ON A PARABOLIC FREE BOUNDARY EQUATION MODELING PRICE FORMATION †

DEDICATED TO LUIS CAFFARELLI AT THE OCCASION OF HIS 60TH BIRTHDAY

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Abstract. We discuss existence and uniqueness of solutions for a one dimensional parabolic evolution equation with a free boundary. This problem was introduced by J.-M. Lasry and P.-L. Lions as description of the dynamical formation of the price of a trading good.
Short time existence and uniqueness is established by a contraction argument. Then we discuss the issue of global-in.time-extension of the local solution which is intimately connected to the regularity of the free boundary. We also present numerical results.

Key words.
mean field models in economy, free boundary problems, parabolic equations.

1. Introduction This paper is concerned with a mean field model in Economics and Finance, which, among others, was introduced in a series of papers by J.-M. Lasry and P.-L. Lions [8, 5, 6, 10, 9, 11, 7]. Given a (large) group of buyers and a (large) group of vendors the non-linear free boundary evolution model describes the dynamical formation of the price of a trading good under negotiation between the two groups.
The groups are described by two non-negative density functions f_B and f_V, which satisfy the parabolic system

$$\frac{\partial f_B}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 f_B}{\partial x^2}=\lambda(t)\delta(x-p(t)+a), \quad \text{for } x<p(t) \tag{1.1a}$$
$$f_B\geq 0,\ f_B(x,t)=0 \quad \text{for } x\geq p(t)$$

and

$$\frac{\partial f_V}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 f_V}{\partial x^2}=\lambda(t)\delta(x-p(t)-a), \quad \text{for } x>p(t) \tag{1.1b}$$
$$f_V\geq 0,\ f_V(x,t)=0 \quad \text{for } x\leq p(t),$$

where

$$\lambda(t)=-\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}(p(t),t)=\frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}(p(t),t) \tag{1.1c}$$

is the transaction rate and $x=p(t)$ denotes the price. The variable t denotes time and the space-like variable $x\in\mathbb{R}$ stands for the possible value of the price. The positive

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parameter a measures the bid-ask spread (assumed to be equal to $2a$) and $\sigma > 0$ the randomness. A natural property of the model is that the total numbers of the buyers and vendors is preserved, i.e.

$$\frac{d}{dt}\int_{\mathbb{R}} f_B(x,t)dx = 0 \quad \text{and} \quad \frac{d}{dt}\int_{\mathbb{R}} f_V(x,t)dx = 0. \tag{1.2}$$

The preservation property (1.2) holds for the Dirac delta δ as well as for smoothed versions δ_ε with compact support in $(-a,a)$ and $\int \delta_\varepsilon = 1$.
By introducing the function (signed density of buyers-vendors)

$$f(x,t) = \begin{cases} f_B(x,t) & \text{if } x < p(t) \\ -f_V(x,t) & \text{if } x > p(t), \end{cases}$$

system (1.1) can be reduced to the following scalar free boundary value problem, with unknowns $f = f(x,t)$ and $p = p(t)$:

$$\frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = \lambda(t)\left(\delta(x - p(t) + a) - \delta(x - p(t) - a)\right) \tag{1.3a}$$

$$f(x,t) > 0 \text{ if } x < p(t),\ f(x,t) < 0 \text{ if } x > p(t) \tag{1.3b}$$

with

$$f(x,0) = f_I(x),\, p(0) = p_0 \tag{1.3c}$$

(initial conditions). The compatibility conditions

$$f_I(p_0) = 0 \text{ and } f_I(x) > 0 \text{ for } x < p_0 \text{ and } f_I(x) < 0 \text{ for } x > p_0. \tag{1.4}$$

are assumed to hold. We reiterate $\lambda(t) = -\sigma^2 \frac{\partial f}{\partial x}(p(t),t)$. Note that this reduction requires that at $t = 0$, $f'(p_{0+}) = f'(p_{0-})$ or equivalently $(f_B)'(p_0,0) = -(f_V)'(p_0,0)$ (otherwise additional technicalities have to be taken care of). Also, we remark that by the shift $x = p(t) + y$ equation (1.3) is equivalent to

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\partial^2 g}{\partial y^2} - \frac{\partial g}{\partial y}(0,t)\left[\delta(y+a) - \delta(y-a)\right] + \dot{p}(t) g_y \\ \dot{p}(t) &= -\frac{g_{yy}(0,t)}{g_y(0,t)}, \end{aligned}$$

where we set $g(y,t) = f(y + p(t),t)$. Here the time derivative of the free boundary $\dot{p}(t)$ can be interpreted as the constraint that ensures $g(0,t) = 0$. Note that this formulation, based on mapping the free boundary into the line $y = 0$, shows that the problem under consideration is highly nonlinear.

Existence and asymptotic behaviour in case of equally distributed buyers and vendors, i.e. $f_B(p_0 - x, t = 0) = f_V(p_0 + x, t = 0)$ for all $x \in \mathbb{R}$ has been addressed by Gonzalez and Gualdani in [4] recently. In this special case the price $p(t)$ is constant in time, i.e. $p(t) = p_0$ and the free boundary disappears from the problem, which becomes a linear parabolic IVP. For this special case they verified existence and proved exponential convergence of the solution towards its stationary state. An extension of their analysis to problems with initial condition close to equilibrium, on

bounded domains, has been presented in [3], based on linearisation and semigroup techniques.
In [6] a strategy for carrying out an existence proof (by a time stepping argument, in the framework of nonlinear semigroups, introduced by Crandall and Liggett in [1]) is outlined, we shall however follow an entirely different 'direct' approach, based on classical solutions. Indeed we shall comment on how these two approaches relate at the end of Section 2.

This work is organized as follows. In Section 2 we show local existence of (1.3) for general initial data and discuss the maximal extension of the solution. Finally we illustrate the behaviour of solutions with numerical experiments in Section 3.

1.1. Stationary Solution - Bounded Interval As a model we consider the stationary problem (1.3), posed on the bounded domain $(0,A)$, $(A>0)$, subject to homogeneous Neumann boundary conditions:

$$\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = -\frac{\sigma^2}{2}\frac{\partial f}{\partial x}(p,t)\left(\delta(x-p+a)-\delta(x-p-a)\right)$$
$$\frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial x}(A) = 0.$$

The solution, as given by J.-M. Lasry and P.-L. Lions in [7], satisfies:

$$\frac{\partial f}{\partial x} = 0 \text{ if } x<p-a \text{ or if } x>p+a \tag{1.5a}$$
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(p) \text{ if } p-a<a<p+a. \tag{1.5b}$$

For the equilibrium price p they obtained the algebraic equation

$$p = \frac{2M_-A - a(M_- - M_+)}{2(M_- + M_+)}, \tag{1.5c}$$

where $M_- = \int_0^p f dx$ (number of buyers) and $M_+ = \int_p^A (-f)dx$ (number of vendors), if the parameter are such that $p \in [a, A-a]$. Note that the price depends explicitely on the ratio of $\frac{M_-}{M_+}$. If the number of buyers increases or the number of vendors go down the price goes up, which is reasonable from an economical viewpoint.
We reiterate that the corresponding dynamic free boundary problem with close to equilibrium initial data and homogeneous Neumann boundary conditions has been analyzed in [3]. We shall consider the case of a bounded price domain in our numerical experiments in Section 3.

2. (Classical) Existence Theory for the Whole Space Problem

Note that throughout this Section we use the letter C as well as C_1, C_2, ... for generic, not necessarily equal constants. When needed we shall specify on which parameters the constants depend. W.r.o.g. we set $\frac{\sigma^2}{2}=1$ in the remaining parts of this paper.
We start by discussing the existence and uniqueness of solutions of the FBP (1.3).

2.1. Preliminaries At the beginning we would like to reiterate a classical estimate for the first derivative (with respect to x) of the solution of the heat equation;

$$u_t = u_{xx}, \quad x\in\mathbb{R}, t>0$$
$$u(x,t=0) = u_I(x), \quad x\in\mathbb{R}$$

with $u_I \in L^2(\mathbb{R})$. Multiplying by u and integrating, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u^2\,dx = -\int_{-\infty}^{\infty} u_x^2\,dx$$

and thus

$$u_x \in L^2((0,\infty);L^2(\mathbb{R})), u_t \in L^2((0,\infty);H^{-1}(\mathbb{R})), u \in L^\infty((0,\infty);L^2(\mathbb{R})).$$

We now localize by choosing

$$\varphi \in \mathcal{C}_0^\infty(\mathbb{R}). \tag{2.1}$$

Then $v = u\varphi$ satisfies

$$v_t = v_{xx} + h \tag{2.2}$$

where

$$h = -2u_x\varphi_x - u\varphi_{xx} \in L^2((0,T);L^2(\mathbb{R})).$$

Thus by multiplying (2.2) by v_{xx} and integrating with respect to x we deduce

$$-\int_{\mathbb{R}} v_{tx}v_x\,dx = \int_{\mathbb{R}} (v_{xx})^2\,dx + \int_{\mathbb{R}} hv_{xx}\,dx,$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (v_x)^2\,dx = -\int_{\mathbb{R}} (v_{xx})^2\,dx - \int_{\mathbb{R}} hv_{xx}\,dx.$$

Therefore, by integrating with respect to t, we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} (v_x)^2\,dx + \frac{1}{2}\int_0^T \int_{\mathbb{R}} (v_{xx})^2\,dx \\ \leq \frac{1}{2}\int_0^T \int_{\mathbb{R}} h^2\,dx + \frac{1}{2}\int_{\mathbb{R}} \left(\frac{\partial v_I}{\partial x}\right)^2\,dx,\end{aligned}$$

where v_I denotes the corresponding initial datum. We conclude

$$v_x \in L^\infty\left((0,T),L^2(\mathbb{R})\right),\ v_{xx} \in L^2((0,T);L^2(\mathbb{R})).$$

if $\frac{\partial v_I}{\partial x} \in L^2(\mathbb{R})$. Using (2.2) we obtain that

$$v_t \in L^2((0,T);L^2(\mathbb{R})).$$

Iterating the above procedure with

$$z = v\psi, \quad \psi \in \mathcal{C}_0^\infty(\mathbb{R})$$

with $\operatorname{supp}\psi$ compactly contained in $\operatorname{supp}\psi$ and so on we obtain, after a simple exercise, for $\beta \in \mathbb{N}$:

$$\begin{aligned}\|u\|_{L^\infty((0,T);H^\beta(l_2))} &\leq C\left(\|u\|_{L^2(((0,T));H^1(\mathbb{R}))} + \|u_I\|_{H^\beta(l_1)}\right) \\ &\leq C\left(\|u_I\|_{L^2(\mathbb{R})} + \|u_I\|_{H^\beta(l_1)}\right)\end{aligned} \tag{2.3}$$

where l_1 and l_2 are real open intervals with l_2 compactly contained in l_1. Here C only depends on l_1 and l_2. Now we pose the following assumption

(A1) The initial data f_I satisfies

$$f_I \in L^2(\mathbb{R}) \cap H^4(\Lambda) \cap L^1(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}),$$

where $\Lambda = (p_0 - r_0, p_0 + r_0)$ for some $0 < r_0 < a$ and p_0 is such that $f_I(p_0) = 0$, $f_I > 0$ for $x < p_0$ and $f_I < 0$ for $p > p_0$.

