The Principle of Least Action in Dynamics

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Abstract

The mathematical laws of nature can be formulated as variational principles. This paper surveys the physics of light rays using Fermat's Principle and the laws of motion of bodies using the Principle of Least Action. The aim is to show that this approach is suitable as a topic for enriching 6th-form mathematics and physics. The key ideas, calculations and results are presented in some detail.

1 Introduction

In everyday life, some of our activities are directed towards optimising something. We try to do things with minimal effort, or as quickly as possible. Here are a couple of examples: We may plan a road journey to minimise the time taken, and that may mean taking a longer route to travel along a section of motorway. Figure 1 is a schematic road map between towns A and B. Speed on the ordinary roads is 50 mph, on the motorway it is 70 mph. Show that the quickest journey time is along the route AFGB.

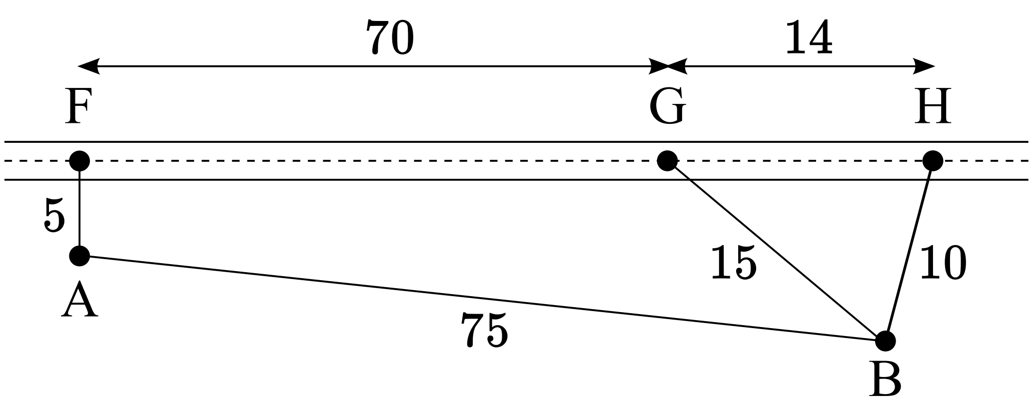

Figure 1: Road map with distances in miles. Speed on ordinary roads is 50 mph, on the motorway 70 mph.

The second example involves flight routes. Find a globe and see that the shortest flight path from London to Los Angeles goes over Greenland and Hudson's Bay. This is the route that planes usually take, to minimise the time and fuel used. Now look at a map of the world in the usual Mercator projection, and notice that the shortest straight line path appears to go over New York. The Mercator map is misleading for finding shortest paths between points on the Earth's surface. The shortest path is part of a *great circle* on the globe.

Remarkably, many natural phenomena proceed by optimising some quantity. A simple example is the way light travels along rays in a medium like air. The light ray between two points A and B is a straight line, which is the shortest path from A to B. In a given medium, light travels at a constant, very fast but finite speed. The shortest path is therefore also the path that minimises the travel time from A to B. We shall see later that the principle that the *travel time is minimised* is the fundamental one. It is called Fermat's Principle.

The motion of massive bodies, for example, a heavy ball thrown through the air, or a planet's motion around the sun, also minimises a certain quantity, called the *action*, which involves the body's energy. This is called the Principle of Least Action. The equations of motion can be derived from this principle. Essentially all the laws of physics, describing everything from the smallest elementary particle to the motion of galaxies in the expanding universe can be understood using some version of this principle. It has been the goal of physicists and mathematicians to discover and understand what, in detail, the action is in various areas of physics.

The Principle of Least Action says that, in some sense, the true motion is the *optimum* out of all possible motions, The idea that the workings of nature are somehow optimal, suggests that nature is working in an efficient way, with minimal effort, to some kind of plan. But nature doesn't have a brain that is "trying" to optimise its performance. There isn't a plan. It just works out that way[1].

These action principles are not the only way to state the laws of physics. Usually one starts directly with the equations of motion. For the motion of bodies and particles, these are Newton's laws of motion. They can be extended to give the laws of motion of fluids, which are more complicated. There are also equations of motion for fields – Maxwell's equations for electric and magnetic fields, Einstein's equations for gravitational fields, and the rather complicated equations for the fields that describe elementary particles. These field equations have some important simple solutions, like the electrostatic field surrounding a charged particle.

Surprisingly, the Principle of Least Action seems to be more fundamental than the equations of motion. The argument for this is made, in a lively manner, in one of the famous Feynman lectures [2]. A key part of the argument is that the Principle of Least Action is not just a technique for obtaining classical equations of motion of particles and fields. It also

[1] This issue has been much discussed. See, e.g. the introductory section of [1].

plays a central role in the quantum theory.

