

TANGENTIAL TOUCH BETWEEN FREE AND FIXED BOUNDARIES IN A PROBLEM FROM SUPERCONDUCTIVITY

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ABSTRACT. In this paper we study regularity properties of the free boundary problem

$$\Delta u = \chi_{\{|\nabla u| \neq 0\}} \text{ in } B_1^+, \quad u = 0 \text{ on } B_1 \cap \{x_1 = 0\},$$

where $B_1^+ = \{|x| < 1, x_1 > 0\}$ and $B_1 = \{|x| < 1\}$. If the origin is a free boundary point, then we show that the free boundary touches the fixed boundary $\{x_1 = 0\}$ tangentially.

1. INTRODUCTION

The aim of this paper is to analyze the regularity of solutions and the behavior of the free boundary near the fixed one for a certain type of free boundary problem. Mathematically the problem is formulated as follows. Suppose we are given a function u such that

$$(1.1) \quad \begin{cases} \Delta u = \chi_{\{|\nabla u| \neq 0\}} & \text{in } B_1^+, \text{ in the sense of distributions} \\ u = 0 & \text{on } \Pi \cap B_1, \end{cases}$$

where $B_1^+ = \{|x| < 1, x_1 > 0\}$, $B_1 = \{|x| < 1\}$ and $\Pi = \{x_1 = 0\}$. Let us denote $\Omega = \Omega(u) = \{|\nabla u| \neq 0\}$, $\Lambda = \Lambda(u) = \{|\nabla u| = 0\}$, $\Gamma(u) = \{x : |\nabla u(x)| = 0\} \cap \partial\Omega$ the free boundary and $\Gamma^*(u) = \Gamma(u) \cap \Pi$ is the set of contact points (see Figure1).

Note that from classical elliptic regularity theory we have that the solutions of the problem (1.1) are in the space $C^{1,\alpha}$, for some $0 < \alpha < 1$.

The main point of interest in this paper is to investigate the question, “How do the free and fixed boundaries meet?”

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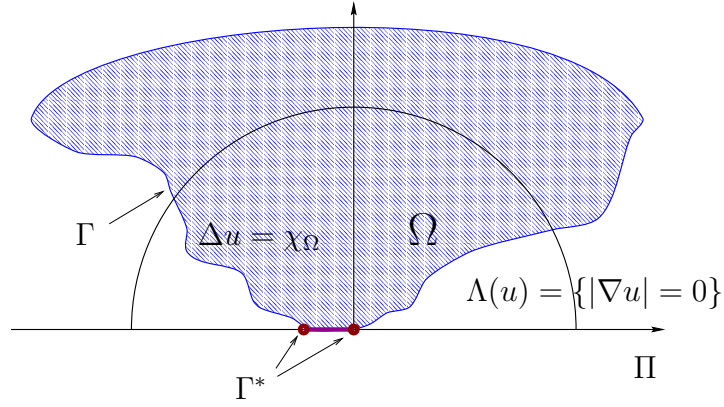


FIGURE 1

Notations. We will use the following notations:

\mathbf{R}_+^n	$\{x \in \mathbf{R}^n : x_1 > 0\}$,
\mathbf{R}_-^n	$\{x \in \mathbf{R}^n : x_1 < 0\}$,
$B(z, r)$	$\{x \in \mathbf{R}^n : x - z < r\}$,
$B^+(z, r)$	$\{x \in \mathbf{R}_+^n \cap B(z, r)\}$,
$B^-(z, r)$	$\{x \in \mathbf{R}_-^n \cap B(z, r)\}$,
B_r	$B(0, r)$,
B_r^+	$B^+(0, r)$,
B_r^-	$B^-(0, r)$,
$\Pi, \Pi(z, r), \Pi_r$	$\{x \in \mathbf{R}^n : x_1 = 0\}, \quad \Pi \cap B(z, r), \quad \Pi(0, r)$,
$\ \cdot\ _\infty$	supremum norm,
e_1, \dots, e_n	standard basis in \mathbf{R}^n ,
ν, e	arbitrary unit vectors,
$D_\nu, D_{\nu e}$	first and second directional derivatives,
v_+, v_-	$\max(v, 0), \max(-v, 0)$,
χ_D	the characteristic function of the set D ,
∂D	the boundary of the set D ,
$\Omega = \Omega(u)$	$\{ \nabla u \neq 0\}$,
$\Lambda = \Lambda(u)$	$\{ \nabla u = 0\}$,
$\Gamma = \Gamma(u)$	$\{x : \nabla u(x) = 0\} \cap \partial\Omega$,
$\Gamma^*(u)$	$\Gamma(u) \cap \Pi$.