Note that (A1) can certainly be weakened, as far as the local regularity close to p_0 is concerned, at the expense of additional technicalities.
By a simple Min-Max-principle argument the solution f of (1.3) has a unique zero $x = p(t)$ for all t in its maximal interval of existence. In addition $\lambda(t) \geq 0$ (as long as the solution exists). The Maximum-Minimum principle implies:

$$f \geq u_1,\, x < p(t);\; f \leq u_2,\, x > p(t), \tag{2.4a}$$

where

$$\frac{\partial}{\partial t} u_1 = \frac{\partial^2}{\partial x^2} u_1,\, x < p(t);\; u_1(-\infty, t) = u_1(p(t), t) = 0,\, u_1(t=0) = f_I \tag{2.4b}$$

$$\frac{\partial}{\partial t} u_2 = \frac{\partial^2}{\partial x^2} u_2,\, x > p(t);\; u_2(+\infty, t) = u_2(p(t), t) = 0,\, u_2(t=0) = f_I \tag{2.4c}$$

By classical arguments we shall now derive a fixed-point formulation of the system (1.3) that will be used to prove local and - later on - global existence. Let Γ denote the fundamental solution of the one-dimensional heat equation

$$\Gamma(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}. \tag{2.5}$$

Then the (mild) solution of (1.3) can be expressed using the Duhamel's principle

$$f(x,t) = \int_{\mathbb{R}} \Gamma(x-y,t) f_I(y)\, dy + \int_0^t \int_{\mathbb{R}} \Gamma(x-y, t-s) \lambda(s) q(y,s) ds dy, \tag{2.6}$$

where $q(x,t) = (\delta(x - p(t) + a) - \delta(x - p(t) - a))$. The explicit formulation of (2.6) is given by

$$\begin{aligned} f(x,t) = &\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} f_I(y) dy}_{=: f_1(x,t)} \\ &+ \underbrace{\int_0^t \frac{\lambda(s)}{\sqrt{4\pi(t-s)}} \left(e^{-\frac{|x-p(s)+a|^2}{4(t-s)}} - e^{-\frac{|x-p(s)-a|^2}{4(t-s)}} \right) ds}_{=: f_2(x,t)} \\ =&: F[p,\lambda](x,t). \end{aligned} \tag{2.7}$$

By differentiation of (2.6) with respect to x and evaluation at $x = p(t)$ we obtain the Volterra integral equation of the second kind for λ (given by (1.1c) with $\sigma^2 = 1$)

$$\lambda(t) = \lambda_0(t) + \int_0^t \lambda(s) K[p](s,t) ds, \tag{2.8}$$

where $K[p](s,t)$ is the difference of the x-derivatives of the heat kernel (2.5), evaluated at $x = p(t) - p(s) \mp a$, i.e.

$$K[p](s,t) = \frac{1}{2\sqrt{4\pi(t-s)}} \left(\frac{p(t)-p(s)+a}{(t-s)} e^{-\frac{|p(t)-p(s)+a|^2}{4(t-s)}} - \frac{p(t)-p(s)-a}{(t-s)} e^{-\frac{|p(t)-p(s)-a|^2}{4(t-s)}} \right).$$

The function λ_0 is given by

$$\lambda_0(t) = -\int_{\mathbb{R}} \Gamma_x(p(t)-y,t) f_I(y)\, dy =: \lambda_0[p](t).$$

Since $f(p(t),t) = 0$, we conclude that

$$-\dot{p}(t)\lambda(t) + f_{xx}(p(t),t) = 0. \tag{2.9}$$

Here we use the fact that near the free boundary f satisfies the heat equation and therefore replace f_t by f_{xx}.
Throughout our calculations we shall use

$$\lim_{z\to 0+} z^{-\gamma} e^{-\frac{\beta}{z}} = 0 \qquad \forall \gamma,\beta > 0. \tag{2.10}$$

Using the above considerations, we can write (1.3) as the following fixed-point problem:

1. Given $p = p(t)$ appropriately, we define
$$S[p](\lambda)(t) = \lambda_0[p](t) + \int_0^t \lambda(s) K[p](s,t) ds, \tag{2.11}$$
where F is given by (2.7) and we prove that $S[p]$ has a unique fixed point λ in an appropriate set.
2. Given $\lambda = \lambda(t)$ from step 1, we define
$$L(p) = p_0 + \int_0^t \frac{F[p,\lambda](x = p(\tau),\tau)}{\lambda(\tau)}\, d\tau, \tag{2.12}$$
where f is given by (2.7) and prove the existence of a locally unique fixed point p.

Lemma 2.1 (Volterra equation). *Let $p = p(t)$ and $\lambda_0 = \lambda_0(t)$ be in $\mathcal{C}([0,T])$ and let (A1) hold. Then there exists a unique solution $\lambda = \lambda(t)$ in $\mathcal{C}([0,T])$ of the second kind Volterra integral equation* (2.8)*. The $L^\infty((0,T))$ norm of the solution depends only on the modulus of continuity of p and on $\|\lambda_0\|_{L^\infty}$.*

Proof. The kernel $K[p](s,t)$ is continuous for $s \le t$. An explicit upper bound is easily found due to (2.10) if $\max|p(t)-p(s)| < a$. Since by assumption λ_0 is continuous on $[0,T]$, the result follows from the standard theory of Volterra integral equations of the second kind (by Picard iteration). ∎

Remark 2.1. *Clearly, $\lambda_0(t)$ is the x-derivative of the solution of the heat equation with initial datum $-f_I(x)$ evaluated at $(x,t) = (p(t),t)$. Thus, assumption (A1), giving* *local x-Lipschitz continuity of the solution of the heat equation with initial datum $-f_I$, implies that λ_0 is bounded in $\mathcal{C}([0,T])$, if $\|p-p_0\|_{\mathcal{C}([0,T])}\leq r_0$. In particular, note that by the localisation procedure leading to estimate (2.3) a bound on $\|\lambda_0\|_{\mathcal{C}([0,T])}$, which only depends on $\|f_I\|_{L^2(\mathbb{R})}+\|f_I\|_{H^2(\Lambda)}$ can be established if $\|p-p_0\|_{\mathcal{C}([0,T])}\leq r_0$.*

LEMMA 2.2. *Let $p_1,p_2\in\mathcal{C}([0,T])$ be such that*

$$\|p_i-p_0\|_{\mathcal{C}([0,T])}\leq r_0, \quad i=1,2.$$

Then

$$\|\lambda_1-\lambda_2\|_{\mathcal{C}([0,T])}\leq C_1\left(\|f_{Ixx}\|_{L^\infty((p_0-r_0,p_0+r_0))}+\|f_I\|_{L^2(\mathbb{R})}+T\nu\right)e^{C_2T}\|p_1-p_2\|_{C([0,T])},$$

where $\lambda_1=S[p_1]\lambda_1$, $\lambda_2=S[p_2]\lambda_2$. The constants C_1,C_2 may depend on r_0 and ν is an upper bound for λ_1 in $L^\infty((0,T))$.

Proof. We have

$$\begin{aligned}\lambda_1-\lambda_2=&\lambda_0[p_1](t)-\lambda_0[p_2](t)\\&+\int_0^t(K[p_1]-K[p_2])\lambda_1(s)ds\\&+\int_0^t K[p_2](\lambda_1(s)-\lambda_2(s))ds.\end{aligned}$$

Using the Lipschitz continuity of the x-derivative of the solution of the heat equation with initial datum $-f_I$ we obtain

$$\begin{aligned}|(\lambda_1-\lambda_2)(t)|\leq& C_1\left(\|f_I\|_{H^3(\Lambda)}+\|f_I\|_{L^2(\mathbb{R})}\right)\|p_1-p_2\|_{C([0,T])}\\&+\nu C_2T\|p_1-p_2\|_{C([0,T])}+C_3\int_0^t(\lambda_1(s)-\lambda_2(s))ds.\end{aligned}\tag{2.13}$$

Here we used the localisation estimate (2.3), the boundedness of $K[p](t,s)$ as long as $s<t$ and $\|p-p_0\|\leq r_0$ and the uniform Lipschitz continuity property

$$|K[p_1](s,t)-K[p_2](s,t)|\leq C_3\|p_1-p_2\|_{C([0,T])},$$

if $s<t$, $\|p_i-p_0\|\leq r_0$ for $i=1,2$. The Gronwall Lemma gives the result. □

REMARK 2.2. *Using the same arguments we easily obtain the bound (as long as $\|p-p_0\|_{\mathcal{C}([0,T])}\leq r_0$):*

$$\|\lambda\|_{\mathcal{C}([0,T])}\leq C_3\left(\|f_{Ix}\|_{L^\infty((p_0-r_0,p_0+r_0))}+\|f_I\|_{L^2(\mathbb{R})}\right)e^{C_4T},\tag{2.14}$$

where C_3,C_4 may depend on r_0, too.