What are the advantages of the Principle of Least Action? The first is conceptual, as it appears to be a fundamental and unifying principle in all areas of physical science. A second is that its mathematical formulation is based on the geometry of space and time, and on the concept of energy. Velocity is more important than acceleration, and force, the key quantity that occurs in equations of motion, becomes a secondary, derived concept. This is helpful, as velocity is simpler than acceleration, and energy is something that is intuitively better understood than force. With Newton's equations, one always wonders how the forces arise, and what determines them. A third advantage is that there are fewer action principles than equations of motion. All three of Newton's laws follow from one principle. Similarly, although we will not show this, all four of Maxwell's equations follow from one action principle.

So what are the disadvantages? The main (pedagogical) disadvantage is that one needs fluency in calculus. The action is an integral over time of a combination of energy contributions, and the equation of motion that is derived from the Principle of Least Action is a differential equation. This still needs to be solved. Furthermore, the standard method by which one derives the equation of motion is the calculus of variations, which is calculus in function space – not elementary calculus.

There is also apparently a physics issue, in that the equation of motion derived from a Principle of Least Action has no friction term, and energy is conserved. Friction needs to be added separately. At a fundamental level, this is a good thing, as it expresses the fact that energy really is conserved. Friction terms are a phenomenological way of dealing with energy dissipation, the transfer of energy to microscopic degrees of freedom outside the system being considered.

These disadvantages may seem great. Indeed, they are the reason why the Principle of Least Action is left for university courses. The aim of this paper, however, is to show that it can be made more accessible. In Section 2 we make a start by showing that in a number of examples, in particular involving light rays, the calculus of variations is not needed. One can obtain physically important results using geometry alone, coupled with elementary calculus, i.e. differentiating to find a minimum. In Section 3 we summarise Newton's laws of motion for bodies. In Section 4 we present the Principle of Least Action for a body moving in a potential in one dimension, and rederive Newton's second law of motion. By carefully studying the example of motion in a linear potential, where there is a constant force, we can again avoid using the calculus of variations, and still derive the equation of motion in a general potential. For completeness, we also give the calculus of variations derivation.

In Section 5 we discuss the motion of two interacting bodies, which leads to Newton's third law, and Momentum Conservation. We also show that a composite body, with two or more parts, has a natural notion of its Centre of Mass, This emerges by considering the body's total momentum. We also make some brief remarks about motion in 2 or 3 dimensions.

Section 6 deals with Energy Conservation.

In what follows we are largely following the inspiring book Perfect Form, by Lemons [3], and also the first few sections of the classic text Mechanics, by Landau and Lifshitz [4].

2 Light rays – reflection and refraction

The simplest example of a physical principle that involves minimisation is Fermat's Principle in the ray theory of light. Light rays are infinitesimally thin beams of light. Narrow beams of light which are close to ideal light rays can be obtained using a light source and screens with narrow slits, or using mirrors, as in a pocket torch. Even if light is not restricted by narrow slits, it is still made up of a collection of rays, travelling in different directions.

A light ray traces out a straight path or a bent path, or possibly a curved path, as it passes through various media. A fundamental assumption is that in a given medium, the light ray has a definite, finite speed.

Fermat's Principle says that the actual path taken by a light ray between two given points, A and B, is the path that minimises the total travel time. In a uniform medium, for example air or water, or a vacuum, the travel time is the length of the path divided by the light speed. Since the speed is constant, the path of shortest time is also the path of shortest distance, and this is the straight line path from A to B. (This can be mathematically proved using the calculus of variations, but we take it as obvious.) Therefore, in a uniform medium, light travels along straight lines.

This can be verified experimentally. A beam that heads off in the right direction from A will arrive at B. More convincingly, consider a light source at A that emits light in all directions. A small obstacle anywhere along the straight line between A and B will prevent light reaching B, and will cast a shadow at B.

Fermat's Principle can be used to understand two basic laws of optics, those of reflection and refraction. Let us consider reflection first. Suppose we have a long flat mirror, and a light source at A. Let B be the light receiving point, on the same side of the mirror (see Figure 2). Consider all the possible light rays from A to B that bounce off the mirror once. If the time is to be minimised, we know that the paths before and after the reflection must be straight. What we are trying to find is the reflection point. We use the coordinates in the figure, with the x-axis along the mirror, and $x = X$ the reflection point. Let c be the light speed, which is the same for both segments of the ray.