Free boundary problems, where $\Delta u = \chi_{\{|\nabla u| \neq 0\}}$, appear for instance in connection with super-conductivity (see [6], [10]). In [8], [9] the authors investigated the problem for the “interior case,” i.e. when the problem is considered in the full ball and there is no fixed boundary.

A similar problem but with a restriction $u = 0$ on $\{|\nabla u| = 0\}$ has been considered earlier by H.Shahgholian and N.N.Uraltseva in [14].

There the authors have used and developed further a technique, which mainly uses global analysis as in [5], [7] and allows one to gain stronger results in problems with no sign assumption on the solutions.

The elimination of the condition $u = 0$ on $\{|\nabla u| = 0\}$ generates a number of difficulties in the application of the technique in [7], [14]. One practical difference is that we no longer have u vanishing on the free boundary, and this appears in the technical parts of the proofs. A simple example is that when scaling we have to take into account the value of the function at the free boundary points. Also, some of the most crucial tools for the application of the methods required new proofs. One example is Lemma 2.5. Another example, which is probably the most important part of this paper, is the second part of the proof of Theorem B. Here new geometrical ideas has to be employed, and these ideas are illustrated in the proof of Lemma 4.2.

Definition 1.1. (Local Solutions) A function u belongs to the class $P_r^+(M)$, if u satisfies:

- (1) $\Delta u = \chi_{\{|\nabla u| \neq 0\}}$ in B_r^+ , in the sense of distributions,
- (2) $u = 0$ on Π_r ,
- (3) $\|u\|_{\infty, B_r^+} \leq M$.

Observe that $P_r^+(M)$ is invariant under rotations of coordinate system, that leave e_1 unchanged.

Definition 1.2. (Global Solutions) A function u belongs to the class $P_\infty^+(M)$ if u satisfies:

- (1) $\Delta u = \chi_{\{|\nabla u| \neq 0\}}$ in \mathbf{R}_+^n in the sense of distributions,
- (2) $u = 0$ on Π ,
- (3) $|u(x)| \leq M(|x| + 1)^2$.

We will also need the definition of solution in the whole ball.

Definition 1.3. A function u belongs to the class $P_r(M)$, if u satisfies:

- (1) $\Delta u = \chi_{\{|\nabla u| \neq 0\}}$ in B_r , in the sense of distributions,
- (2) $\|u\|_{\infty, B_r} \leq M$.

Let us introduce the following notations:

$$\begin{aligned} P_r^+(0, M) &:= \{u \in P_r^+(M) : 0 \in \Gamma\}, \\ P_\infty^+(0, M) &:= \{u \in P_\infty^+(M) : 0 \in \Gamma\}, \\ P_r(0, M) &:= \{u \in P_r(M) : 0 \in \Gamma\}. \end{aligned}$$

In the first section we prove $C^{1,1}$ -regularity of the solutions up to $B_{1/2} \cap \Pi$ (see Theorem A). Then we classify global solutions in $\mathbf{R}_+^n := \{x_1 > 0\}$. Here we encounter some surprises, in contrast to the problem

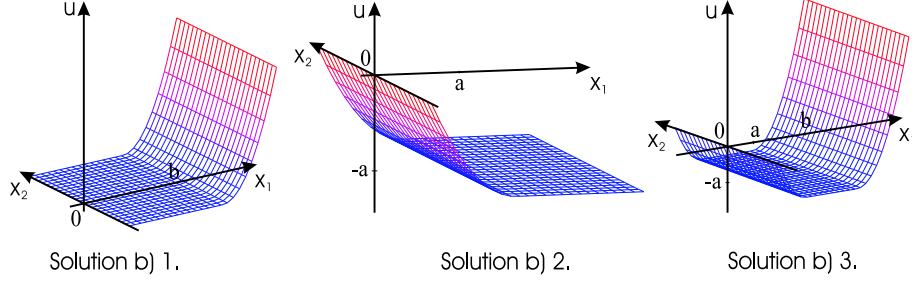


FIGURE 2. This figure illustrates three examples of global solutions for the problem considered in \mathbf{R}^2 , which we get under the conditions of Theorem B b), that is for the case when $\overline{\Omega} \neq \mathbf{R}_+^2$

studied in [14]. We show that global solutions are either polynomials (depending on two variables) or one dimensional. The latter will itself give three different types of solutions (see Theorem B). Finally we prove our main result, which asserts that the free boundary touches the fixed one tangentially.