LEMMA 2.3 (Positivity of λ for short time intervals). *Let (A1) hold and assume $\|p-p_0\|_{\mathcal{C}([0,T])}\leq r_0$. Then there exists a time $T=T(f_I)$, such that $\lambda=\lambda(t)$, the fixed-point of $S[p]$ on $\mathcal{C}([0,T])$, is positive for all $t\in[0,T]$.*

Proof. Note that $\lambda_0(0)=-\frac{\partial f_I}{\partial x}(p_0)$ is positive, we write

$$\lambda_0(t)=\lambda_0(0)+(w(p(t),t)-w(p_0,0)),\tag{2.15}$$

where w, as said before, solves the heat equation

$$w_t=w_{xx},\ x\in\mathbb{R}, t>0,\tag{2.16a}$$

$$w(x,t=0)=-\frac{\partial f_I(x)}{\partial x}(x),\ x\in\mathbb{R}.\tag{2.16b}$$

Here w is locally Lipschitz in x and t, therefore we conclude that $\lambda_0(t) > \frac{\lambda_0(0)}{2}$ as long as $T = T(f_I)$ is sufficiently small. Note that T depends on f_I only through r_0, $\frac{1}{f_{Ix}(p_0)}$, $\|f_I\|_{L^2(\mathbb{R})}$ and $\|f_I\|_{H^3(\Lambda)}$. Simple calculations show that $K[p](t,s) > 0$ for $|p(s) - p(t)| < a$ and $0 < s < t$. Therefore, choosing T small enough, we ensure that $\lambda_0(t)$ is strictly positive for all $t \le T$. □

Theorem 2.3 (Local Existence and Uniqueness). *Under the assumption (A1), the system*

$$\lambda(t) = \lambda_0[p](t) + \int_0^t \lambda(s) K[p](s,t)\, ds \tag{2.17a}$$

$$0 = -\dot{p}(t)\lambda(t) + F[p,\lambda]_{xx}\,(x = p(t), t) \tag{2.17b}$$

supplemented by $p(0) = p_0$, has a unique solution $(\lambda, p) \in \mathscr{A} \times \mathcal{C}([0,T])$ on some time interval $[0,T]$, where $\mathscr{A}$ is given by

$$\mathscr{A} = \left\{ p \in \mathcal{C}([0,T]) \mid \quad \|p - p_0\|_{L^\infty((0,T))} \le \frac{r_0}{2} \right\}.$$

Then the solution f of (1.3) *is uniquely determined on the same time interval.*

Proof. The local existence proof is based on Banach's Fixpoint theorem. Using Lemma 2.1 we conclude that the operator $S[p]$, given by (2.11), has a unique fixed-point. The function λ, being a fixed-point of the operator $S[p]$, is then used in the definition of the operator L given by (2.12).

First, we have to check that the operator L is a self-mapping of $\mathscr{A}$. Integration of (2.17b) gives

$$p(t) - p_0 = \int_0^t \frac{F[p,\lambda]_{xx}(p(\tau),\tau)}{\lambda(\tau)}\, d\tau, \tag{2.18}$$

where $f = F[p,\lambda](x,t)$ is given by (2.7). Using estimate (2.10) we obtain for $t \in [0,T]$

$$|(f_1(p(t),t))_{xx}| = |w_x(p(t),t)| \le C\left(\|f_I\|_{L^2(\mathbb{R})} + \|f_I\|_{H^3(\Lambda)} \right) =: M, \tag{2.19}$$

where w solves the IVP (2.16a). The second derivative of f_2 with respect to x is given by

$$\begin{aligned} f_{2xx}(x,t) = \int_0^t \Bigg(& \frac{\lambda(\tau)}{4\sqrt{\pi(t-\tau)^3}} \left(\frac{|x - p(\tau) + a|^2}{2(t-\tau)} - 1 \right) e^{-\frac{|x-p(\tau)+a|^2}{4(t-\tau)}} \\ & - \frac{\lambda(\tau)}{4\sqrt{\pi(t-\tau)^3}} \left(\frac{|x - p(\tau) - a|^2}{2(t-\tau)} - 1 \right) e^{-\frac{|x-p(\tau)-a|^2}{4(t-\tau)^2}} \Bigg)\, d\tau \end{aligned} \tag{2.20}$$

Inserting (2.19) and (2.20) into (2.18) and setting $x = p(\tau)$ we obtain the following estimate

$$\begin{aligned} |L(p)(t) - p_0| \le & \int_0^t \frac{M}{\lambda(\tau)} d\tau \\ & + \int_0^t \frac{1}{\lambda(\tau)} \int_0^\tau \left| \frac{\lambda(s)}{4\sqrt{\pi(\tau-s)^3}} \left(\frac{|p(\tau) - p(s) + a|^2}{4(\tau - s)} - 1 \right) e^{-\frac{|p(\tau)-p(s)+a|^2}{2(\tau-s)}} \right| ds d\tau \\ & + \int_0^t \frac{1}{\lambda(\tau)} \int_0^\tau \left| \frac{\lambda(s)}{4\sqrt{\pi(\tau-s)^3}} \left(\frac{|p(\tau) - p(s) - a|^2}{4(\tau - s)} - 1 \right) e^{-\frac{|p(\tau)-p(s)-a|^2}{2(\tau-s)^2}} \right| ds d\tau \\ \le & \left(M + Ct \max_{s \in [0,t]} \lambda(s) \right) \int_0^t \frac{d\tau}{\lambda(\tau)}, \end{aligned}$$

since λ is uniformly bounded away from 0 on $[0,T]$. By choosing T sufficiently small we ensure the self-mapping property of L. The contraction property of L follows from

$$\begin{aligned} \|L(p_2)-L(p_1)\|_{L^\infty((0,T))} = \\ &= \max_{t\in[0,T]} \left| \int_0^t \left(\frac{1}{\lambda_1(\tau)} F[p_1,\lambda_1](p_1(\tau),\tau) - \frac{1}{\lambda_2(\tau)} F[p_2,\lambda_2](p_2(\tau),\tau) \right) d\tau \right| \\ &\le CT\|p_2-p_1\|_{L^\infty((0,T))}, \end{aligned} \tag{2.21}$$

where λ_i is the fixed point of $S[p_i]$ for $i=1,2$. In (2.21) we used Lemma 2.2, simple estimates on the derivatives of the heat kernel and in particular (2.10). Once λ and p are known the linear parabolic equation (1.3a) can easily be integrated. □

Lemma 2.4. *Let $\lambda \in L^\infty((0,T))$, $p\in\mathcal{C}([0,T])$ and let (A1) hold. Then the solution f of (1.3) is in $\mathcal{C}((0,T];H^\beta(\mathbb{R}))$ for every $\beta<\frac{3}{2}$. Moreover the estimate*

$$\|f\|_{\mathcal{C}((0,T];H^\beta(\mathbb{R}))} \le C_1\|f_I\|_{L^2(\mathbb{R})} + C_2\|\lambda\|_{L^2(0,T)}$$

holds with generic constants C_1 and C_2.

The proof follows from a simple computation based on the Fourier transformed equation (1.3).

Remark 2.4. *In Theorem 2.3, the time T, determining the length of the local existence interval, can easily be traced to depend only on the quantities $\|f_I\|_{H^4(\Lambda)}$, $\|f_I\|_{L^2(\mathbb{R})}$, r_0 and $1/|\frac{\partial}{\partial x}f_I(p_0)|$. In fact, T can be chosen universally, if $\|f_I\|_{L^2(\mathbb{R})}$, $\|f_I\|_{H^4(\Lambda)}$ and $1/|\frac{\partial}{\partial x}f_I(p_0)|$ vary in a bounded subset of the non-negative reals and if r_0 is bounded away from 0.*

Global Existence. To discuss global existence we make use of a blow-up alternative. For this we need to apply the local existence result on a sequence of time intervals

$$[0,T_0],[T_0,T_1],[T_1,T_2],\dots,[T_{n-1},T_n],\dots,$$

with $T_n > T_{n-1}$. Proceeding by induction, assume that the n-th time step has been carried out, giving a solution on $[T_{n-2},T_{n-1}]$ ($T_{n-2}:=0$). We have to verify that $f(T_{n-1})$ satisfies the assumptions (A1) with $r=rn-1>0$, such that the local existence-uniqueness Theorem 2.3 can be reapplied to extend the solution on $[T_{n-2},T_{n-1}]$. First of all we note that by construction

$$\sup_{t,s\in(T_{j-1},T_j)} |p(t)-p(s)| < a$$

for $j=1,\dots,n-1$, so we can iterate the estimate (2.14) and obtain $\lambda\in L^\infty(0,T_{n-1})$. Thus by Lemma 2.4, $f\in\mathcal{C}((T_{n-2},T_{n-1}];H^\beta(\mathbb{R}))$ for all $\beta<\frac{3}{2}$, so in particular $f(T_{n-1})\in L^2(\mathbb{R})\cap\mathcal{C}(\mathbb{R})$.

Mass conservation (actually conservation of M_+ and M_-) then follows from the solution representation (2.7) by dominated convergence. Thus $f(T_{n-1})\in L^1(\mathbb{R})$. To verify local regularity of $f(T_{n-1})$ around $x=p(T_{n-1})$, we recall the well known localisation estimate for solutions of the one-dimensional heat equation

$$u_t = u_{xx},\ x\in\mathbb{R},\ t>0,$$

which states that there are constants $D_{l,k}>0$ such that

$$\sup_{(x,t)\in C_{\frac{r}{2}}(x_0,t_0)} |\frac{\partial^{l+k}u}{\partial t^l \partial x^k}(x,t)| \le D_{l,k}\frac{1}{r^{k+2l+3}}\|u\|_{L^1(C_r(x_0,t_0))}$$

for all $k,l\in\mathbb{N}\cup\{0\}$. Here the parabolic downward cylinder $C_r(x_0,t_0)$ centered at (x_0,t_0) is given by

$$C_r(x_0,t_0)=\{(x,t)\,|\,|x-x_0|\le r,\ t_0-r^2\le t\le t_0\},$$

see [2, Section 2.3, p. 61]. Now, having constructed $f=f(x,t),\ t\le T_{n-1}$, we choose $0<r_{n-1}<a$ such that

$$C_{r_{n-1}}(p(T_{n-1}),T_{n-1})\subseteq\{(x,t)\,|\,p(t)-a<x<p(t)+a,0\le t\le T_{n-1}\}.$$

In particular, let $w=w_t(\delta),\delta>0$, be the modulus of continuity of $p=p(t)$ at time t. Therefore it suffices to set $r_n=\frac{q_n}{2}$, where

$$w_{T_{n-1}}(q_n^2)=a-q_n$$

(chose the minimal solution). Then f satisfies the heat equation in an open cylinder, which contains $C_{r_{n-1}}(p(T_{n-1}),T_{n-1})$, and

$$\sup_{x\in\left(p(T_{n-1})-\frac{r_{n-1}}{2},p(T_{n-1})+\frac{r_{n-1}}{2}\right)}\left|\frac{\partial f(x,T_{n-1})}{\partial x^k}\right|\le D_{l,k}\frac{M_+ + M_-}{r_{n-1}^{k+1}},\ k\in\mathbb{N}\cup\{0\} \tag{2.22}$$

holds due to mass conservation.
We conclude that (A1) is satisfied at $t=T_{n-1}$, so the local existence and uniqueness Theorem 2.3 can be applied and f can be extended to $[0,T_n]$ for some $\Delta T_{n-1}:=T_n-T_{n-1}>0$ sufficiently small.
Either $T_n\to+\infty$, which implies global in time existence of a unique solution of (1.3) or $T_n\to T_{\max}<\infty$. In the latter case either, possibly after extraction of a subsequence:

(C1) $\gamma_n:=\|f(T_n)\|_{L^2(\mathbb{R})}+\|f(T_n)\|_{H^4((p(T_n)-r_n,p(T_n)+r_n))}\to\infty$ as $n\to\infty$

or, again after possible extraction of a subsequence,

(C2) $\left|\frac{\partial}{\partial x}f(p(T_n),T_n)\right|=\lambda(T_n)\to 0$ as $n\to\infty$

or after maybe extracting another subsequence

(C3) $r_n\to 0$ as $n\to\infty$.