Concentrate on the various lengths in the figure, and ignore the angles θ and ϕ for now. Using Pythagoras' theorem to find the path lengths, we find the time for the light to travel from A to B is

$$T = \frac{1}{c}\Big((a^2 + X^2)^{1/2} + (b^2 + (L - X)^2)^{1/2}\Big) . \qquad (2.1)$$

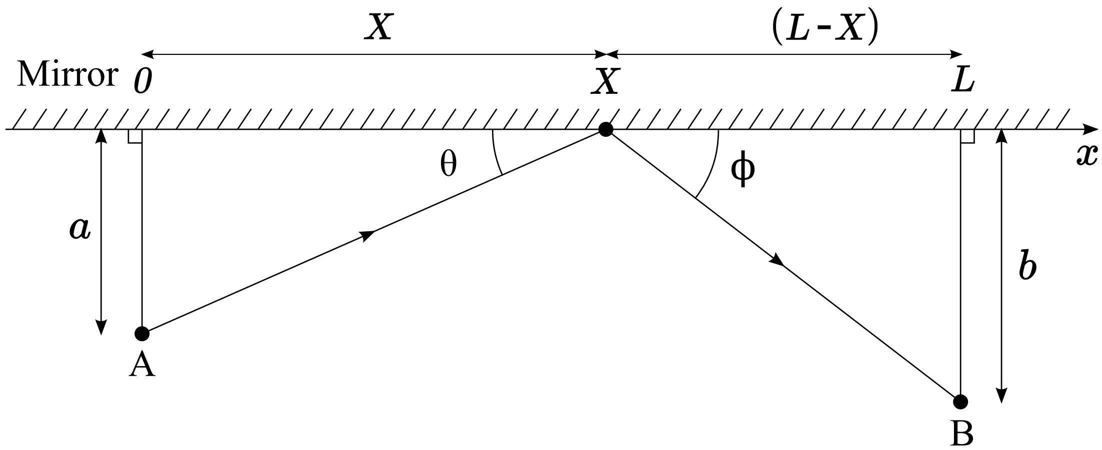

Figure 2: Reflection from a mirror.

The derivative of T with respect to X is

$$\frac{dT}{dX} = \frac{1}{c}\left(\frac{X}{(a^2+X^2)^{1/2}} - \frac{L-X}{(b^2+(L-X)^2)^{1/2}}\right). \tag{2.2}$$

The time is minimised when this derivative vanishes, giving the equation for X

$$\frac{X}{(a^2+X^2)^{1/2}} = \frac{L-X}{(b^2+(L-X)^2)^{1/2}}. \tag{2.3}$$

Now the angles come in handy, as (2.3) says that

$$\cos\theta = \cos\phi\,. \tag{2.4}$$

Therefore θ and ϕ are equal. We haven't explicitly found X, but that does not matter. The important result is that the incoming and outgoing light rays are at equal angles to the mirror surface. This is the fundamental law of reflection. [In fact, eq.(2.3) can be simplified to $X/a = (L-X)/b$, and then X is easily found.]

Now let's consider refraction. Here, light rays pass from a medium where the speed is c_1 into another medium where the speed is c_2. The geometry of refraction is different from that of reflection, but not very much, and we use similar coordinates (see Figure 3). By Fermat's Principle, the path of the actual light ray from A to B is the one that minimises the time taken. Note that (unless $c_1 = c_2$) this is definitely not the same as the *shortest* path from A to B, which is the straight line between them. The path of minimal time has a kink, just like the route via the motorway that we considered at the beginning.

We can assume the rays from A to X and from X to B are straight, because only one medium, and one light speed, is involved in each case. The total time for the light to travel from A to B is

$$T = \frac{1}{c_1}(a^2+X^2)^{1/2} + \frac{1}{c_2}(b^2+(L-X)^2)^{1/2}\,. \tag{2.5}$$

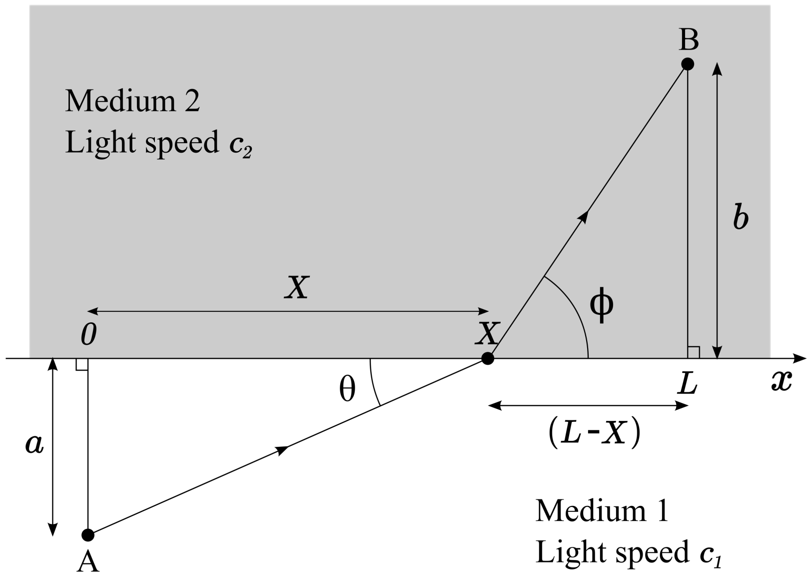

Figure 3: Ray refraction

The minimal time is again determined by differentiating with respect to X:

$$\frac{dT}{dX} = \frac{1}{c_1}\frac{X}{(a^2+X^2)^{1/2}} - \frac{1}{c_2}\frac{L-X}{(b^2+(L-X)^2)^{1/2}} = 0\,. \tag{2.6}$$

This gives the equation for X,

$$\frac{1}{c_1}\frac{X}{(a^2+X^2)^{1/2}} = \frac{1}{c_2}\frac{L-X}{(b^2+(L-X)^2)^{1/2}}\,. \tag{2.7}$$

We do not really want to solve this, but rather to express it more geometrically. In terms of the angles θ and ϕ in Figure 3, the equation becomes

$$\frac{1}{c_1}\cos\theta = \frac{1}{c_2}\cos\phi\,, \tag{2.8}$$

or more usefully

$$\cos\phi = \frac{c_2}{c_1}\cos\theta\,. \tag{2.9}$$

This is the desired result. It is one version of Snell's law of refraction. It relates the angles to the ratio of the light speeds c_2 and c_1. Snell's law can be tested experimentally even if the light speeds are unknown, by plotting a graph of $\cos\phi$ against $\cos\theta$. The light beam must be allowed to hit the surface at varying angles to find this relationship, so A and B are no longer completely fixed. The resulting graph should be a straight line through the origin.

If c_2 is less than c_1, then $\cos\phi$ is less than $\cos\theta$, so ϕ is greater than θ. An example is when light passes from air into water. The speed of light in water is less than in air, so light rays are bent into the water and towards the perpendicular to the surface, as in Figure 3.

There are many consequences of Snell's law, including the phenomenon of total internal reflection for rays that originate in the medium with the smaller light speed, and hit the surface at angles in a certain range. There are also practical applications to lens systems and light focussing, but we will not discuss these further.

[Historically, the law of refraction was given in terms of a ratio of refractive indices on the right hand side of (2.9). It was through Fermat's Principle that the ratio was understood as a ratio of light speeds. Later, the speed of light in various media could be directly measured. It was found that the speed of light is maximal in a vacuum, with the speed in air only slightly less, and the speed in denser materials like water and glass considerably less, by $20\% - 40\%$. The speed of light in vacuum is an absolute constant, approximately $300,000,000 \mathrm{~m\,s^{-1}}$, but the speed in dense media can depend on the wavelength, which is why a refracted beam of white light splits into beams of different colours when passing through a glass prism or water drop. The rays of different colours are bent by different angles as they pass from air into glass or water.]

3 Motion of bodies – Newton's laws

In this Section we summarise Newton's laws of motion, and give a couple of examples of the motion of bodies. This is a prelude to the next Section, where we discuss how Newton's laws can be derived from the Principle of Least Action.

Newton's laws describe the motion of one or more massive bodies. A single body has a definite mass, m. The body's internal structure and shape can initially be neglected, and the body can be treated as a point particle with a definite position. As it moves, its position traces out a curve in space. Later we will show that composite bodies, despite their finite size, can be treated as having a single central position, called the Centre of Mass.

The first law states that motion of a body at constant velocity is self-sustaining, and no force is needed. Velocity is the time derivative of the position of the body. If it is constant, and non-zero, the body moves at constant speed along a straight line. If the velocity is zero, the body is at rest.

The second law states that if a force acts, then the body accelerates. The relation is

$$ma = F \,. \tag{3.1}$$

The acceleration a and force F are parallel vectors. The acceleration is the time derivative of velocity, and hence the second time derivative of position. If there is no force, then the acceleration is zero and the velocity is constant, which is a restatement of the first law.

For eq.(3.1) to have predictive power, we need an independent understanding of the force. We have that understanding for electric and magnetic forces on charged particles (using the notions of electric and magnetic fields), and for the gravitational force on a body due to other massive bodies. Forces due to springs, and various types of contact forces, describing collisions and friction, are also understood.

