iii) *The value* $t^* := \sup\{t : z = (x,t) \in \Sigma_r$ *and* $u(z) = 0\}$ *satisfies the inequality*

$$t^* \geqslant t^0 - \varepsilon(2n+1)r^2. \tag{5.3}$$

Proof. Suppose that there is a point $z^{(1)} = (x^{(1)}, t^{(1)}) \in Q_{7r}^+(z^0) \cap \{x_1 < r\}$ such that $u(z^{(1)}) > 0$; otherwise we already have (5.1) with $N(n) = 0$.

Then for $\rho := r - x_1^{(1)}$ we deduce from Fact 2.1 and Lemma 5.1 the inequalities

$$\frac{\rho^2}{2n+1} \leqslant \sup_{Q_\rho^+(z^{(1)})} u - u(z^{(1)}) \leqslant \sup_{Q_{8r}^+(z^0)\cap\{x_1<r\}} |u| \leqslant \varepsilon r^2,$$

which are impossible if $\rho > r\sqrt{(2n+1)\varepsilon}$. Therefore, for all $z = (x,t) \in Q_{7r}^+(z^0)$ with $x_1 < r(1 - \sqrt{(2n+1)\varepsilon})$ we have $u(z) \leqslant 0$. Choosing $N(n) = \sqrt{2n+1}$ we arrive at (5.1).

Finally, with (5.1) at hand, application of the maximum principle to the subcaloric function u implies (5.2), since $u(\widehat{z}) = 0$.

Now we can prove (5.3). If $t^* = t^0$ then (5.3) is true. Otherwise, there exists $\rho > 0$ such that for $z^{(1)} := (x_1^0/2, (x^0)', t^0)$ we have $u < 0$ in $Q_\rho(z^{(1)})$ and $Q_\rho(z^{(1)}) \subset \Sigma$. It suffices to show that

$$\rho \leqslant r\sqrt{\varepsilon(2n+1)}. \tag{5.4}$$

It is easy to see that item (iv) from Fact 2.1 and Lemma 5.1 imply the inequalities

$$\frac{\rho^2}{2n+1} \leqslant |u(z^{(1)})| \leqslant \varepsilon r^2,$$

which are only possible if (5.4) holds. □

Lemma 5.3. *Let* N_0 *and* N_τ *(with* $\tau = 2, \dots, n$*) be arbitrary constants satisfying*

$$|N_0| \leqslant \frac{1}{32(2n+1)M}, \qquad \sum_{\tau=2}^{n} |N_\tau| \leqslant 1. \tag{5.5}$$

There exists $\delta = \delta(n, M) > 0$ *such that if* $u \in P_2^+(M)$ *and* $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/8}^+ \cap \{x_1 < \delta\}$ *then for* $r = x_1^0$ *we have*

$$v := rD_1 u + r\sum_{\tau=2}^{n} N_\tau D_\tau u + r^2 N_0 \partial_t u - u \geqslant 0 \quad in \quad K_r(z^0), \tag{5.6}$$

$$u \geqslant 0 \quad in \quad K_{3r/2}^+(z^0). \tag{5.7}$$

Proof. We take $\varepsilon := \frac{1}{32(2n+1)}$ and set $\delta = \min\{\delta_0, \delta_1\}$, where $\delta_0 = \delta_0(n, M)$ and $\delta_1 = \delta_1(n, \varepsilon, M)$ are the constants defined in Lemmas 4.3 and 5.2, respectively. The proof of (5.6) falls naturally into four parts.

$\boxed{\text{Step 1.}}$

Consider the function $v_s := rD_1 u - u$ that is the special case of v with $N_0 = N_\tau = 0$, $\tau = 2, \ldots, n$. It is easy to see that

$$v_s \geqslant 0 \quad \text{in} \quad Q_{r/2}(z^0), \tag{5.8}$$

$$v_s \geqslant 0 \quad \text{in} \quad K_{3r/2}^+(z^0) \cap \{t \leqslant t^*\}, \tag{5.9}$$

where t^* is the same as in Lemma 5.2. Inequalities (5.8) and (5.9) follow from Lemma 7.6 [5] and Lemma 2.5 [3], respectively. For the reader's convenience and for the completeness, we recall the main arguments.

Note that the function $\psi(x) = \frac{((x_1 - r)_+)^2}{2}$ satisfies for $x_1 \leqslant 3r$ the inequality

$$rD_1\psi - \psi \geqslant 0. \tag{5.10}$$

This together with Lemma 5.1 imply the following estimate in $K_{2r}^+(z^0)$

$$v_s \geqslant -2\varepsilon r^2. \tag{5.11}$$

Suppose, towards a contradiction to (5.8), that there exists a point $z^{(1)} = (x^{(1)}, t^{(1)}) \in Q_{r/2}(z^0)$ with $v_s(z^{(1)}) < 0$. Then we consider the auxiliary function

$$w(x,t) = v_s(x,t) + \frac{|x - x^{(1)}|^2 + t^{(1)} - t}{2n+1}.$$

It is evident that w is caloric in $\Omega(u)$, continuous in $\overline{\Omega(u)}$, and $w(z^{(1)}) < 0$. Hence, applying the maximum principle to the function w in $\Omega(u) \cap Q_{r/2}(z^0)$ and taking into account (5.11) together with the estimate $w \geqslant 0$ on $\partial\Omega(u)$, we get the inequalities

$$-2\varepsilon r^2 + \frac{r^2}{4(2n+1)} \leqslant w\big|_{\Omega(u)\cap\partial' Q_{r/2}(z^{(1)})} \leqslant \inf_{\Omega(u)\cap Q_{r/2}(z^{(1)})} w \leqslant w(z^{(1)}) < 0,$$

which are not possible if $\varepsilon \leqslant \frac{1}{8(2n+1)}$.