Otherwise the local solution argument could be restarted in $T-\epsilon_1$, with $\epsilon_1>0$ sufficiently small and - due to Remark 2.4 - a solution could be obtained on $[T_{\max}-\epsilon_1,T_{max}+\epsilon_2]$ for some $\epsilon_2>0$.
Before we proceed with discussing global versus local existence we prove that $f_x(p(t),t)<0$ on $[0,T]$, assuming that $p=p(t)$ is in $\mathcal{C}([0,T])$. Note that every solution f of the heat equation in a cylinder $D:=(a,b)\times[T_1,T]$ is analytic in the spatial variable, for each time $t\in(T_1,T]$ (where the non-empty interval (a,b) is contained in,

say, $(p(T)-\frac{a}{4},p(T)+\frac{a}{4})$ and $T_1<T$ but sufficiently close to T, see [2, p. 62]. From this we conclude, that if all spatial derivatives of f at $z_0:=(p(T),T)$ are zero we obtain that $f(x,T)=0$ for all $x\in(a,b)$. This implies that f vanishes identically in D, according to the min-max principle, applied in the small downward cylinders D_1, D_2 to the left and resp., to the right of $p(T)$, such that the free boundary does not intersect the cylinders D_1 and D_2. Without loss of generality we may drop this last case. Thus there exists $K\in\mathbb{N}$ such that the K-th order spatial derivative of f is not zero at $(p(T),T)$:

$$\frac{\partial^K f}{\partial x^K}(p(T),T))\neq 0,$$
$$\frac{\partial^l f}{\partial x^l}(p(T),T))=0, \text{ for } l=1,2,...,K-1.$$

Now let us consider $f\in\mathcal{C}^{2K+1}$ (using again the localisation estimate) in both x and t coordinates and

$$f_{xx}=f_t \qquad \text{in } D:=(a,b)\times[T_1,T], \tag{2.23a}$$
$$f_x<0 \qquad \text{on } B:=\{(x,t)\in D:\ f(x,t)=0\}. \tag{2.23b}$$

We know that B is a graph of a function $p(t)$ in D and:

$$\begin{cases} f(x,t)>0 & \text{when } x<p(t),\\ f(x,t)=0 & \text{when } x=p(t),\\ f(x,t)<0 & \text{when } x>p(t). \end{cases} \tag{2.24}$$

Lemma 2.5. *We have $f_x(p(T),T)<0$, if p is continuous up to $t=T$.*

Remark 2.5. *For the proof of Lemma 2.5 it is sufficient to show that p is differentiable at $t=T^-$. Then the parabolic version of the Hopf Lemma can be applied and gives precisely $f_x(p(T),T)<0$ (since f is negative for $x<p(t)$ and positive for $x>p(t)$).*

Proof. We state the proof in four steps:

I. We know that at time $\{t=T\}$

$$f>0 \text{ for } x<p(T) \text{ and } f<0 \text{ for } x>p(t).$$

Therefore we conclude that

$$K=2N+1 \quad \text{for some } N\in\mathbb{N}$$
$$\frac{\partial^{2N+1}}{\partial x^{2N+1}}f(p(T),T)<0.$$

II. Differentiation of equation (2.23a) with respect to time yields

$$f_{tt}=(f_{xx})_t=(f_{xx})_{xx}=f_{xxxx}.$$

Reiterating the above equation for the mixed derivatives we obtain:

$$\frac{\partial^l f}{\partial t^l}=\frac{\partial^{2l} f}{\partial x^{2l}},$$
$$\frac{\partial^{l+k} f}{\partial t^l\partial x^k}=\frac{\partial^{2l+k} f}{\partial x^{2l+k}},$$

for $l,k\in 0\cup\mathbb{N}$ such that $2l+k=0,1,...,K$.

III. Let us write the Taylor expansion of f at z_0 with the mean-value Lagrange remainder of $(N+1)$-order.

$$\begin{aligned} f(x,t) = & \sum_{\substack{0\le l,k: \\ l+k\le N}} \frac{1}{l!k!}\frac{\partial^{l+k}}{\partial t^l \partial x^k} f(p(T),T)(x-p(T))^k (t-T)^l \\ & + \sum_{\substack{0\le l,k \\ l+k=N+1}} \frac{1}{l!k!} a_{l,k}(x,t)(x-p(T))^k (t-T)^l \end{aligned} \tag{2.25}$$

where

$$a_{l,k}(x,t) = \frac{\partial^{l+k}}{\partial t^l \partial x^k} f(\xi_l(x,t),\mu_k(x,t)) = \frac{\partial^{2l+k}}{\partial x^{2l+k}} f(\xi_l(x,t),\mu_k(x,t)).$$

Here $\xi_l(x,t)$, $\mu_k(x,t)$ are some intermediate points in (a,b) and $(0,T)$ respectively.

Let us note, that $l+k\le N$ implies $2l+k\le 2N$ since $l\le N$. Therefore we conclude that

$$\frac{\partial^{l+k} f}{\partial t^l \partial x^k} f(p(T),T) = 0$$

since all spatial derivatives of order less than $K=2N+1$ vanish at that point. Thus the first sum in (2.25) vanishes and we are left to deal with the remainder, where $2l+k=N+1+l$ and $l\le N+1$.

IV. We calculate the remainder:

$$\begin{aligned} f(x,t) = & \frac{a_{N+1,0}(x,t)}{(N+1)!}(t-T)^{N+1} + \frac{a_{N,1}(x,t)}{N!}(x-p(T))(t-T)^N \\ & + \sum_{\substack{0\le l<N \\ 1<k\le N+1 \\ l+k=N+1}} \frac{1}{l!k!} a_{l,k}(x,t)(x-p(T))^k (t-T)^l \end{aligned} \tag{2.26}$$

Dividing both sides of (2.26) by $(t-T)^{N+1}$, evaluating at $x=p(t)$ and keeping in mind that $f(p(t),t)=0$ we obtain:

$$\begin{aligned} 0 = \frac{f(p(t),t)}{(t-T)^{N+1}} = & \frac{1}{(N+1)!} a_{N+1,0}(p(t),t) + \frac{1}{N!} a_{N,1}(p(t),t)\frac{p(t)-p(T)}{t-T} \\ & + \sum_{1<k\le N+1} \frac{1}{k!(N+1-k)!} a_{N+1-k,k}(p(t),t)\left(\frac{p(t)-p(T)}{t-T}\right)^k \end{aligned} \tag{2.27}$$

Next we take a close look at terms in the sum:

(i) When $l=N+1$ and $k=0$, then $2l+k=2N+2$:

$$a_{N+1,0}(x,t) = \frac{\partial^{N+1}}{\partial t^{N+1}} f(x,t) = \frac{\partial^{2N+2}}{\partial x^{2N+2}} f(x,t) = O(1). \tag{2.28}$$

(ii) When $l=N$ and $k=1$, then $2l+k=2N+1$:

$$a_{N,1}(x,t) = \frac{\partial^{2N+1}}{\partial x^{2N+1}} f(\xi_N(x,t),\mu_1(x,t)).$$

thus there exist positive constants ε,δ such that:

$$a_{N,1}(x,t) \le -\epsilon < 0 \quad \text{in } \mathbb{K}_\delta := \{(x,t)\,|\,|(x,t)-(p(T),T)| < \delta\}. \tag{2.29}$$

(iii) When $l<N$, then $2l+k=N+1+l<2N+1$ and we have:

$$a_{l,k}(x,t)=o(1).$$

Let

$$z(t):=\frac{p(t)-p(T)}{t-T},$$

then (2.27) can be written as

$$\begin{aligned}0=&\frac{1}{(N+1)!}a_{N+1,0}(p(t),t)+\frac{1}{N!}a_{N,1}(p(t),t)z(t)\\&+\sum_{1<k\le N+1}\frac{1}{k!(N+1-k)!}a_{N+1-k,k}(p(t),t)z^k(t)\end{aligned}\tag{2.30}$$

Since all coefficients $a_{N+1-k,k}(p(t),t)$ in the last part of the above sum vanish as t goes to T, we conclude (using the implicit function theorem) that

$$\begin{aligned}\frac{p(t)-p(T)}{t-T}=z(t)&=-\frac{1}{N+1}\frac{a_{N+1,0}(p(t),t)}{a_{N,1}(p(t),t)}+o(1)\\&=-\frac{1}{N+1}\frac{\frac{\partial^{2N+2}}{\partial x^{2N+2}}f(p(T),T)}{\frac{\partial^{2N+1}}{\partial x^{2N+1}}f(p(T),T)}+o(1),\end{aligned}$$

as $t\to T$ and $x\to p(T)$. Thus $p(t)$ left-differentiable at $t=T$, and thus, by the parabolic version of the Hopf Lemma, $f(p(T),T)<0$ and $N=1$ follows.