Newton famously discovered the inverse square law gravitational force that one body exerts on another. This simplifies for bodies near the Earth's surface, subject to the Earth's gravitational pull. The force on a body of mass m is downwards, with magnitude mg. g is a positive constant related to the Earth's mass, Newton's universal gravitational constant G, and the radius of the Earth. In this case, (3.1) reduces to

$$ma = -mg \tag{3.2}$$

where a is the upward acceleration. m cancels in the equation, so $a = -g$. This is why g is called *the* acceleration due to gravity. The upward acceleration, which is $-g$, is the same for all bodies, and independent of their height.

Newton's second law is intimately tied to calculus. With a given force, (3.1) becomes a second order differential equation for the position of the body as a function of time. Sometimes this is easy to solve, and sometimes not.

Let us look more closely at the example (3.2), where there is a constant force that doesn't depend on the body's height, and assume the motion is purely vertical. As a differential equation, and after cancelling m, (3.2) takes the form

$$\frac{d^2z}{dt^2} = -g\,, \tag{3.3}$$

with z the vertical position, or height of the body, above some reference level. The solution is

$$z(t) = -\frac{1}{2}gt^2 + ut + z_0\,, \tag{3.4}$$

where z_0 and u are the height and upward velocity at time $t = 0$. The graph of z against t, for any z_0 and u, is a parabola, or part of a parabola if the time interval is finite (see Figure 4).

We can consider non-vertical motion too. Suppose the body moves in a vertical plane, with z the vertical coordinate, and x the horizontal coordinate. Since there is no horizontal component to the gravitational force, the body has no horizontal acceleration, so x is linearly related to t. In the simplest case, x is just a constant multiple of t, and let's assume the multiple is non-zero. On the other hand, the vertical part of the motion is the same as before, with solution (3.4). Rather than plotting z against t, we can now plot z against x. This just requires a rescaling of the t axis, because x is a multiple of t. The graph now shows the parabolic trajectory of the body in the x-z plane, rather than the height as a function of time.

Newton's third law is important but less fundamental. It states that for every force there is a reaction force acting the other way. If one body exerts a force F on a second body, then

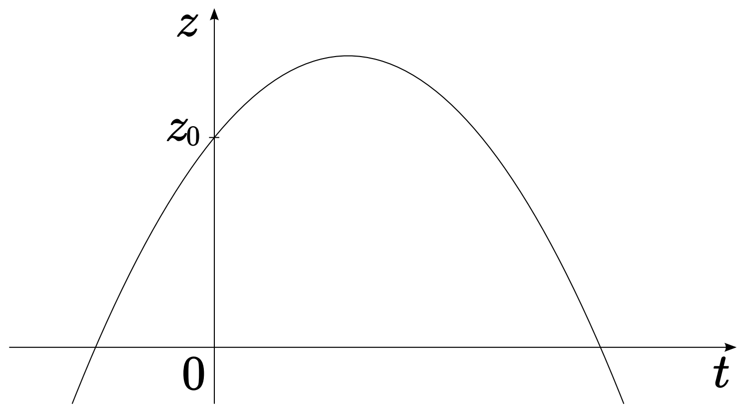

Figure 4: Motion under gravity.

at the same time the second body exerts a force $-F$ on the first body. It is not obvious that this is true, nor is it obvious why it is true. For example, every massive body near the Earth's surface, whether falling or stationary, exerts a gravitational pull on the Earth, opposite to the pull of the Earth on the body, but the result cannot be measured. One test of the third law is provided by the motion of celestial bodies of comparable mass, like binary stars. We shall see below, in the context of the Principle of Least Action, how the third law actually follows from a simple geometrical assumption.

4 Principle of Least Action

Let us now see how the Principle of Least Action can be used to derive Newton's laws of motion. It is simplest to consider the motion of a single body in one dimension, along, say, the x-axis. Let $x(t)$ be a possible path of the body, not necessarily the one actually taken. The velocity of the body is

$$v = \frac{dx}{dt}\,, \tag{4.1}$$

which is also a function of t.

To set up the Principle of Least Action, one must postulate that a moving body has two types of energy. The first is *kinetic energy* due to its velocity. Intuitively, this doesn't depend on the direction of motion, so is the same for velocity v and velocity $-v$, and suggests that kinetic energy is a multiple of v^2. Intuitively also, the kinetic energy of several bodies is the sum of the kinetic energies of the individual bodies. A group of N equal bodies, moving together with the same velocity, has N times the kinetic energy of one body. It also has N times the mass. So kinetic energy is proportional to mass. One is led to asserting that the kinetic energy K of a body of mass m and velocity v is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2\,. \tag{4.2}$$

The factor $\frac{1}{2}$ is convenient to connect with Newton's laws.