Theorem A. *If $u \in P_1^+(0, M)$, then there is a constant $C = C(n)$ such that*

$$(1.2) \quad \sup_{B_{1/2}^+} |D_{ij}u| \leq CM.$$

Theorem B. *Let $u \in P_\infty^+(M)$. Then, in some rotated system of coordinates which leaves e_1 unchanged, the following holds*

a) *If $\overline{\Omega} = \overline{\mathbf{R}_+^n}$, then*

$$u(x) = \frac{x_1^2}{2} + ax_1x_2 + \alpha x_1, \quad \text{with } a, \alpha \in \mathbf{R}.$$

b) *If $\overline{\Omega} \neq \mathbf{R}_+^n$, then u depends only on x_1 and has one of the following representations:*

$$(1) \quad u(x) = \frac{(x_1-b)_+^2}{2}, \quad \text{for } b > 0;$$

$$(2) \quad u(x) = \frac{(x_1-a)_-^2 - a^2}{2}, \quad \text{for } a > 0;$$

$$(3) \quad u(x) = \frac{(x_1-a)_-^2 + (x_1-b)_+^2 - a^2}{2}, \quad \text{for some } 0 < a < b.$$

Theorem C. *There exists $r_0 = r_0(n, M) > 0$ and a modulus of continuity $\sigma(\sigma(0^+) = 0)$ such that if $u \in P_1^+(0, M)$, then*

$$(1.3) \quad \partial\Omega \cap B_{r_0} \subset \{x : x_1 \leq \sigma(|x|)|x|\}.$$

2. SOME USEFUL TOOLS

Monotonicity Formula

As is common for these types of problems, the following monotonicity formula will be very useful for us. For a function v let us define

$$I(r, v, x^0) = \int_{B(x^0, r)} \frac{|\nabla v(x)|^2}{|x - x^0|^{n-2}} dx.$$

Theorem 2.1. [1] *Let h_1, h_2 be two non-negative continuous subsolutions of $\Delta u = 0$ in $B(x^0, R)$. Assume further that $h_1 h_2 = 0$ and that $h_1(x^0) = h_2(x^0) = 0$. Then the following function is monotone in r ($0 < r < R$)*

$$(2.1) \quad \varphi(r, h_1, h_2, x^0) = \frac{1}{r^4} I(r, h_1, x^0) I(r, h_2, x^0).$$

Moreover, if any of the sets $\text{supp}(h_i) \cap \partial B(x^0, r)$ digresses from a spherical cap by a positive area, then either $\varphi'(r) > 0$ or $\varphi(r) = 0$.

We use the abbreviated notations $\varphi(r) = \varphi(r, h_1, h_2) = \varphi(r, h_1, h_2, 0)$.

Odd Reflection

In order to be able to use the monotonicity formula, in some cases we extend $P_r^+(M)$ to the class $P_r^*(M)$ of functions that are defined in the whole B_r :

$$(2.2) \quad \tilde{u}(x_1, x_2, \dots, x_n) = \begin{cases} u(x_1, x_2, \dots, x_n) & \text{for } x_1 \geq 0, \\ -u(-x_1, x_2, \dots, x_n) & \text{for } x_1 < 0, \end{cases}$$

We also set

$$\begin{aligned} \Omega^- &= \{(x_1, x_2, \dots, x_n) \in B_r^- : (-x_1, x_2, \dots, x_n) \in \Omega\}, \\ \Omega' &= \Omega \cup \Omega^-. \end{aligned}$$

Definition 2.2. A function u (not identically zero) belongs to the class $P_r^*(M)$, if u satisfies:

- (1) $\Delta u = \chi_\Omega - \chi_{\Omega^-}$ in B_r , in the sense of distributions ,
- (2) $|\nabla u| = 0$ in $B_r \setminus \Omega'$,
- (3) $u = 0$ on Π_r ,
- (4) $\|u\|_{\infty, B_r} \leq M$.

We also define $P_r^*(0, M)$ as a subclass of $P_r^*(M)$ for which the origin belongs to the free boundary.