□

THEOREM 2.6. *Let $T_{\max}<\infty$. Then either*

(a) $\limsup_{t\to T_{max}^-} p(t)=\infty$

or

(b) $\liminf_{t\to T_{max}^-} p(t)=-\infty$

or

(c) $-\infty<\limsup_{t\to T_{max}^-} p(t)<\liminf_{t\to T_{max}^-} p(t)<\infty.$

Proof. Assume that $T_{\max}<\infty$ and $p\in\mathcal{C}([0,T_{\max}])$. Returning to the blow up alternative we can exclude (C2) because of Lemma 2.5. Also, since p is uniformly continuous on $[0,T_{\max}]$ we conclude from the above construction of the sequence r_n that $r_n\ge R$, where R satisfies $R=\frac{Q}{2}$ and Q is the minimal solution of

$$w(Q^2)=a-Q.$$

Here $w(\delta):=\sup_{t\in(0,T)}w_t(\delta)$ is the global modulus of continuity of $p=p(t)$ on $[0,T]$. Therefore (C3) is also excluded. To exclude (C1) it is sufficient to iterate the estimates (2.14) and (2.22) on the intervals $[T_{n-2},T_{n-1}]$, $n=1,2,3,\ldots$. $T_{\max}=\infty$ follows, which is a contradiction. Thus $p=p(t)$ is discontinuous at $T_{\max}$ if $T_{\max}<\infty$, which implies (a),(b) or (c). □

Next we state an interesting result on the large time decay of the function $\lambda=\lambda(t)$.

Lemma 2.6 (Unboundedness of λ in the L^1 norm). *Let assumption (A1) be satisfied and let $T_{max}=+\infty$. Then*

$$\int_0^\infty \lambda(s)\,ds=\infty$$

holds.

Proof. First we derive the following formulas for the (negative and positive) mass.

$$\begin{aligned}
M_- &= \int_{-\infty}^{p(t)} f(x,t)\,dx = \int_{-\infty}^{p(t)}\int_{-\infty}^{\infty}\Gamma(x-y,s)f_I(y)dyds \\
&\quad + \frac{1}{\sqrt{\pi}}\int_0^t \lambda(s)\int_{-\infty}^{\frac{p(t)-p(s)+a}{2\sqrt{t-s}}} e^{-u^2}\,duds - \frac{1}{\sqrt{\pi}}\int_0^t \lambda(s)\int_{-\infty}^{\frac{p(t)-p(s)-a}{2\sqrt{t-s}}} e^{-u^2}\,duds \\
&= \int_{-\infty}^{p(t)}\int_{-\infty}^{\infty}\Gamma(x-y,s)f_I(y)dyds + \frac{1}{\sqrt{\pi}}\int_0^t \lambda(s)\int_{\frac{p(t)-p(s)-a}{2\sqrt{t-s}}}^{\frac{p(t)-p(s)+a}{2\sqrt{t-s}}} e^{-u^2}\,duds \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{p(t)-y}{\sqrt{4t}}} e^{-u^2}\,du f_I(y)dy + \frac{1}{\sqrt{\pi}}\int_0^t \lambda(s)\int_{\frac{p(t)-p(s)-a}{2\sqrt{t-s}}}^{\frac{p(t)-p(s)+a}{2\sqrt{t-s}}} e^{-u^2}\,duds.
\end{aligned} \tag{2.31}$$

and

$$M_+ = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{\frac{p(t)-y}{\sqrt{4t}}}^{\infty} e^{-u^2}\,du f_I(y)dy - \frac{1}{\sqrt{\pi}}\int_0^t \lambda(s)\int_{\frac{p(t)-(s)-a}{2\sqrt{t-s}}}^{\frac{p(t)-p(s)+a}{2\sqrt{t-s}}} e^{-u^2}\,duds, \tag{2.32}$$

(obtained by analogous calculations). We write

$$\begin{aligned}
M_- = &\frac{1}{\sqrt{\pi}}\int_{-\infty}^{p_0}\int_{-\infty}^{\frac{p(t_n)-y}{\sqrt{4t_n}}} e^{-u^2}\,du\,f_I(y)\,dy \\
&-\frac{1}{\sqrt{\pi}}\int_{p_0}^{\infty}\int_{-\infty}^{\frac{p(t_n)-y}{\sqrt{4t_n}}} e^{-u^2}\,du\,|f_I(y)|\,dy + A(t_n)
\end{aligned}$$

and

$$\begin{aligned}
-M_+ = &\frac{1}{\sqrt{\pi}}\int_{-\infty}^{p_0}\int_{\frac{p(t_n)-y}{\sqrt{4t_n}}}^{\infty} e^{-u^2}\,du\,f_I(y)\,dy \\
&-\frac{1}{\sqrt{\pi}}\int_{p_0}^{\infty}\int_{\frac{p(t_n)-y}{\sqrt{4t_n}}}^{\infty} e^{-u^2}\,du\,|f_I(y)|\,dy - A(t_n).
\end{aligned}$$

where $A(t)$ is given by

$$A(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty \lambda(s)\mathbb{1}_{s<t}\int_{\frac{p(t)-p(s)-a}{2\sqrt{t-s}}}^{\frac{p(t)-p(s)+a}{2\sqrt{t-s}}} e^{-u^2}\,duds.$$

Next we assume that $\int_0^\infty \lambda(s)\,ds < \infty$ and choose a sequence t_n such that $t_n \to \infty$ as $n \to \infty$. Thus

$$A(t_n) = \frac{1}{\sqrt{\pi}} \int_0^\infty \lambda(s) \mathbb{1}_{s<t_n} \int_{\frac{p(t_n)-p(s)-a}{2\sqrt{t_n-s}}}^{\frac{p(t_n)-p(s)+a}{2\sqrt{t_n-s}}} e^{-u^2}\,duds$$
$$\leq \frac{1}{\sqrt{\pi}} \int_0^\infty \lambda(s) \mathbb{1}_{s<t_n} \int_{-\infty}^{\infty} e^{-u^2}\,duds.$$

. Now if $s < t_n$ we obtain;

$$\frac{p(t_n)-p(s)-a}{\sqrt{t_n-s}} = \underbrace{\frac{p(t_n)}{\sqrt{t_n-s}}}_{=:\alpha_n(s)} - \underbrace{\frac{p(s)+a}{\sqrt{t_n-s}}}_{=:\beta_n}$$

$$\frac{p(t_n)-p(s)+a}{\sqrt{t_n-s}} = \frac{p(t_n)}{\sqrt{t_n-s}} - \underbrace{\frac{p(s)-a}{\sqrt{t_n-s}}}_{=:\gamma_n}$$

since $\lim_{n\to\infty} \beta_n(s) = \lim_{n\to\infty} \gamma_n(s) = 0$ for all $s > 0$ we have:

$$\int_{\frac{p(t_n)-p(s)-a}{2\sqrt{t_n-s}}}^{\frac{p(t_n)-p(s)+a}{2\sqrt{t_n-s}}} e^{-u^2}\,duds = \int_{\alpha_n(s)+\gamma_n(s)}^{\alpha_n(s)+\beta_n(s)} e^{-u^2}\,duds$$
$$= e^{-\xi_n^2}(\beta_n - \gamma_n) \to 0 \,\text{as}\, n \to \infty,$$

for some $\xi_n \in (\alpha_n(s)+\gamma_n(s), \alpha_n(s)+\beta_n(s))$. From dominated convergence we conclude

$$A(t_n) \to 0 \text{ as } n \to \infty.$$

Furthermore we note that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{p(t_n)-y}{\sqrt{4t_n}}} e^{-u^2}\,du = \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{p(t_n)}{\sqrt{4t_n}}} e^{-u^2}\,du}_{=:\sigma_n} + \frac{1}{\sqrt{\pi}} \int_{\frac{p(t_n)}{\sqrt{4t_n}}}^{\frac{p(t_n)-y}{\sqrt{4t_n}}} e^{-u^2}\,du.$$

The second integral is bounded from above by

$$\int_{-\infty}^{\infty} e^{-u^2}\,du$$

and thus, by dominated convergence, pointwise tends to zero as $n \to \infty$. Now, using again (2.31) and (2.32), we can write

$$M_- = \sigma_n M_- - \sigma_n M_+ + B_n,$$

where B_n tends to 0 as n tends to infinity. We obtain in the limit $n \to \infty$ (after maybe passing to a subsequence), with $\sigma = \lim_{n\to\infty} \sigma_n$,

$$(1-\sigma)M_- + \sigma M_+ = 0$$

which is contradiction as $0 \leq \sigma \leq 1$, $M_- > 0$, $M_+ > 0$. □
To conclude this section, we present a result about the regularity of the free boundary for equation (1.3) when the delta-distributions in the parabolic equation are replaced by smoothed approximations D. We assume the following:

(A2) D in $\mathcal{C}_0^\infty(-a,a)$, $D\geq 0$ and $\int_{-a}^{a} D\,dx = 1$.
and consider the (smoothed) FBP:

$$\begin{aligned}&\frac{\partial f}{\partial t}-\frac{\partial^2 f}{\partial^2 x}=\lambda(t)\left(D(x-p(t)+a)-D(x-p(t)-a)\right)\\&f(x,t)>0 \text{ if } x<p(t),\ f(x,t)<0 \text{ if } x>p(t)\end{aligned}\tag{2.33}$$

with

$$f(x,0)=f_I(x),\, p(0)=p_0. \tag{2.34}$$

Note that we do not go through the local existence arguments previously discussed for the smoothed model, the arguments are very similar to the case with the delta distributions.