The second type of energy of a body is *potential energy.* This depends on the environment and is independent of velocity. It depends on the presence of other bodies and the way they interact with the first one (electrically, gravitationally etc.). The potential energy of a body is a function of its position, $V(x)$. We only really need to know V at the location of the body at each time t, which is $x(t)$, so to be precise we write $V(x(t))$; however it is important that V is defined everywhere that the body *might* be, which is all x in some range. We often say that the body is moving in the potential V.

We now consider the motion of the body between an initial position x_0 at time t_0 and a final position x_1 at a later time t_1. We assume that these are the endpoints of an actual motion $x(t) = X(t)$, although we do not yet know what that motion is. The action S is defined to be[2]

$$S = \int_{t_0}^{t_1} \left(\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x(t)) \right) dt \,. \tag{4.3}$$

The integrand is the kinetic energy minus the potential energy of the body at time t. The minus sign is important, and explains why we spoke earlier about two types of energy. They are distinguished because one depends on velocity and the other doesn't. The action is sometimes written in the condensed form

$$S = \int_{t_0}^{t_1} (K - V)\, dt \,, \tag{4.4}$$

or more shortly still as

$$S = \int_{t_0}^{t_1} L\, dt \,, \tag{4.5}$$

where $L = K - V$ is called the Lagrangian. The action is the time integral of the Lagrangian, and this is the case not just for one body moving in one dimension, but much more generally.

The form of the potential energy, $V(x)$, depends on the physical situation. It must be known in order to proceed, just as with Newton's second law the force must be known in order to work out the body's motion. $V(x)$ sometimes has a simple form. For example, suppose the body is free, and has no significant interaction with the environment. Then V is independent of position – it is just a constant V_0. We shall see later that the value of this constant has no effect. For a body close to the Earth's surface we know intuitively that it takes energy to lift a body. The body's potential energy increases with height. Lifting a body through a height h requires a certain energy, and lifting it through a further height h requires the same energy again. Also, lifting two bodies of mass m through height h requires twice the energy needed to lift one body of mass m. The conclusion is that the increase in potential energy when a body is raised by height h is mgh, proportional to mass and height,

[2]This is Hamilton's definition of the action, the one now most commonly considered. Historically there were other definitions.

and multipled by a constant g, which we will see later is the acceleration due to gravity. The complete potential energy of the body at height x above some reference level is therefore

$$V(x) = V_0 + mgx\,, \tag{4.6}$$

where V_0 again has no effect. (In this Section, for consistency, we use x as the coordinate denoting height, rather than z as before.) For a body attached to a stretched spring, the potential $V(x)$ is a quadratic function of x, and there are other cases where the form of V is known or can be postulated.

The Principle of Least Action now states that among all the possible paths $x(t)$ that connect the fixed endpoints, the actual path taken by the body, $X(t)$, is the one that makes the action S minimal[3].

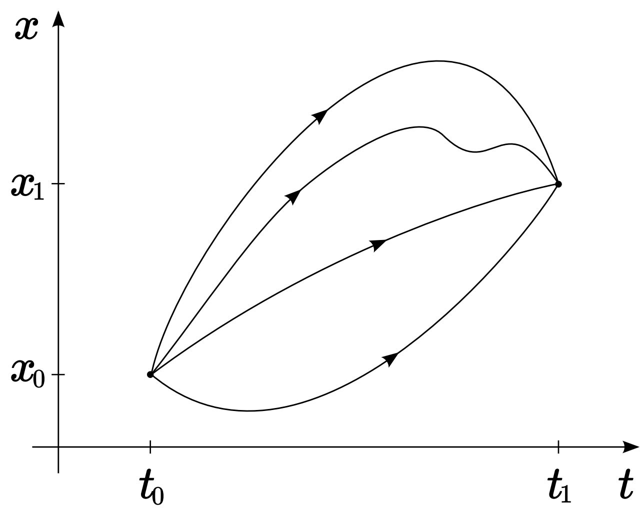

Figure 5: Possible paths $x(t)$.

Note that we are not minimising with respect to just a single quantity, like the position of the body at the mid-time $\frac{1}{2}(t_0 + t_1)$. Instead, we are minimising with respect to the infinite number of variables that characterise all possible paths, with all their possible wiggles. We must however assume that paths $x(t)$ have some smoothness. Acceptable paths are those for which the acceleration remains finite, so the velocity is continuous. Some typical acceptable paths are shown in Figure 5.

Now is a good time to explain why V_0, either as a constant potential, or as an additive contribution to a non-constant potential as in (4.6), has no effect. Its contribution to the action S is simply $(t_1 - t_0)V_0$, which is itself a constant, independent of the path $x(t)$. Finding the path $X(t)$ that minimises S is unaffected by this constant contribution to S. From now on, therefore, we will sometimes drop V_0.

[3]This is usually the case, but sometimes the action is stationary rather than minimal. The equation of motion is unaffected by this difference.

