

# BEHAVIOR OF THE FREE BOUNDARY NEAR CONTACT POINTS WITH THE FIXED BOUNDARY FOR NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. The aim of this paper is to study a free boundary problem for a uniformly elliptic fully non-linear operator. Under certain assumptions we show that free and fixed boundaries meet tangentially at contact points.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper we consider a free boundary problem for a uniformly elliptic fully non-linear operator  $F$  in the following setting:

$$(1.1) \quad \begin{cases} F(D^2u) = \chi_\Omega & \text{in } B_1^+, \text{ for an open set } \Omega = \Omega(u) \subset B_1^+ \text{ defined by} \\ u = |\nabla u| = 0 & \text{in } B_1^+ \setminus \Omega, \\ u = 0 & \text{on } \Pi := \{x_1 = 0\}. \end{cases}$$

where  $n \geq 2$  and the PDE holds in the viscosity sense:

The following conditions are imposed on  $F$  throughout the paper:

- (1)  $F$  is uniformly elliptic with ellipticity constants  $\lambda$  and  $\Lambda$ , i.e.,

$$\lambda\|N\| \leq F(A + N) - F(A) \leq \Lambda\|N\|$$

where  $A$  and  $N$  are arbitrary  $n \times n$  symmetric matrices with  $N \geq 0$ .

- (2)  $F$  is homogeneous of degree one, i.e.,

$$F(tA) = tF(A),$$

for all real numbers  $t$  and matrices  $A$ .

- (3)  $F$  is convex.

- (4)  $F$  is  $C^1$ .

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**Definition 1.1.** [CC] A continuous function  $u$  is a viscosity solution of equation

$$F(D^2u) = \chi_\Omega$$

in  $B_1^+$ , when the following condition holds: for any  $x^0 \in B_1^+$  and quadratic polynomials  $\varphi, \psi$  such that  $u - \varphi$  has a local maximum at  $x^0$  and  $u - \psi$  has a local minimum at  $x^0$ , the following holds

$$\begin{aligned} F(D^2\varphi(x^0)) &\geq \chi_\Omega(x^0), \\ F(D^2\psi(x^0)) &\leq \chi_\Omega(x^0). \end{aligned}$$

Let us denote the free boundary

$$\{x : u = |\nabla u(x)| = 0\} \cap \partial\Omega$$

by  $\Gamma = \Gamma(u)$  and the complement of  $\Omega$

$$\Lambda = \Lambda(u) = B_1^+ \setminus \Omega(u) = \{x \in B_1^+ : u(x) = |\nabla u(x)| = 0\}.$$

It is well known that viscosity solutions of fully nonlinear uniformly elliptic PDEs have the usual maximum/minimum principle as well as compactness properties. Furthermore, they are uniformly  $C^{1,\alpha}$  when

$$|F(D^2u)| \leq C,$$

and  $C^{2,\alpha}(B(0, 1/2))$  when

$$F(D^2u) = 1 \quad \text{in } B(0, 1).$$

For the details we refer to [CC].

In the future we shall use the following notations:

$\mathbf{R}_+^n$	$\{x \in \mathbf{R}^n : x_1 > 0\}$ ,
$\mathbf{R}_-^n$	$\{x \in \mathbf{R}^n : x_1 < 0\}$ ,
$B(z, r)$	$\{x \in \mathbf{R}^n :  x - z  < r\}$ ,
$B^+(z, r)$	$\{x \in \mathbf{R}_+^n \cap B(z, r)\}$ ,
$B^-(z, r)$	$\{x \in \mathbf{R}_-^n \cap B(z, r)\}$ ,
$B_r, B$	$B(0, r), B_1$ ,
$B_r^+, B^+$	$B^+(0, r), B_1^+$ ,
$B_r^-, B^-$	$B^-(0, r), B_1^-$ ,
$\Pi, \Pi(z, r), \Pi_r$	$\{x \in \mathbf{R}^n : x_1 = 0\}, \Pi \cap B(z, r), \Pi(0, r)$ ,
$\ \cdot\ _\infty$	canonical norm,
$e_1, \dots, e_n$	standard basis in $\mathbf{R}^n$ ,
$\nu, e$	arbitrary unit vectors,
$D_\nu, D_{\nu e}$	first and second directional derivatives,
$v_+, v_-$	$\max(v, 0), \max(-v, 0)$ ,
$\chi_D$	the characteristic function of the set $D$ ,
$\partial D$	the boundary of the set $D$ ,
$\Omega = \Omega(u)$	$B^+ \setminus \{x : u(x) =  \nabla u(x)  = 0\}$ ,
$\Omega^+(u)$	$\{x \in \Omega : u(x) > 0\}$ ,
$\Omega^-(u)$	$\{x \in \Omega : u(x) < 0\}$ ,
$\Lambda = \Lambda(u)$	$\{x \in B_1^+ : u(x) =  \nabla u(x)  = 0\}$ ,
$\Gamma = \Gamma(u)$	$\{x : u =  \nabla u(x)  = 0\} \cap \partial\Omega$ the free boundary,
$\Gamma^*(u)$	$\Gamma(u) \cap \Pi$ the set of contact points.

We define the density function  $V_r$  as

$$V_r(z, u) = \frac{\text{vol}(\Omega^-(u) \cap B^+(z, r))}{r^n}.$$

Observe that by continuity of  $u$ , if  $V_r(0, u) = 0$  for all  $r$ , then  $u \geq 0$ .

Assume that the set  $\{u < 0\}$  is small enough near the origin, we shall prove the quadratic growth of  $|u|$  near the origin (Theorem A). But since in Theorem A we only assume  $V_r(z, u) \leq C_0$  for  $z = 0$ , we cannot prove that  $u$  is  $C^{1,1}$  near the origin. However, Theorem A still allows us to re-scale  $u$  quadratically such that  $((u(rx)/r^2))$  and remain bounded as  $r$  tends to zero.

In Theorem B we give the classification of positive global solutions (solutions in the half space  $\mathbb{R}_+^n$ ). The latter we use in the proof of Theorem C, which concerns the tangential approach of the free boundary  $\Gamma$  to the fixed boundary  $\Pi$ . In the special case of Theorem C, global solutions obtained by the procedure of the blowup (see below) are non-negative due to the zero-density assumption on the set  $\{u < 0\}$  at the origin.

The main difficulty in proving this result in the nonlinear case, without any a priori conditions on the solution, lies in the lack of the monotonicity lemma, which is heavily used in the case of the Laplacian.

To deal with the complications described above, we use the technique developed in [LS].

**Definition 1.2** (Local solution). We say a continuous function  $u$  belongs to the class  $P_r^+(z, M)$  if  $u$  satisfies

- (1)  $F(D^2u) = \chi_\Omega$  in  $B^+(z, r)$  in the viscosity sense, for some open set  $\Omega$ ,
- (2)  $u = |\nabla u| = 0$  in  $B^+(z, r) \setminus \Omega$ ,
- (3)  $\|u\|_{\infty, B^+(z, r)} \leq M$ ,
- (4)  $u = 0$  on  $\Pi_1$ ,
- (5)  $z \in \partial\Omega$ .

**Definition 1.3** (Global solutions). We say a continuous function  $u$  belongs to the class  $P_\infty^+(z, M)$  if  $u$  satisfies

- (1)  $F(D^2u) = \chi_\Omega$  in  $\mathbb{R}_+^n$  in the viscosity sense, for some open set  $\Omega$ ,
- (2)  $u = |\nabla u| = 0$  in  $\mathbb{R}_+^n \setminus \Omega$ ,
- (3)  $|u(x)| \leq M(|x|^2 + 1)$ ,
- (4)  $u = 0$  on  $\Pi$ ,
- (5)  $z \in \partial\Omega$ .

Our first result asserts that local solutions have quadratic growth if the set  $\Omega^-$  is sufficiently small. See [CKS, KS] for a similar type of result for the Laplacian case.