Lemma 2.3. *If u is a solution of problem (1.1), then for all $x^0 \in \Gamma$ and $0 < r < \text{dist}(x^0, \partial B)$ we have*

$$(2.3) \quad \sup_{B^+(x^0, r)} u > u(x^0).$$

Proof. For $x^0 \in \Gamma \setminus \Gamma^*$ we may apply the strong maximum principle to u in $\Omega \cap B^+(x^0, r)$ to obtain the result (u cannot be constant in $B^+(x^0, r)$ since $x^0 \in \partial\Omega$). Next let $x^0 \in \Gamma^*$. We know that $\Delta u \geq 0$ in $B^+(x^0, r)$. If $\sup_{B^+(x^0, r)} u \leq u(x^0)$, then by Hopf's lemma type argument we have $\frac{\partial u}{\partial x_1}(x^0) < 0$, which contradicts to $|\nabla u| = 0$ on Γ^* . Here we have used that $u \in C^{1,\alpha}(\overline{B_1^+})$, for some $0 < \alpha < 1$. \square

Blow-Up and Non-Degeneracy

For a function u , point $x^0 \in \Gamma(u)$ and $r > 0$ we consider the following scaling

$$u_r(x) := \frac{u(rx + x^0) - u(x^0)}{r^2}.$$

Remark 2.4. If $u \in P_r^+(x^0, M)$, $\|D_{ij}u\| \leq CM$ then

$$\begin{aligned} u_s(x) &\in P_1^+(0, M), \\ \Omega(u_s) &= \Omega_s(u), \\ \Lambda(u_s) &= \Lambda_s(u), \\ \Gamma(u_s) &= \Gamma_s(u), \end{aligned}$$

where $E_s = \{x : sx + x^0 \in E\}$ for any set E .

The uniform limit of u_{r_j} when $r_j \rightarrow 0$, is called blow-up of u .

Obviously, if $u(x) = u(x^0) + o(|x - x^0|^2)$ then the blow-up limit of u will degenerate to be identically zero. The following lemma shows, that in our problem we have **non-degeneracy**:

Lemma 2.5. [8] *If $u \in P_1^+(0, M)$, $x^0 \in \overline{\Omega} \cap B_{1/2}$ such that $u(x^0) \geq 0$, then*

$$(2.4) \quad \sup_{B^+(x^0, r)} u \geq u(x^0) + Cr^2, \quad \text{for all } r < \text{dist}(x^0, \partial B_1),$$

where $C = C(n)$. If $u(x^0) < 0$ then (2.4) holds with smaller C_n , provided $B(x^0, r) \subset B_1^+$.

Proof. We will consider the cases when $u(x^0) \geq 0$ and $u(x^0) < 0$ separately.

Case 1. $u(x^0) \geq 0$. We can assume that $x^0 \in \Omega \cap B_{1/2}$, since if (2.4) holds for all $x^0 \in \Omega \cap B_{1/2}$ then it will be true also for all $x^0 \in \overline{\Omega} \cap B_{1/2}$.

Let us set

$$(2.5) \quad v(x) = u(x) - u(x^0) - \frac{1}{2n}|x - x^0|^2.$$

There exists $x^1 \in \overline{B^+}(x^0, r)$ such that the following holds:

$$(2.6) \quad v(x^1) = \sup_{B^+(x^0, r)} v.$$

To prove this case, it is enough to prove the following two steps:

- $v(x^1) \geq 0$,
- $x^1 \in \partial B^+(x^0, r) \setminus \Pi(x^0, r)$.

The first step simply follows from the fact that $v(x^1) \geq v(x^0) = 0$. To prove the second step assume $x^1 \in B^+(x^0, r)$. Then from (2.6) we have $|\nabla v|(x^1) = 0$. Thus by (2.5)

$$(2.7) \quad (\nabla u)(x^1) = \frac{1}{n}(x^1 - x^0).$$

Now, if $x^1 \neq x^0$, then $(\nabla u)(x^1) \neq 0$, i.e., $x_1 \in \Omega$. But $\Delta v \geq 0$ in Ω and (2.6) together with maximum principle gives us that $v(x) \equiv \text{constant} =: C$ in $\overline{\Omega} \cap \overline{B^+}(x^0, r)$. Particularly, $C = v(x^0) = 0$ so we have $u(x) = u(x^0) + \frac{1}{2n}|x - x^0|^2$ and $(\nabla u)(x) = \frac{1}{n}(x - x^0)$ in $\overline{\Omega} \cap \overline{B^+}(x^0, r)$. Without loss of generality we may assume that there exists $y \in \partial\Omega \cap B^+(x^0, r)$. Indeed, if such an y does not exist, then we have $B^+(x^0, r) \subset \Omega$ and thus $u(x) = u(x^0) + \frac{1}{2n}|x - x^0|^2$ in $B^+(x^0, r)$, which implies (2.4).