4.1 A simple example and a simple method

A simple example is where the potential energy $V(x)$ is linear in x, i.e. $V(x) = kx$ with k constant. Let us try to determine the motion of the body over the time interval $-T \le t \le T$, assuming that the initial position is $x(-T) = -X$ and the final position is $x(T) = X$. This choice of initial and final times and positions may look rather special, but it simplifies the calculations, and can always be arranged by choosing the origin of t and of x to be midway between the initial and final times and positions, as they are here.

Next, consider a very limited class among the possible paths $x(t)$ from the initial to final positions. Assume that the graph of $x(t)$ is a parabola passing through the given endpoints (see Figure 6). $x(t)$ is therefore a quadratic function, of the form At^2+Bt+C. This has three parameters, but here there are two endpoint constraints, so there is only one *free* parameter. The form of $x(t)$ must be

$$x(t) = \frac{X}{T}t + \frac{1}{2}a(t^2 - T^2)\,. \tag{4.7}$$

a is the free parameter, and it is the (constant) acceleration, as $\frac{d^2x}{dt^2} = a$. The term involving a vanishes at the endpoints, so $x(-T) = -X$ and $x(T) = X$, as required.

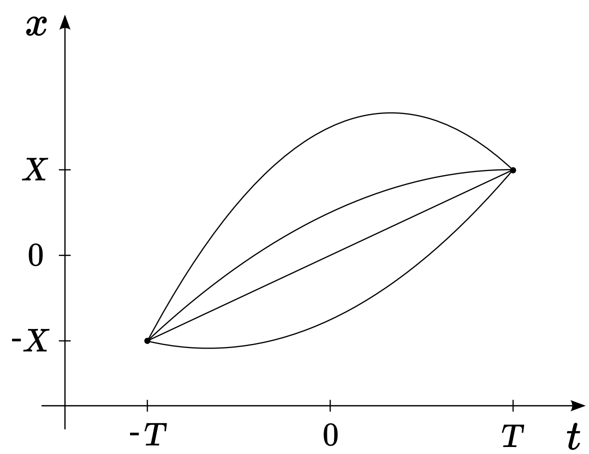

Figure 6: Parabolic paths with various accelerations.

The Principle of Least Action requires us to determine the value of a for which the action S is minimised. We need to calculate S, and find the minimum by differentiation. For the path (4.7), the velocity is

$$\frac{dx}{dt} = \frac{X}{T} + at \tag{4.8}$$

so the kinetic energy is $K = \frac{1}{2}m\left(\frac{X}{T} + at\right)^2$. The potential energy kx, at time t, is k times the expression (4.7). Therefore

$$S = \int_{-T}^{T} \left\{ \frac{1}{2}m\left(\frac{X}{T} + at\right)^2 - k\frac{X}{T}t - \frac{1}{2}ka(t^2 - T^2) \right\} dt\,, \tag{4.9}$$

with the integrand quadratic in t. Expanding out, and integrating, we obtain after simplifying,

$$S = m\frac{X^2}{T} + \frac{1}{3}ma^2T^3 + \frac{2}{3}kaT^3 \,. \tag{4.10}$$

The derivative with respect to a is

$$\frac{dS}{da} = \frac{2}{3}maT^3 + \frac{2}{3}kT^3 \,, \tag{4.11}$$

and setting this to zero gives the relation

$$ma = -k \,. \tag{4.12}$$

The acceleration a that minimises S is therefore $-k/m$, and the true motion is

$$x(t) = \frac{X}{T}t - \frac{k}{2m}(t^2 - T^2) \,. \tag{4.13}$$

A less important result is that the minimum of the action is $S = \frac{mX^2}{T} - \frac{k^2T^3}{3m}$.

The interpretation of (4.12) is as follows. The linear potential $V(x) = kx$ results in a constant force $-k$, and (4.12) is Newton's second law in this case. The acceleration a is constant, and equal to $-k/m$. The result is the same for $V(x) = V_0 + kx$.

Our method has determined the true motion in this example. However, the method looks quite incomplete, because we have not minimised S over all paths through the endpoints, but only over the subclass of parabolic paths, for which the acceleration is constant. The next step is to show that the method is better than it appears, and that it can lead us to the correct equation of motion for completely general potentials $V(x)$.