Suppose now that inequality (5.9) is not true in a point $z^{(1)} = (x^{(1)}, t^{(1)}) \in K_{3r/2}^+(z^0) \cap \{t \leqslant t^*\}$. In this case we repeat the above arguments and take into account that $w \geqslant 0$ on the set $\Pi \cap \{t \leqslant t^*\}$ as well.

Step 2.

We claim that $t^* = t^0$ (here t^* is the parameter occurring in (5.9)). Indeed, if $t^* < t^0$ and

$$\operatorname{int}\Big\{\Lambda(u)\cap\{t^* < t < t^0\}\cap K_{r/4}(z^0)\Big\} = \emptyset, \tag{5.12}$$

then, in view of item (v) in Fact 2.4, z^0 is a high energy point, which contradicts to Lemma 4.3.

Suppose now that $t^* < t^0$ and (5.12) is false. The latter means that there exist a point $\hat{z} = (\hat{x}, \hat{t})$ and some cylinder $Q_\rho(\hat{z})$ satisfying

$$Q_\rho(\hat{z}) \subset \Lambda(u)\cap\{t^* < t < t^0\}\cap K_{r/4}(z^0).$$

From here, using (5.8) we may conclude that

$$u \leqslant 0 \qquad \text{in} \quad Q' := \{|x' - (\widehat{x})'| < \rho\}\times\{\frac{r}{2} \leqslant x_1 \leqslant \widehat{x}_1\}\times\{\widehat{t} - \rho^2 < t \leqslant \widehat{t}\}.$$

Since u is a subcaloric, we say that $u \equiv 0$ in Q'. Combining the last equality with (5.2) we get

$$u \equiv 0 \qquad \text{in} \quad \Sigma_r \cap \{t \leqslant \widehat{t}\},$$

which contradicts to definition of t^*.

Thus, we have proved that $t^* = t^0$ and, consequently,

$$u \equiv 0 \qquad \text{in} \quad \Sigma_r. \tag{5.13}$$

Step 3.

We claim that all the points $z = (x,t) \in \partial\Omega(u) \cap K^+_{3r/2}(z^0)$ are low energy points, i.e. $\omega(x,t,u) = A$. Indeed, inequalities (5.9) and (5.13) guarantee the nonnegativity of the function u on the set $K^+_{3r/2}(z^0)$, that is the desired inequality (5.7). Hence we have no zero energy points on this set (see item (iii) in Fact 2.4).

It remains only to note that, in view of Lemma 4.3, the set $K^+_{3r/2}(z^0)$ does not contain the high energy points as well.

Step 4.

Using Step 3 and Fact 2.5 we conclude that $\partial_t u$ is continuous in $K^+_{3r/2}(z^0)$. This allows us to prove the statements of the lemma in the full form. Namely, we consider the function

$$v = rD_1 u + r\sum_{\tau=2}^{n} N_\tau D_\tau u + r^2 N_0 \partial_t u - u$$

with constants N_0 and N_τ satisfying conditions (5.5).

Inequalities (5.5) and (5.10) guarantee for v the estimate

$$v \geqslant -\frac{r^2}{8(2n+1)} \qquad \text{in} \quad K^+_{2r}(z^0).$$

Then repeating the arguments from Step 1 finishes the proof of (5.6). ☐

6. Regularity properties of the free boundary

Lemma 6.1. *Let* $u \in P_2^+(M)$*, let* $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q^+_{1/8} \cap \{x_1 < \delta\}$*, where* $\delta = \delta(n, M)$ *is the constant occurring in Lemma 5.3, and let* $r := x_1^0$*.*

Then the inclusion $z \in \partial\Omega(u) \cap K^+_{2r}(z^0)$ *implies* $z \in \Gamma(u)$*. Moreover, there exists a positive constant* $C = C(n, M)$ *such that for a cone*

$$\mathcal{K} := \left\{(x,t) \in \mathbb{R}^{n+1}_+ : x_1 > \sqrt{|x'|^2 + C^2 r^{-2} t^2}\right\}$$

and an arbitrary point $z \in \Gamma(u) \cap K_r(z^0)$ *we have*

$$K_r(z^0) \cap \{z - \mathcal{K}\} \subset \Lambda(u). \tag{6.1}$$

Proof. Let us take $C = 32(2n+1)M$.

The first statement of the lemma was already proved in Step 3 of the proof of Lemma 5.3.

Further, from Lemma 5.3 it follows that for an arbitrary unit vector $e \in \mathcal{K}$ the inequality $D_e u \geqslant 0$ holds in $K_r(z^0)$. Then, evidently, $D_{-e} u \leqslant 0$ in $K_r(z^0)$. The latter together with the assumptions $z \in \Gamma(u)$ and (5.7) gives the desired inclusion (6.1). ☐

Proof of Theorem II. First we observe that the estimate for the function u was proved already in Lemma 5.3. To prove the second statement we set $r = x_1^0$ and define f as follows:

$$f(x', t) = \sup\{x_1 \in [0, 2r] : u(x_1, x', t) = 0\}.$$

We claim that f is a Lipschitz function. Indeed, Lemma 6.1 guarantees for every point $z \in \Gamma(u) \cap Q^+_{1/8} \cap K_r(z^0)$ the existence of standard cone having top at z and lying in $\Lambda(u)$. This implies the corresponding space-time Lipschitz regularity of $\Gamma(u) \cap K_r(z^0)$. The theorem is proved. ☐

Proof of Theorem III. The statement of the theorem is an obvious consequence of Theorem II, Lemma 6.1, and the result of I. Athanasopoulos and S. Salsa proved in [4]. ☐

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