Lemma 2.7 (Local boundedness of the free boundary). *Let $p=p(t)$ be the free boundary, assume that $\frac{d}{dx}f_I$ is in $L^2(\mathbb{R})$ and that (A1) holds. Then there is $C>0$ such that*

$$|p(t)|\leq C,\, t\in[0,T_{max}).$$

Proof. We start by deriving a bound on $\lambda(t)=-f_x(p(t),t)$, where f satisfies (2.33). Differentiating the equation w.r.t. x, multiplying by f_x and integrating over $\mathbb{R}$ leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(f_x)^2\,dx &= -\int_{\mathbb{R}}(f_{xx})^2\,dx+\int_{\mathbb{R}}\lambda(t)f_x(x,t)\left(D'(x-p(t)+a)-D'(x-p(t)-a)\right)dx\\ &\leq -\int_{\mathbb{R}}(f_{xx})^2\,dx+K\lambda(t)\|f_x\|_{L^2(\mathbb{R})}\\ &\leq -\int_{\mathbb{R}}(f_{xx})^2\,dx+K\|f_x\|_{L^\infty(\mathbb{R})}\|f_x\|_{L^2(\mathbb{R})},\end{aligned}$$

where $K=2\|D'\|_{L^2(\mathbb{R})}<\infty$, as $D(x)$ and its derivatives are bounded. Next, we reiterate the estimate

$$\|f_x\|_{L^\infty}\leq C\left(\|f_x\|_{L^2}+\|f_{xx}\|_{L^2}\right) \tag{2.35}$$

and conclude

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(f_x)^2\,dx &\leq -\int_{\mathbb{R}}(f_{xx})^2\,dx+CK\|f_x\|^2_{L^2(\mathbb{R})}+CK\|f_x\|_{L^2(\mathbb{R})}\|f_{xx}\|_{L^2(\mathbb{R})}\\ &\leq -\int_{\mathbb{R}}(f_{xx})^2\,dx+CK\|f_x\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\|f_{xx}\|^2_{L^2(\mathbb{R})}+\frac{(CK)^2}{2}\|f_x\|_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(f_x)^2\,dx+\frac{1}{2}\int_{\mathbb{R}}(f_{xx})^2\,dx\leq C_2\int_{\mathbb{R}}f_x^2\,dx \tag{2.36}$$

with some constant $C_2\geq\max\left\{CK,\frac{(CK)^2}{2}\right\}$. Integration w.r.t. t results in

$$\begin{aligned}\int_{\mathbb{R}}(f_x(t))^2\,dx+\frac{1}{2}\int_0^T\int_{\mathbb{R}}(f_{xx}(s))^2\,dxds\leq C(T)\int_{\mathbb{R}}(f_x(t=0))^2\,dx.\\ \forall t\in[0,T],\, T\leq T_{max}.\end{aligned}\tag{2.37}$$

From this we conclude

$$f_x \in L^\infty((0,T);L^2(\mathbb{R})),\ f_{xx} \in L^2((0,T);L^2(\mathbb{R})).$$

The estimate (2.35) gives

$$f_x \in L^2((0,T);H^1(\mathbb{R})).$$

As $\lambda(t) = -f_x(p(t),t)$ this also means $\lambda \in L^2((0,T))$.
We continue by stating an explicit formula for f, using again Duhamel's principle.

$$\begin{aligned} f(x,t) = & \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} f_I(y)dy \\ & + \int_0^t \frac{\lambda(s)}{\sqrt{4\pi(t-s)}} \left(\int_{\mathbb{R}} D(x-p(s)+a) e^{-\frac{|x-y|^2}{4(t-s)}}\, dy \right. \\ & \qquad \left. - \int_{\mathbb{R}} D(x-p(s)-a) e^{-\frac{|x-y|^2}{4(t-s)}}\, dy \right) ds \end{aligned}$$

Now we use this representation of f to calculate the masses M_+ and M_-. In analogy to (2.31) and (2.32) we obtain:

$$\begin{aligned} M_- = & \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{p(t)-y}{\sqrt{4t}}} e^{-u^2} du f_I(y) dy \\ & + \frac{1}{\sqrt{\pi}} \int_0^t \lambda(s) \int_{\mathbb{R}} D(y') \int_{\frac{p(t)-p(s)-a-y'}{\sqrt{4(t-s)}}}^{\frac{p(t)-p(s)+a-y'}{\sqrt{4(t-s)}}} e^{-\frac{u^2}{2}} du dy' ds \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} M_+ = & \int_{p(t)}^{\infty} f(x,t)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \int_{\frac{p(t)-y}{\sqrt{4t}}}^{\infty} e^{-u^2} du f_I(y) dy \\ & + \frac{1}{\sqrt{\pi}} \int_0^t \lambda(s) \int_{\mathbb{R}} D(y') \int_{\frac{p(t)-p(s)-a-y'}{\sqrt{4(t-s)}}}^{\frac{p(t)-p(s)+a-y'}{\sqrt{4(t-s)}}} e^{-\frac{u^2}{2}} du dy' ds \end{aligned} \tag{2.39}$$

Assume now that $D \in L^1(\mathbb{R})$ and that there exists a sequence $t_n \nearrow T_{max}$ as $n \to \infty$ such that $p(t_n) \to \infty$ as $n \to \infty$. Since $f_I \in L^1(\mathbb{R})$ and since $\lambda \in L^2((0,T_{max}))$ we conclude, by using dominated convergence in both integrals of (2.39) that $M_+ \to 0$ as $n \to \infty$. This contradicts mass conservation on $[p(t_n),\infty)$. If $p(t_n) \to -\infty$ as $t_n \nearrow T_{max}$ we proceed analogously with the formula (2.38) for M_-. ☐

THEOREM 2.8. *Let (A1), (A2) hold, $\frac{d}{dx} f_I$ in $L^2(\mathbb{R})$. Then either*
1.) $T_{max} = \infty$,
2.) $-\infty < p^- := \liminf_{t \nearrow T_{max}} p(t) < \limsup_{t \nearrow T_{max}} p(t) =: p^+ < +\infty$.
Furthermore if 2.) holds, then $f \equiv 0$ in the interval (p^-, p^+).

Proof. We first note that all previous results for equation (1.3) also hold for (2.33). Thus combining Theorem 2.6 with Lemma 2.7 we conclude

$$-\infty < \liminf_{t \nearrow T_{max}} p(t) < \limsup_{t \nearrow T_{max}} p(t) < +\infty \tag{2.40}$$

if T_{max} is finite. We continue by showing that $f\in\mathcal{C}([0,T_{max}];\mathcal{C}(\mathbb{R}))$. First we reiterate that $f_x\in L^2((0,T_{max});H^1(\mathbb{R}))$. Next we notice $f_{xx}\in L^2((0,T_{max});L^2(\mathbb{R}))$ from which we conclude $f_{xxx}\in L^2((0,T_{max});H^{-1}(\mathbb{R}))$ and thus

$$f_{xt}=f_{xxx}+\lambda(t)(D'(x-p(t)+a)-D'(x-p(t)-a))\in L^2((0,T_{max});H^{-1}(\mathbb{R})).$$

We now use Theorem 3 from [2, Section 5.9, p. 287] to conclude

$$f_x\in\mathcal{C}([0,T_{max}];L^2(\mathbb{R})).$$

Since it is easy to show that $f\in\mathcal{C}([0,T_{max}];L^2(\mathbb{R}))$ we obtain

$$f\in\mathcal{C}([0,T_{max}];H^1(\mathbb{R})).$$

As in one dimension the space H^1 can be embedded (via Morrey's inequality) into a space of (Hölder) continuous functions, we finally conclude

$$f\in\mathcal{C}([0,T_{max}];\mathcal{C}(\mathbb{R})). \tag{2.41}$$

To prove that $f(x,T_{max})=0$ in (p^-,p^+), we first fix $x\in(p^-,p^+)$ and choose a sequence t_n such that $t_n\nearrow T_{max}$ as $n\to\infty$ and $f(x,t_n)<0$ for all n (note that there is a sequence $\tau_n\nearrow T_{max}$ as $n\to\infty$, such that $p(\tau_n)=x$ and $f(x,t)>0$ for $t\in(\tau_{2k+1},\tau_{2k+2})$, $f(x,t)<0$ for $t\in(\tau_{2l},\tau_{2l+1})$). Then we conclude, by the continuity of f, that

$$f(x,T_{max})\le 0.$$

Analogously, we obtain $f(x,T_{max})\ge 0$ and thus

$$f(x,T_{max})=0\quad \forall x\in(p^-,p^+).$$

□

Corollary 2.9. *Let (A1), (A2) hold and additionally $f_I\in H^2(\mathbb{R})$. Then, $\lambda(t)\to 0$ as $t\nearrow T_{max}$.*

Proof. Is is a simple exercise to show that $f\in\mathcal{C}([0,T_{max}];H^2(\mathbb{R}))$. Thus $f\in\mathcal{C}([0,T_{max}];\mathcal{C}^1(\mathbb{R}))$. Since $f(x,T_{max})=0$ for $x\in(\liminf_{t\nearrow T_{max}},\limsup_{t\nearrow T_{max}})$ the statement follows. □

The existence-uniqueness theory presented in this paper does not exclude the occurence of a 'fat' free boundary in finite time. Although $f(t)$ approaches $f(T_{max})$ in a very smooth way, the local existence theorem cannot be restarted at $t=T_{max}$ since no uniquely defined initial value for the free boundary p can be found to solve the integral-differential system (2.17a), (2.17b). In fact, this can be dealt with by looking for weaker solutions in the framework of nonlinear semigroups, i.e. by employing an implicit Euler-type time discretization of the form

$$\frac{f^{n+1}-f^n}{\Delta t}=f_{xx}^{n+1}+\lambda^{n+1}\left(D(x-p^{n+1}+a)-D(x-p^{n+1}-a)\right), \tag{2.42}$$

where $f^n\approx f(t_n)$, $\lambda^n\approx\lambda(t_n)$ and $p^n\approx p(t_n)$ with $t_n:=n\Delta t$ for some $\Delta t>0$. It has to be shown that - given g^n appropriately - the elliptic equation (2.42) can be solved for f^{n+1}, p^{n+1} with $\lambda^{n+1}:=-f_x^{n+1}$. Thus we consider (following [7]) the stationary problem, for $\kappa>0$:

$$\kappa^2 f-\frac{d^2f}{dx^2}=g+\lambda(D(x-p+a)-D(x-p-a)) \tag{2.43a}$$

$$f(x)>0\,\text{if}\,x<p,\, f(x)<0\,\text{if}\,x>p. \tag{2.43b}$$

Here, $\kappa^2 = \frac{1}{\Delta t}$, where D denotes either the Dirac delta or an approximation (as above). Furthermore g is a given smooth function (which can be thought of as the result of the previous iteration). We proceed as in [7] and write down the solution of equation (2.43a) via convolution with the Green's function of $-\frac{d^2}{dx^2}+\kappa^2$, i.e.

$$\frac{1}{2\kappa}e^{-\kappa|x|}. \tag{2.44}$$

In the case where $D=\delta$, we obtain

$$f = G + \frac{\lambda}{2\kappa}\left(e^{-\kappa|x-p+a|} - e^{-\kappa|x-p-a|}\right), \tag{2.45}$$

with $G = \frac{1}{2\kappa}e^{-\kappa|x|} * g$. We notice $f(p) = G(p)$ and therefore p can be determined uniquely as long as G has exactly one zero. To determine $\lambda = -\frac{\partial f}{\partial x}(p)$ we differentiate (2.45) and set $x = p$. Thus

$$\frac{\partial f}{\partial x}(p) = \frac{dG}{dx}(p) + e^{-\kappa a}\frac{df}{dx}(p). \tag{2.46}$$

This equation can be solved to obtain $\frac{df}{dx}(p)$, as $e^{-\kappa a} < 1$.