Thus we get $|\nabla u(y)| = \frac{1}{n}|y - x^0| \neq 0$, which is a contradiction, since $|\nabla u| = 0$ on $\partial\Omega$.

If $x^1 = x^0$, then again $x^1 = x^0 \in \Omega$ which contradicts to (2.7). Thus we have $x^1 \in \partial B^+(x^0, r)$.

Finally, if $x^1 \in \Pi(x^0, r)$, then because $u(x^0) \geq 0$, we get the following contradiction $0 > v(x^1) \geq v(x^0) = 0$.

Case 2. $u(x^0) < 0$. The proof of this case is essentially the same as the proof of the previous one, except we do not have $x^1 \in \Pi(x^0, r)$, which was the only occasion when we used the nonnegativity of $u(x^0)$. \square

3. PROOF OF THEOREM A

First let us extend u from the class P_1^+ to the class P_1^* by odd reflection, as in (2.2). Set

$$S_j(z, u) = \max_{B_{2^{-j}}(z)} |u(x) - u(z)|.$$

It is enough to prove the following lemma:

Lemma 3.1. *There exist a constant C_0 depending only on n , such that for every $u \in P_1^*(0, M)$, $j \in \mathbb{N}$ and $z \in \Gamma(u) \cap B_{1/2}$*

$$(3.1) \quad S_{j+1}(z, u) \leq \max\{S_j(z, u)2^{-2}, C_0M2^{-2j}\}.$$

Proof. If the conclusion in the lemma fails, then there exist sequences $\{u_j\} \subset P_1^*(0, M)$, $\{z_j\} \subset \Gamma(u_j) \cap B_{1/2}$, $\{k_j\} \subset \mathbb{N}$, $k_j \nearrow \infty$ such that

$$S_{k_j+1}(z_j, u_j) > \max\{S_j(z_j, u_j)2^{-2}, M_j2^{-2k_j}\} \quad \forall j \in \mathbb{N}.$$

Observe that $u_j \in P_1^*(0, M)$ implies

$$\Delta u_j = \begin{cases} \chi_{\Omega_j} & \text{if } x_1 > 0, \\ -\chi_{\Omega_j^-} & \text{if } x_1 < 0. \end{cases}$$

Now consider the following scalings

$$\tilde{u}_j(x) = \frac{u_j(z_j + 2^{-k_j}x) - u_j(z_j)}{S_{k_j+1}(z_j, u_j)} \quad \text{in } B_1.$$

The following results can be obtained by computation like in [14]:

- $\|\tilde{u}_j\|_{\infty, B} = \frac{S_{k_j}(z_j, u_j)}{S_{k_j+1}(z_j, u_j)} \leq 4$,
- $\|\tilde{u}_j\|_{\infty, B_{1/2}} = 1$,
- $\tilde{u}_j(0) = |\nabla \tilde{u}_j(0)| = 0$,
- $\|\Delta \tilde{u}_j\|_{\infty, B} \leq \frac{4}{j} \rightarrow 0$, when $j \rightarrow \infty$.

By compactness there exists a subsequence of $\{\tilde{u}_j\}$ converging to a function u_0 in $W^{2,p}(B_{1/2}) \cap C^{1,\alpha}(B_{1/2})$. We have $u_0(0) = |\nabla u_0(0)| = 0$. For the renamed converging subsequence \tilde{u}_j set

$$v = D_e u_0, \quad v_j = D_e u_j, \quad \tilde{v}_j = D_e \tilde{u}_j,$$

where e is a fixed direction orthogonal to e_1 . Obviously we have that v is the $C^{0,\alpha}$ limit of the sequence \tilde{v}_j in B_1 , $v_j^\pm(0) = 0$ and $\Delta v_j^\pm = 0$. Next we will use the monotonicity formula (2.1) for the sequence $\{v_j^\pm\}$ to get