**Theorem A.** *There is a universal constant  $C_0 = C_0(n, \lambda, \Lambda)$  such that, for all  $r < 1$  and  $u \in P_1^+(z, M)$  we have*

$$\sup_{B^+(z, r)} |u| \leq \frac{M}{C_0} r^2 \text{ for } r < 1,$$

provided  $V_r(z, u) \leq C_0$ , for all  $r < 1$ .

*Remark 1.4.* Let  $d(x, \partial\Omega)$  be the distance from  $x$  to  $\partial\Omega$ . From Theorem A we have that if  $V_r(y, u) \leq C_0$  for all  $y \in B^+(0, 1/2) \cap \partial\Omega$ , and  $r < 1/2$ , then for  $u \in P_1(0, M)$

$$|u(x)| \leq \frac{M}{C_0} d^2(x, \partial\Omega).$$

The interior  $C^{2,\alpha}$  estimates (see [CC]) coupled with the above remark imply that  $u \in C^{1,1}(B^+(0, \frac{1}{4}))$  uniformly for the class  $P_1^+(0, M)$ , provided the assumption  $V_r(y, u) \leq C_0$  holds for all  $y \in B^+(0, 1/2) \cap \partial\Omega$ , and  $r < 1/2$ . For details we refer to the proof of theorem 1.1 in [CKS].

**Theorem B.** *Let  $u \in P_\infty^+(0, M)$ ,*

$$u \geq 0, \quad \text{and} \quad |\nabla u(0, x_2, \dots, x_n)| = 0.$$

*Then  $\Lambda(u) = \Pi$ .*

In order to state our last theorems we will need the following definitions.

**Definition 1.5.** Let  $\sigma(r)(\sigma(0^+) = 0)$  be a modulus of continuity. Then we define  $P_r^+(0, M, \sigma)$  as the subset of all functions  $u$  in  $P_r^+(0, M)$  with the property

$$V_r(0, u) \leq \sigma(r) \quad \text{and} \quad |\nabla u(0, x_2, \dots, x_n)| \leq |x|\sigma(|x|).$$

We believe that both these properties are superfluous and they could be relaxed. However, at this moment, the lack of techniques such as monotonicity formulas forces us to restrict ourselves to such cases.

The idea with these assumption is that after scaling the limit functions will be non-negative, and the gradient will be zero on the fixed boundary.

**Theorem C.** *Let  $M > 0$  and  $\sigma$  be modulus of continuity. Then there exists  $r_0 = r_0(n, M) > 0$  and a modulus of continuity  $\sigma_1(\sigma_1(0^+) = 0)$  such that if  $u \in P_1^+(0, M, \sigma)$ , then*

$$(1.2) \quad \partial\Omega \cap B_{r_0} \subset \{x : x_1 \leq \sigma_1(|x|)|x|\}.$$

The interior case of problem (1.1) (i.e. the problem in the whole ball  $B_1$ ) has been considered earlier in [LS]. When  $F$  is the Laplacian operator, the problem is considered in [SU], and for the parabolic operator in [ASU1], [ASU2], [CPS]. See also the pioneering work of L. A. Caffarelli [C1].

## 2. PROOF OF THEOREM A

We will show that, the solution  $u$  grows away from the free boundary at most with a quadratic rate. We follow the main idea given in [LS], which is to use a homogeneous stretching of the solution by the maximum of  $u$  over the ball  $B^+(0, r)$ . Then we will have a control over the growth of these functions, and we can consider their limit as  $r$  tends to zero.

We define

$$S_j(z, u) = \sup_{B^+(z, 2^{-j})} |u|.$$

In view of the results of [LS] it will be sufficient to prove the following lemma.

**Lemma 2.1.** *There exist a constant  $C_0$  depending only on  $n$ , such that for every  $u \in P_1^+(z, M)$ ,  $j \in \mathbb{N}$  and  $z \in \Gamma(u) \cap B_{1/2}$*

$$(2.1) \quad S_{j+1}(z, u) \leq \max\{S_j(z, u)2^{-2}, C_0M2^{-2j}\}$$

provided

$$V_{2^{-j}}(z, u) \leq C_0.$$

The constant  $C_0$  depends on  $n$ ,  $\lambda$ , and  $\Lambda$ .