Remark 2.10. *In the case where D is an approximation of the Dirac delta, equation* (2.46) *has to be replaced by*

$$\frac{\partial f}{\partial x}(p) = \frac{dG}{dx}(p) \underbrace{- \frac{1}{2\kappa}\int_{-a}^{a} D_x(z)\left(e^{\kappa(z-a)} - e^{-\kappa(z+a)}\right)dx}_{=:A(\kappa)} \frac{\partial f}{\partial x}(p). \tag{2.47}$$

$A(\kappa)$ is, for $\kappa > 0$ strictly less than 1 as $A(0) = 1$, $\lim_{\kappa \nearrow +\infty} A(\kappa) = 0$ (since $\int_{\mathbb{R}} D(x)\,dx = 1$) and A is decreasing. Thus, this equation can be solved to obtain $\frac{\partial f}{\partial x}(p)$.

What is left is to show that G really has only one zero, even if g has a 'fat' zero(-interval). The case when g has a unique zero in $\mathbb{R}$ was dealt with in [7]. We proceed similarly but assume that there are $-\infty \le \underline{p} < \overline{p} < +\infty$ and $g(x) > 0$ for $x < \underline{p}$, $g(x) = 0$ for $\underline{p} \le x \le \overline{p}$ and $g(x) < 0$ for $x > \overline{p}$. We calculate:

$$2\kappa G(x) = \int_{-\infty}^{\underline{p}} e^{-\kappa|x-y|} g_+(y)dy - \int_{\overline{p}}^{\infty} e^{-\kappa|x-y|}|g_-(y)|dy$$

For $x > \overline{p}$ we obtain

$$\begin{aligned} &\int_{-\infty}^{\underline{p}} e^{-\lambda|x-y|} g_+(y)dy - \int_{\overline{p}}^{\infty} e^{-\lambda|x-y|}|g_-(y)|dy \\ &= e^{-\lambda x} \underbrace{\left(\int_{-\infty}^{\underline{p}} e^{\lambda y} g_+(y)dy - \int_{\overline{p}}^{x} e^{\lambda y}|g_-(y)|dy - e^{2\lambda x}\int_{x}^{\infty} e^{-\lambda y}|g_-(y)|dy\right)}_{=:S_+(x)} \end{aligned}$$

For $x < \underline{p}$ analogous calculations lead to

$$\int_{-\infty}^{\underline{p}} e^{-\lambda|x-y|} g_+(y)dy - \int_{\overline{p}}^{\infty} e^{-\lambda|x-y|}|g_-(y)|dy$$
$$= e^{\lambda x} \underbrace{\left(e^{-2\lambda x} \int_{-\infty}^{x} e^{\lambda y} g_+(y)dy + \int_{x}^{\underline{p}} e^{-\lambda y} g_+(y)dy - \int_{\overline{p}}^{\infty} e^{-\lambda y}|g_-(y)|dy \right)}_{=:S_-(x)}.$$

We have

$$S_-'(x) = -2\lambda e^{-2\lambda x} \int_{-\infty}^{x} e^{\lambda y} g_+(y)dy$$
$$S_+'(x) = -2\lambda e^{2\lambda x} \int_{x}^{\infty} e^{-\lambda y}|g_-(x)|dy$$

and thus $S_-(x)$ is decreasing for $x < \underline{p}$, $S_+(x)$ for $x > \overline{p}$. Furthermore, we have

$$S_-(\underline{p}) = e^{-2\lambda \underline{p}} \int_{-\infty}^{\underline{p}} e^{\lambda y} g_+(y)dy - \int_{\overline{p}}^{\infty} e^{-\lambda y}|g_-(y)|dy \tag{2.48}$$
$$S_+(\overline{p}) = \int_{-\infty}^{\underline{p}} e^{\lambda y} g_+(y)dy - e^{2\lambda \overline{p}} \int_{\overline{p}}^{\infty} e^{-\lambda y}|g_-(y)|dy \tag{2.49}$$

Now, we can state

Lemma 2.11. *Let $D(x) = \delta(x)$ or let (A2) hold and*

$$g(x) \begin{cases} > 0, & x < \underline{p}, \\ = 0, & x \in [\underline{p}, \overline{p}], \\ < 0, & x > \overline{p}. \end{cases}$$

Then, under the additional assumption

$$\int_{-\infty}^{\underline{p}} e^{-\lambda y} g_+(y)\,dy - \int_{\overline{p}}^{\infty} e^{-\lambda y}|g_-(y)|\,dy > 0 > \int_{-\infty}^{\underline{p}} e^{\lambda y} g_+(y)\,dy - \int_{\underline{p}}^{\infty} e^{\lambda y}|g_-(y)|\,dy \tag{2.50}$$

there exactly exists one $\tilde{p} \in (-\infty, \infty)$ such that

$$G(\tilde{p}) = 0.$$

Proof. First we notice that (2.50) means precisely $S_-(-\infty) > 0 > S_+(\infty)$. Aslo, note that for $x \in (\underline{p}, \overline{p})$ we have

$$2\kappa G(x) = \int_{-\infty}^{\underline{p}} e^{-\lambda|x-y|} g_+(y)dy - \int_{\overline{p}}^{\infty} e^{-\lambda|x-y|}|g_-(y)|dy \tag{2.51}$$
$$= e^{-\lambda x} \int_{-\infty}^{\underline{p}} e^{\lambda y} g_+(y)\,dy - e^{\lambda x} \int_{\overline{p}}^{\infty} e^{-\lambda y}|g_-(y)|\,dy \tag{2.52}$$
$$=: H(x). \tag{2.53}$$

By differentiation we conclude $H' < 0$. Using (2.48) and (2.49) we find

$$S_-(\underline{p}) = e^{-\lambda \underline{p}} H(\underline{p}), \tag{2.54}$$

$$S_+(\overline{p}) = e^{\lambda \overline{p}} H(\overline{p}), \tag{2.55}$$

and thus

$$\operatorname{sgn} H(\underline{p}) = \operatorname{sgn} S_-(\underline{p}), \tag{2.56}$$

$$\operatorname{sgn} H(\overline{p}) = \operatorname{sgn} S_+(\overline{p}). \tag{2.57}$$

Now we consider the following cases

case 1 $S_-(\underline{p}) \le 0$.
We imediately conclude (as S_- is decreasing) that there exists a $\tilde{p} \in (-\infty, \underline{p}]$ sucht that

$$S_-(\tilde{p}) = 0 \quad \text{and} \quad H(\underline{p}) \le 0. \tag{2.58}$$

Since $H' < 0$ we conclude $H(\overline{p}) < 0$ and thus $S_+(\overline{p}) < 0$. Since $S_+' < 0$ on $(\overline{p}, \infty)$ we finally obtain that there exists exactly one $\tilde{p}$ such that $G(\tilde{p}) = 0$.

case 2 $S_-(\underline{p}) > 0$.
First we notice that in this case $S_- > 0$ on $(-\infty, \underline{p})$ and that $H(\underline{p}) > 0$. We also reiterate $H' < 0$ on $(\underline{p}, \overline{p})$. Now there are two possibilities:

case 2a $H(\overline{p}) > 0$ and thus $S_+(\overline{p}) > 0$.
Since S_+ decays on $(\overline{p}, \infty)$ and since $S_+(\infty) < 0$ we conclude again that there exists a $\tilde{p} \in [\overline{p}, \infty)$ such that

$$S_+(\tilde{p}) = G(\tilde{p}) = 0. \tag{2.59}$$

case 2b $H(\overline{p}) \le 0$. Then we conclude in the same way as in 2a) that there is a $\tilde{p} \in [\underline{p}, \overline{p}]$ such that $G(\tilde{p}) = 0$.

Putting all these cases together we finally obtain that there exist a $\tilde{p} \in (-\infty, \infty)$ such that

$$G(\tilde{p}) = 0.$$

□

Thus, at least in the implicit Euler discretized framework, a fat free boundary is smoothed out after a single time step. This is the basis for proving that the solution of the free boundary problem (2.23), (2.24) can be extended beyond, albeit as a mild solution according to nonlinear semigroup theory. To be precise, it has to be shown that the constraint (2.50) is maintaned by the discrete evolution and that the discretisation converges to a mild solution of the FBP. This programme has already been outlined in [7].

3. Numerical Results For actual numerical computations there is no need to use the fully implicit scheme presented in the previous section. Instead, we use an implicit-explicit scheme to solve (1.3) on a bounded domain. Let f_I and p_0 denote the initial data satisfying the compatibility condition (1.4). Then

1. Solve (1.3) for $f(x, t_j)$, given $p(t_{j-1})$,
2. Update the free boundary $p(t_j)$ such that $f(p(t_j), t_j) = 0$
3. Set $j = j + 1$, go to 1.

Equation (1.3) is discretized in space using a finite difference method, where the convection term $\frac{\partial f}{\partial x}(p(t),t)$ is approximated by its upwind difference quotient. The resulting ODE system is solved using an implicit Euler method, resulting in the following numerical discretization at $x=x_n$ and $t=t_{j+1}$

$$\begin{aligned}\frac{f_n^{j+1}-f_n^j}{\tau}=\frac{\sigma^2}{2}\frac{f_{n+1}^{j+1}-2f_n^{j+1}+f_{n-1}^{j+1}}{h^2}-\\ -\frac{\sigma^2}{2}\frac{1}{h}\left[-q(x_n)^+f_{k-1}^{j+1}+|q(x_n)|f_k^{j+1}+q(x_n)^-f_{k+1}^{j+1}\right]\end{aligned} \tag{3.1}$$

where $q(x_n)=(\delta(x_n-p(t_{k-1})-a)-\delta(x_n-p(t_{k-1})+a))$, $q^+=\max(q,0)$, $q^-=\min(q,0)$ and k denotes the index such that $f(p(t_{k-1}),t_j)=0$. Here h denotes the mesh size, τ the time steps of the implicit Euler method. The Dirac δ is approximated by a Gaussian

$$\delta_\varepsilon(x)=\frac{1}{\varepsilon\sqrt{\pi}}e^{-\frac{x^2}{\varepsilon^2}}$$

where ε is chosen such that $\delta(a)=\delta(-a)=10^{-6}$.