$$(3.2) \quad \frac{1}{r^{2n}} \int_{B_r} |\nabla v_j^+|^2 \int_{B_r} |\nabla v_j^-|^2 \leq C \quad \forall r, j,$$

where C depends only on M . From here, using Poincare inequality and letting j go to infinity we obtain

$$(3.3) \quad \int_{B_1} |\tilde{v}^+ - M^+|^2 \int_{B_1} |\tilde{v}^- - M^-|^2 = 0,$$

where M^\pm is the mean value of v^\pm in B_1 . Since $v(0) = 0$, from (3.3) we have that either of v^\pm is 0. Using the maximum principle we get $D_e u_0 = v \equiv 0$. This means that u_0 depends only on the x_1 direction. Since it is harmonic and has a second order growth, we obtain $u_0(x) = ax_1 + b$. Finally $u_0(0) = u'_0(0) = 0$ brings us to the statement that $u_0 \equiv 0$, which contradicts $\|\tilde{u}_j\|_{\infty, B_{1/2}} = 1, \forall j$. \square

From this lemma we have the following inequality:

$$|u(x)| \leq CMd(x)^2, \quad d(x) = \text{dist}(x, \partial\Omega),$$

for the points in $B_{1/2}$ that are close to Γ . This together with elliptic estimates for the points close to $\partial\Omega \cap \Pi$ gives us $\sup_{B_{1/2}^+} |D_{ij}u| \leq CM$.

Remark 3.2. The free boundary has zero Lebesgue measure.

This can be checked similarly as it is done in [5] (see also [7; General remarks]). Only the non-degeneracy and $C^{1,1}$ properties of the solution are used in the proof.

4. PROOF OF THEOREM B

The first part of the proof consists of using the quadratic growth of solutions to show that they are two dimensional. In the second part we solve the problem in two dimensions.

Lemma 4.1. [14] *The global solutions are two dimensional.*

Although the proof is very similar to what is found in [14], for readers convenience we will give an outline here. See [14] for more details.

At first we fix a direction e orthogonal to e_1 and consider $(D_e u)^\pm$. Since $(D_e u)^\pm$ vanish on Π , we can extend them to the entire space \mathbf{R}^n defining them as zero in \mathbf{R}_n^- . Then, using the results of Theorem A, a compactness argument and monotonicity formula (2.1), we obtain that $D_e u$ doesn't change sign. Assume it is non-negative (the non-positive case can be treated similarly), then by strong maximum principle we get that on connected components of Ω , $D_e u$ must be strictly positive or identically zero. If $D_e u$ is zero for all directions orthogonal to e_1 , then u is one dimensional, so we have the representation b). If there is a direction e orthogonal to e_1 such that

$$(4.1) \quad D_e u > 0,$$

then it can be proved that u is two dimensional on every connected component of Ω . Thus it is enough to consider the two dimensional problem. We treat two different cases.

Case a) When $\overline{\Omega} = \overline{\mathbf{R}_+^n}$. Then, since $\partial\Omega$ has zero Lebesgue measure (see Remark 3.2), D_2u is harmonic in upper half space and vanishes on Π , so we can continue it harmonically by reflection into entire space. Using Liouville's theorem and the quadratic growth of u we can conclude that D_2u is linear. Simple calculation then gives us the desired result.

Case b) When $\overline{\Omega} \neq \overline{\mathbf{R}_+^n}$. Then the interior of Λ is non-empty and we can take a ball $B(x^0, 2R) \subset \Lambda(u)$. Denote

$$K(x^0, R) := \{(x_1, x_2 - s) : (x_1, x_2) \in B(x^0, R), s \geq 0\}.$$

We claim

$$\partial\Omega \cap K(x^0, R) = \emptyset.$$

Suppose this fails, and let $y \in \partial\Omega \cap K(x^0, R)$ (see Figure 3). Let also $u \equiv C_1$ in the connected component of $\Lambda(u)$ that contains x^0 . From $D_2u > 0$ (see (4.1)) follows that $u \leq C_1$ in $K(x^0, 2R)$. Using the strong maximum principle (u is subharmonic) and that $u(y) = C_1$ we conclude that $u = C_1$ in $B(y, R)$, which contradicts the assumption $y \in \partial\Omega$. Hence $K(x^0, R) \subset \Lambda(u)$.