*Proof.* If the conclusion in the lemma fails, then there exist sequences

$$\{\Omega_j\}, \{u_j\} \subset P_1^+(0, M), \{z_j\} \subset \Gamma(u_j) \cap B_{1/2}, \{k_j\} \subset \mathbb{N}, k_j \nearrow \infty$$

such that

$$S_{k_{j+1}}(z_j, u_j) > \max\{2^{-2}S_j(z_j, u_j), Mj2^{-2k_j}\} \quad \forall j \in \mathbb{N}.$$

Consider the following scaling

$$\tilde{u}_j(x) = \frac{u_j(z_j + 2^{-k_j}x)}{S_{k_{j+1}}(z_j, u_j)} \quad \text{in } B_1^+.$$

The following results can be obtained by computation:

- $\|\tilde{u}_j\|_{\infty, B} = \frac{S_{k_j}(z_j, u_j)}{S_{k_{j+1}}(z_j, u_j)} \leq 4$ ,
- $\|\tilde{u}_j\|_{\infty, B_{\frac{1}{2}}} = 1$ ,
- $\tilde{u}_j(0) = |\nabla \tilde{u}_j(0)| = 0$ ,
- $V_1(\tilde{u}_j) \leq \frac{1}{j} \rightarrow 0$ .

Also, as in [LS], by ellipticity and degree one homogeneity of  $F$  (where  $F$  itself may vary within the bounds of the condition stated earlier)

$$|F(D^2\tilde{u}_j(x))| \leq \Lambda \frac{(2^{-k_j})^2}{S_{k_{j+1}}(z_j, u_j)} \leq \frac{\Lambda S_{k_j}(z_j, u_j)}{jM S_{k_{j+1}}(z_j, u_j)} \leq \frac{4\Lambda}{jM} \rightarrow 0.$$

Standard elliptic estimates [CC] imply a uniform bound for the  $C^{1,\alpha}$ -norms of  $\tilde{u}_j$ . Therefore a subsequence of  $\{\tilde{u}_j\}$  converges to a function  $\tilde{u}_0$  satisfying

$$\begin{aligned} F(D^2\tilde{u}_0) &= 0 \text{ in } B^+(0, 1), \quad \tilde{u}_0 \geq 0, \\ \tilde{u}_0(0) &= |\nabla \tilde{u}_0(0)| = 0 \quad \text{and} \quad \sup_{B_{1/2}} \tilde{u}_0 = 1. \end{aligned}$$

The above in particular implies that the nonzero solution  $\tilde{u}_0$  of the elliptic equation  $F(D^2\tilde{u}_0) = 0$  has a local minimum at a boundary point and its gradient is zero at that point. Using Hopf type lemma we come to a contradiction.

## 3. NONDEGENERACY

We will be concerned with scaling of the type

$$u_r(x) := \frac{u(rx)}{r^2},$$

and its limit (when it exists)

$$u_0 := \lim_{r_j \rightarrow 0} u_{r_j},$$

called blow-up limit. Hence we need to assure that  $u_0 \neq 0$ , i.e.  $u$  is non-degenerate.

**Lemma 3.1.** *If  $u \in P_R^+(z, M)$ ,  $x^0 \in \overline{\{u > 0\}} \cap B_{R/2}(z)$  then*

$$(3.1) \quad \sup_{B^+(x^0, r)} u \geq u(x^0) + C_0 r^2, \quad \text{for all } r < R - |x_0 - z|,$$

where  $C_0$  is a constant depending only on  $n$  and  $\Lambda$ .

*Proof.* It suffices to consider the case  $x^0 \in \{u > 0\} \cap B_{R/2}(z)$  because if (3.1) holds for all  $x^0 \in \{u > 0\} \cap B_{R/2}(z)$ , then it will be true also for all  $x^0 \in \overline{\{u > 0\}} \cap B_{R/2}(z)$ . Set

$$(3.2) \quad v(x) = u(x) - u(x^0) - \frac{1}{2n\Lambda}|x - x^0|^2.$$

There exists  $x^1 \in \overline{B^+}(x^0, r)$  such that the following holds:

$$(3.3) \quad v(x^1) = \sup_{B^+(x^0, r)} v.$$

To prove the lemma, it is enough to prove the following two steps:

- $v(x^1) \geq 0$ ,
- $x^1 \in \partial B^+(x^0, r) \setminus \Pi(x^0, r)$ .