PROPOSITION 3.1. *Let $p(t_{j-1})$ be given and $f(p(t_{j-1}),t_j)=0$. If $\tau<2k\max(q)$, then the matrix defined by* (3.1) *is strictly diagonally dominant, hence regular.*

Proof. The function $q(x)$ is positive on $[p(t)-2a,0)$, negative on $(0,p(t)+2a]$ and has a compact support on $[p(t)-2a,p(t)+2a]$. Therefore we consider the following different cases. If $x_n<p(t)-2a$ then $q(x_n)$ equals zero and (3.1) is the standard discretization of the heat equation. If $x_n\in[p(t)-2a,0)$ then $q(x_n)>0$ and if $\tau<2h\max(q)$ then

$$|a_{ii}|>\sum_{i\neq j}|a_{ij}|.$$

The same argument holds for $x_n\in[0,p(t)+2a)$, therefore the system matrix is strictly diagonally dominant. ∎

3.1. Numerical Experiments on Large Domains In this Section we present long-time numerical experiments on large domains to illustrate the behaviour of solutions on the unbounded domains. We observe that depending on the initial masses M_1 or M_2, the price $p(t)$ either decreases or increases with a rate proportional to $\sqrt{t}$. We choose $\Omega=[-400,400]$, discretized with a non-equidistant mesh of meshsize $h_1=10^{-3}$ in $x\in[p(t)-4a,p(t)+4a]$ and $h_2=1$ for $x\in[-400,p(t)-4a)$ or $x\in(p(t)+4a,400]$ and $a=1$. We solve the discrete scheme (3.1) on the time interval $[0,400]$, with time steps $\tau=5\times10^{-3}$. The initial datum we chose is

$$f_I(x)=\begin{cases}10^{-6} & \text{for } x\in[-400,-1)\\ 2.2222x^3-0.2222x^2-2.4444x & \text{for } x\in[-1,1.1]\\ -10^{-6} & \text{for } x\in(1.1,400]\end{cases}$$

with initial masses $M_1=0.5927$ and $M_2=0.7642$. The evolution of the price $p(t)$ is depicted in Figure 3.1. In fact the price $p(t)$ is proportional to $\sqrt{t}$.

In the second example we choose an initial guess with $M_2<M_1$, given by $f_I(x)=0.5882x^3+0.0588x^2-0.5294x$ on the interval $(-1,0.9)$ and $f_I(x)=\pm10^{-6}$ outside this interval. All other parameters take the same values as in the previous example. Since

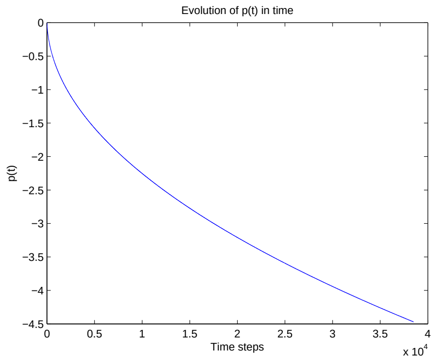


Fig. 3.1. *Evolution of the price $p(t)$ in time for $M_1 < M_2$*

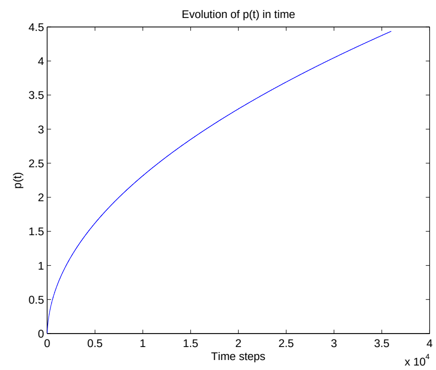


Fig. 3.2. *Evolution of the price $p(t)$ in time for $M_1 > M_2$*

$M_1 = 0.1373$ and $M_2 = 0.1037$, the price increases again proportionally to $\sqrt{t}$, which can be seen in Figure 3.2.
Note that the second derivative of f at $x = p_0$ determines the initial direction of $p(t)$, since

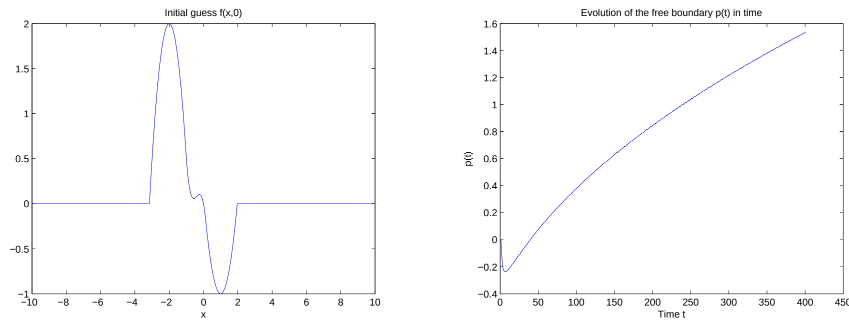
$$\dot{p}(t) = -\frac{f_{xx}(p(t),t)}{f_x(p(t),t)}, \quad f_{Ixx}(p_0) = 0.$$

Therefore we can construct examples, where the price is not monotonously increasing or decreasing in time. We choose the following initial guess

$$f_I(x) = \begin{cases} 10^{-5} & \text{for } x \in [-100, -3.1547) \\ -1.5x^2 - 6x - 4 & \text{for } x \in [-3.1547, -1) \\ -2.5x^3 - 3x^2 - x & \text{for } x \in [-1, 0.1) \\ 1.071x^2 - 2.142x + 0.071 & \text{for } x \in [0.1, 1.96) \\ -10^{-5} & \text{for } x \in [1.96, 100], \end{cases}$$

depicted in Figure 3.3(a). The function f_I is concave at $x = 0$, but $M_1 > M_2$. Due to the negative curvature of f_I at $x = 0$, the price $p(t)$ initially decreases, but since

$M_1 > M_2$ it starts to increase after some iterations. In this example we are only interested in the initial behaviour, therefore we choose $\Omega = [-100, 100]$ and calculate the first 400 time steps with $\tau = 5 \times 10^{-3}$. The evolution of the price $p(t)$ is depicted in Figure 3.3(b).



(a) Zoom of initial guess $f_I(x)$ (b) Evolution of $p(t)$

FIG. 3.3. *Non-monotonous behaviour of* $p(t)$

3.2. Numerical Experiments on Bounded Domains Finally we would like to illustrate the behaviour of solutions on bounded domains. Here the solutions converge quickly to their stationary state, which can be calculated explicitly using (1.5).

We choose $\Omega = [0, 1]$ with an equidistant mesh of meshsize $h = 10^{-3}$ and a temporal meshsize $\tau = 10^{-3}$. The initial data is given by the cubic polynomial

$$f_I(x) = 145.833x^3 - 233.333x^2 + 87.5x,$$

which has the root at $x = 0.6$. Therefore $p_0 = 0.6$ and the initial masses are $M_1 = 3.675$ and $M_2 = 1.2443$. The parameter a is set to 0.1 and $\frac{\sigma^2}{2} = 1$. To ensure the mass preservation property, system (1.3) is supplemented with homogeneous Neumann boundary conditions at $x = 0$ and $x = 1$. Figure 3.4(a) illustrates the evolution of the density $f(x,t)$ in time. The solution converges quickly to the stationary profile given by (1.5). The numerically calculated price "converges" towards $p(t) = 0.709$ (see Figure 3.4(b)), the stationary price calculated from (1.5) is given by $p_{stat} = 0.71$.

In case of symmetric initial data the moving boundary is constant in time, i.e. $p(t) = p_0$ (cf. [4]). In order to test the numerical method we choose $f_I(x) = \sin(2\pi x)$ and the same discretization in space and time as in the previous example. The evolution of $f(x,t)$ is illustrated in Figure 3.5(a). As expected the numerically calculated price $p(t)$ is constant in time, see Figure 3.5(b).

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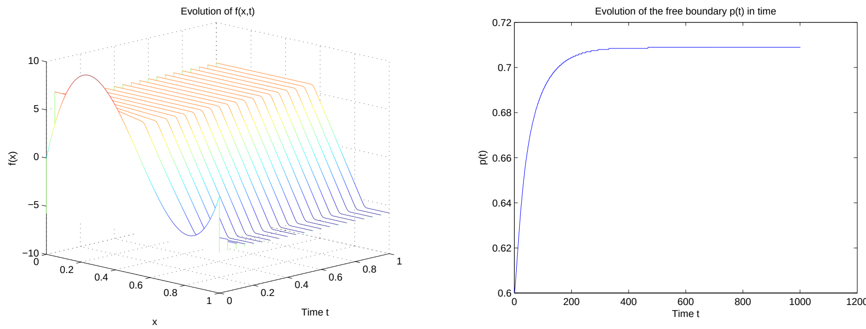
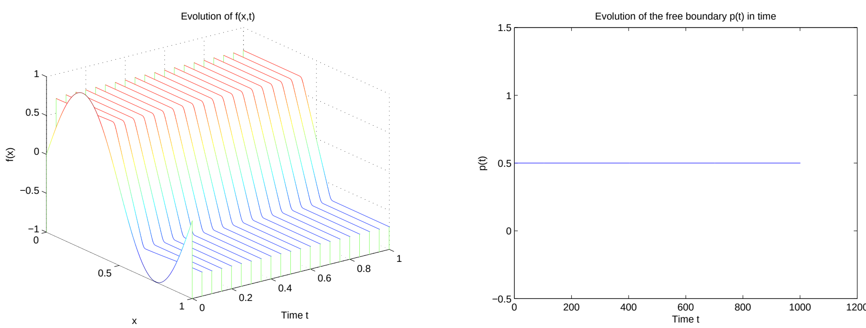


(a) Evolution of $f(x,t)$ (b) Evolution of $p(t)$

Fig. 3.4. *Solution of mean-field equation* (1.3)



(a) Evolution of $f(x,t)$ (b) Evolution of $p(t)$

Fig. 3.5. *Solution of mean-field equation* (1.3) *with symmetric initial data*

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