In order to prove that u is one dimensional, let us extend u to a function \tilde{u} defined in the whole space \mathbf{R}^2 as in (2.2). Then for $D_2\tilde{u}$ we will have:

$$(4.2) \quad D_2\tilde{u}(x_1, x_2) = \begin{cases} D_2u(x_1, x_2) & \text{in } \mathbf{R}_+^2, \\ -D_2u(-x_1, x_2) & \text{in } \mathbf{R}_-^2. \end{cases}$$

We next consider the blow-up of \tilde{u} at ∞ : $\tilde{u}_\infty(x) = \lim_{r_j \rightarrow \infty} \tilde{u}_{r_j}(x)$ where, as usual, $r_j \nearrow \infty$ and $\tilde{u}_{r_j}(x) = \tilde{u}(r_j x)/r_j^2$. Also observe that by the definition of global solutions \tilde{u}_{r_j} is bounded. Writing the monotonicity formula for functions $(D_2\tilde{u})^\pm$ we get:

$$(4.3) \quad \varphi(r, D_2\tilde{u}) \leq \varphi(r_j, D_2\tilde{u}) \leq \lim_{r_j \rightarrow \infty} \varphi(r_j, D_2\tilde{u}) = \varphi(1, D_2\tilde{u}_\infty) = C$$

for $0 < r < r_j$. Next, let us observe that for a fixed $s > 0$ the following holds

$$C = \lim_{r_j \rightarrow \infty} \varphi(sr_j, D_2\tilde{u}) = \lim_{r_j \rightarrow \infty} \varphi(s, D_2\tilde{u}_{r_j}) = \varphi(s, D_2\tilde{u}_\infty).$$

Hence $\varphi(s, D_2\tilde{u}_\infty) = C = \text{constant}$ for all $s > 0$.

In order to complete the proof of the theorem we need the following lemma.

Lemma 4.2. *For any $s > 0$ we have*

$$\varphi(s, D_2\tilde{u}_\infty) = C = 0.$$

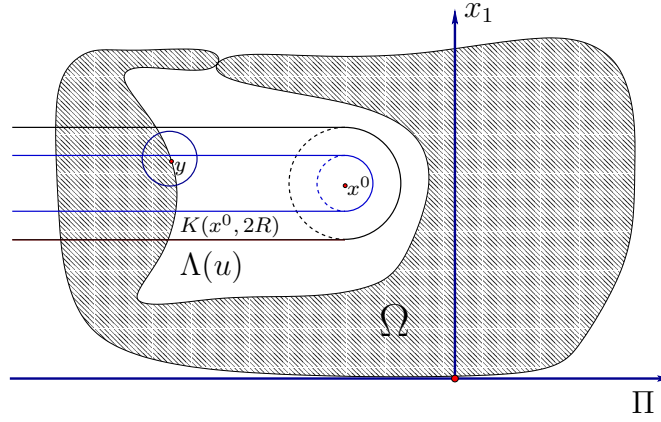


FIGURE 3

Proof. We prove the lemma by a contradictory argument. Let us assume there exists $s > 0$ such that $\varphi(s, D_2\tilde{u}_\infty) = C$ and $C \neq 0$. First we observe that $D_2\tilde{u}_\infty > 0$ in \mathbf{R}_+^2 . Since otherwise (by maximum principle) there exists a ball $B(y_0, t) \subset \{\mathbf{R}_+^2 \setminus \text{supp}(D_2\tilde{u}_\infty)^+\}$, $t > 0$. Also we have that $\text{supp}(D_2\tilde{u}_\infty)^+ \cap \partial B_{|y_0|}(0)$ digresses from a spherical cap by a positive area. Then the monotonicity formula applied for $(D_2\tilde{u})^\pm$ on the ball $B_{|y_0|}(0)$ implies that $\varphi(|y_0|, D_2\tilde{u}_\infty) = C = 0$, which is a contradiction. Also notice that $\text{supp} D_2\tilde{u}_\infty = \overline{\mathbf{R}_+^2}$ implies $\Omega(\tilde{u}_\infty) = \mathbf{R}_+^2$. Just like in case a), we have that $\tilde{u}_\infty(x) = \frac{x_1^2}{2} + ax_1x_2 + \alpha x_1$, where $a, \alpha \in \mathbf{R}$. Also $D_2\tilde{u}_\infty > 0$ implies that $a > 0$. To get a contradiction, it is enough to prove that $D_1\tilde{u}_\infty$ is zero at two different points $(0, x')$, $(0, x'')$ of Π . Indeed, assume $D_1\tilde{u}_\infty(0, x'_2) = 0$ and $D_1\tilde{u}_\infty(0, x''_2) = 0$, then we have $ax' = \alpha = ax''$, and the only possibility is that $a = 0$, contradicting $D_2\tilde{u} > 0$.