The first step simply follows from the fact that

$$v(x^1) \geq v(x^0) = 0.$$

To prove the second step assume  $x^1 \in B^+(x^0, r)$ . Then from (3.3) we have  $|\nabla v|(x^1) = 0$ . Thus by (3.2)

$$(\nabla u)(x^1) = \frac{1}{n\Lambda}(x^1 - x^0).$$

Now, if  $x^1 \neq x^0$ , then  $(\nabla u)(x^1) \neq 0$ , i.e.,  $x_1 \in \Omega$ . We also have

$$F(D^2 v) = F\left(D^2 u - \frac{I}{n\Lambda}\right) \geq F(D^2 u) - \Lambda \frac{1}{\Lambda} = 0 \text{ in } \Omega,$$

and (3.3) together with maximum principle gives us that

$$v(x) \equiv \text{constant} =: C \quad \text{in } \overline{\Omega \cap B^+(x^0, r)}.$$

In particular,  $C = v(x^0) = 0$  so we have

$$u(x) = u(x^0) + \frac{1}{2n\Lambda}|x - x^0|^2$$

and

$$(\nabla u)(x) = \frac{1}{n\Lambda}(x - x^0) \quad \text{in } \overline{\Omega \cap B^+(x^0, r)}.$$

But if we take  $y \in \partial\Omega \cap \overline{B^+(x^0, r)}$  (we may assume it exists without loss of generality) then we get

$$|\nabla u(y)| = \frac{1}{n\Lambda}(y - x^0) \neq 0,$$

which is a contradiction, since  $|\nabla u| = 0$  on  $\partial\Omega$ . So in this case  $x^1 \in \partial B^+(x^0, r)$ .

If  $x^1 = x^0$ , then again  $x^1 = x^0 \in \Omega$  and we have the same contradiction as above.

Finally, if  $x^1 \in \Pi(x^0, r)$ , then because  $u(x^0) \geq 0$ , we get the following contradiction

$$0 > v(x^1) \geq v(x^0) = 0,$$

where the second inequality follows from the definition of  $x^1$ .  $\square$

#### 4. PROOF OF THEOREMS B AND C

##### Proof of Theorem B

Under the conditions imposed on  $u$ , more exactly  $\nabla u = 0$  on  $\{x_1 = 0\}$ , one can give a proof of Theorem B by continuing the function  $u$  as zero to the lower half space  $\mathbb{R}_-^n$  to obtain a solution in whole  $\mathbb{R}^n$  (one can show that in the viscosity sense there is no mass on  $\{x_1 = 0\}$ , since there is no jump in the gradient). Then from the interior result [LS] it follows that the coincidence set is convex, hence we have a halfspace solution.

For completeness we give a detailed proof based on ideas of [LS].

From the convexity of  $F$  in  $\mathbb{R}_+^n$  we can conclude that  $D_{ee}u$  is a supersolution to the linearized problem and hence it has the minimum principle (see [LS]). We will prove that  $u$  is convex using a contradictory argument. Assume there is a direction  $e$  such that

$$-\infty < \inf_{\Omega(u)} D_{ee}u = -C < 0.$$

Then there exists a sequence  $\{x^j\}$  such that

$$D_{ee}u(x^j) \rightarrow -C \text{ as } j \rightarrow +\infty.$$

Let us consider the blowup of  $u$  with  $d_j = \text{dist}(x^j, \partial\Omega) < +\infty$

$$u_j(x) = \frac{u(x^j + d_j x)}{d_j^2}.$$

We remark that by the assumption  $|\nabla u(0, x_2 \cdots, x_n)| = 0$  and Theorem A (since  $u \geq 0$ ) we have  $u_r$  is uniformly bounded. Using compactness argument we get

$$u_j \rightarrow u_0 \quad \text{in } C^{2,\alpha}(B_{1/2}^+)$$

which implies

$$D_{ee}u_0(0) = \lim_j D_{ee}u_j(0) = \lim_j D_{ee}u(x^j) = -C$$

and

$$D_{ee}u_j(x) = D_{ee}u(x^j + d_j x) \geq -C.$$

Thus in  $B_{1/2}^+$  we have

$$D_{ee}u_0(x) \geq -C.$$

By maximum principle  $D_{ee}u_0 \equiv -C$  in  $\Omega'$ , the connected component of  $\Omega(u_0)$  containing the origin. Following the steps in [LS], we rotate the coordinate system such that  $e$  coincides with  $e_1$ . Next we integrate  $D_{11}u_0$  and use non-negativity of  $u_0 \geq 0$  to obtain  $|x_1| \leq G(x'_1)$  for some function  $G$  and all  $x \in \Omega'$ . Now, for fixed  $x$  let us consider  $x^m := (x_1 + m, x_2 \cdots, x_n)$ . There exists  $m$  depending on  $x'_1$  such that

$$u_0(x^m) = |\nabla u_0(x^m)| = 0.$$

Combining the facts that for large  $m$  we have  $D_1u_0(x^m) = 0$ , and  $D_1u_0$  is non-increasing, we get

$$D_1u_0 \leq 0 \text{ in } \Omega'.$$

The latter gives a contradiction to the non-degeneracy, Lemma 3.1.

Now, as we have proved the convexity of  $u$  and hence of the set  $\Lambda = \{u = 0\}$ , we see clearly that  $\Lambda(u) = \Pi$ . Indeed, since  $\Pi \subset \Lambda$ , if there is a point  $x^0 \in \Lambda \cap \mathbf{R}_+^n$ , then by convexity

$$\{x : 0 \leq x_1 \leq x_1^0\} \subset \Lambda,$$

implying that the origin is not a free boundary. This is a contradiction.  $\square$

### Proof of Theorem C

It is enough to check that for every given  $\varepsilon$  there exists  $\rho = \rho_\varepsilon$  such that for all  $x^0 \in \partial\Omega \cap B_{\rho_\varepsilon}^+$

$$(4.1) \quad x^0 \in B_{\rho_\varepsilon}^+ \setminus K_\varepsilon,$$

where

$$K_\varepsilon = \{x : x_1 > \varepsilon(x_2^2 + \dots + x_n^2)^{1/2}\}.$$

Then we may choose  $r_0 = \rho_{\{\varepsilon=1\}}$  and  $\sigma$  given by the inverse of  $\varepsilon \rightarrow \rho_\varepsilon$ . The proof is based on a contradictory argument. If (4.1) fails, than there exists a sequence

$$u_j \in P_1^+(0, M, \sigma), \quad x^j \in \partial\Omega(u_j) \cap B_{\rho_j}^+$$

such that  $\rho_j \rightarrow 0$  and  $x^j \in B_{\rho_j}^+ \cap \overline{K_\varepsilon}$ . Now for every scaled function

$$\tilde{u}_j(x) = u_j(x|x^j|)/|x^j|^2$$

we have a point  $\tilde{x}^j \in \partial B_1^+ \cap \partial\Omega(\tilde{u}_j) \cap K_\varepsilon$ . There exists converging subsequences of  $\tilde{u}_j \rightarrow u_0$  and  $\tilde{x}^j \rightarrow x_0$  such that  $x_0 \in \overline{K_\varepsilon} \cap \partial B_1$ , with  $u(x^0) = 0$ . It follows by the assumption

$$|\nabla u_j(0, x_2, \dots, x_n)| \leq |x|\sigma(|x|),$$

that

$$|\nabla \tilde{u}_j(0, x_2, \dots, x_n)| \leq |x|\sigma(|x||x^j|) \rightarrow 0.$$

In particular (by non-degeneracy lemma)  $u_0$  is a nonzero global solution, satisfying the assumptions of Theorem B, and hence  $\Lambda = \Pi$ , contradicting  $x^0 \in \partial\Omega(u_0)$ .  $\square$

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