Let us now show that there exists two different points on Π , where $D_1\tilde{u}_\infty$ is equal to zero. In fact one can prove even more. Namely $|\nabla\tilde{u}|$ vanishes on $\Pi \cap \{x_2 < 0\}$. Recall that we have $B(x^0, R) \subset K(x^0, R) \subset \Lambda(\tilde{u})$. Denote

$$K_{r_j} := K\left(\frac{1}{r_j}x^0, \frac{R}{r_j}\right) \quad \text{and} \quad l_j := \left\{ \frac{1}{r_j}x^0 - se_2, e_2 = (0, 1), s > 0 \right\}.$$

Fix an $s > 0$ and consider the sequence $y_j := x^0/r_j - se_2$. Obviously, $y_j \in l_j \subset K_{r_j} \subset \Lambda(\tilde{u}_{r_j})$. Recalling that \tilde{u}_{r_j} converges to u_∞ in $(W_{loc}^{2,p} \cap C_{loc}^{1,\alpha})(\mathbf{R}_+^n \cup \Pi)$, we get

$$|\nabla\tilde{u}_\infty(0, -s)| = 0, \quad \forall s > 0.$$

This completes the proof of the lemma. \square

Finally, using (4.3), (4.2) and positivity of D_2u in R_+^2 we have

$$\begin{aligned} 0 = \varphi(1, D_2\tilde{u}_\infty) &\geq \varphi(r, D_2\tilde{u}) = \frac{1}{r^4} I(r, (D_2\tilde{u})^+, x^0) I(r, (D_2\tilde{u})^-, x^0) \\ &= \frac{1}{r^4} \left(\int_{B_r^+} \frac{|\nabla(D_2u)(x_1, x_2)|^2 dx}{|x|^{n-2}} \right) \left(\int_{B_r^-} \frac{|\nabla(D_2u)(-x_1, x_2)|^2 dx}{|x|^{n-2}} \right) \\ &= \frac{1}{r^4} \left(\int_{B_r^+} \frac{|\nabla(D_2u)|^2 dx}{|x|^{n-2}} \right)^2, \end{aligned}$$

for any $r > 0$. This gives us $|\nabla D_2\tilde{u}| \equiv 0$ in \mathbf{R}^2 . Hence $D_2u \equiv \text{constant} = D_2u(0) = 0$. Therefore we get u is one dimensional. Simple calculations combined with $C^{1,\alpha}$ regularity of the solutions accomplish the proof of the theorem. \square

5. PROOF OF THEOREM C

It is enough to check that for every given ε there exists $\rho = \rho_\varepsilon$ such that for all $x^0 \in \partial\Omega \cap B_{\rho_\varepsilon}^+$

$$(5.1) \quad x^0 \in B_{\rho_\varepsilon}^+ \setminus K_\varepsilon,$$

where $K_\varepsilon = \{x : x_1 > \varepsilon(x_2^2 + \dots + x_n^2)^{1/2}\}$. Then we may choose $r_0 = \rho_{\{\varepsilon=1\}}$ and σ given by the inverse of $\varepsilon \rightarrow \rho_\varepsilon$.

Conversely, suppose that (5.1) fails, then there exists a sequence $u_j \in P_1^+(0, M)$, $x^j \in \partial\Omega(u_j) \cap B_{\rho_j}^+$ such that $\rho_j \rightarrow 0$ and $x^j \in B_{\rho_j}^+ \cap \overline{K}_\varepsilon$. Now, for every scaled function $\tilde{u}_j(x) = u_j(x|x^j|)/|x^j|^2$ we have a point $\tilde{x}^j \in K_\varepsilon$. There exists converging subsequences of $\tilde{u}_j \rightarrow u_0$ and $\tilde{x}^j \rightarrow x^0$ such that $x^0 \in \overline{K}_\varepsilon \cap \partial B_1$. Since $x^0 \in \Gamma$, $0 \in \Gamma$, and u_0 is a global solution we have a contradiction to Theorem B. \square

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