4.2 The general equation of motion

Let us consider again the Principle of Least Action in its general form for one-dimensional motion in a potential $V(x)$. The action S is given in (4.3), and the endpoint conditions are that $x(t_0) = x_0$ and $x(t_1) = x_1$. Suppose $X(t)$ is the path that satisfies these endpoint conditions and minimises S. Now note that there is nothing special about t_0 and t_1. The motion may occur over a longer time interval than this. More importantly, if the motion minimises the action between times t_0 and t_1, then it automatically minimises the action over any smaller subinterval of time between t_0 and t_1. If it didn't, we could modify the path inside that subinterval, and reduce the total action. Let us now focus on a very short subinterval, between times T and $T + \delta$, where δ is very small. Suppose the true motion is from $X(T)$ at time T to $X(T + \delta)$ at time $T + \delta$, with $X(T + \delta)$ very close to $X(T)$. Over such small intervals in time and space we can make some approximations. The simplest approximation would be to say that the potential V is constant, and X varies linearly with t. But this is too simple and we don't learn anything. A more refined approximation is to say that $V(x)$ varies linearly with x between $X(T)$ and $X(T+\delta)$, and that the path $X(t)$ is a quadratic function of t, whose graph is a parabola (see Figure 7). The assumption that $V(x)$ is linear means that V has a definite slope equal to the derivative $\frac{dV}{dx}$, which can be regarded as constant between $X(T)$ and $X(T+\delta)$. The assumption that $X(t)$ is quadratic allows for some curvature in the path, in other words, some acceleration.

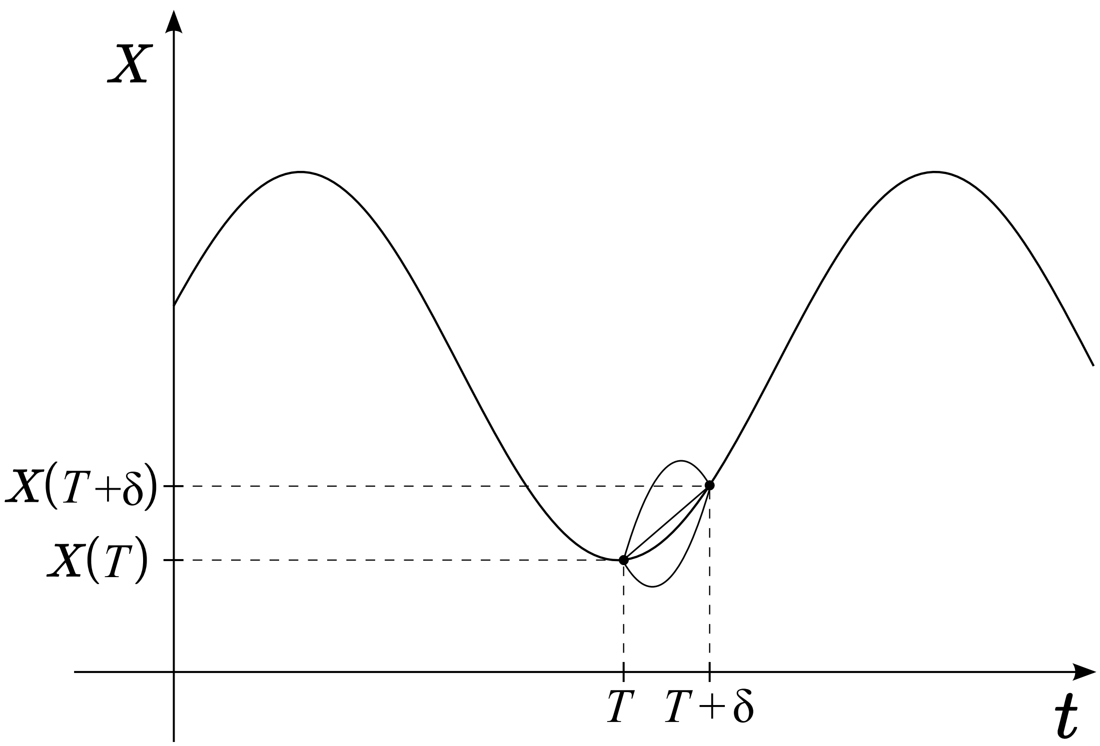

Figure 7: Possible parabolic paths over a very short time interval.

Now we can make use of the simple calculation presented in the last subsection. There we showed that when the potential is $V(x) = V_0 + kx$, i.e. linear with slope k, then in the class of parabolic paths, the path that minimises the action is the one for which the mass times acceleration, ma, is equal to $-k$. Applying this result to the time interval from T to $T+\delta$, we deduce that ma, where a is the acceleration in that time interval, must equal $-\frac{dV}{dx}$, the slope of the potential evaluated at $X(T)$. This is the key result. By approximating the graph of $X(t)$ by a parabola over a short time interval, we have found the acceleration. The result applies not just to one short interval, but to any short interval. $-\frac{dV}{dx}$ will vary from one interval to another, so the acceleration varies too.

Writing the acceleration as the second time derivative of x, we obtain the equation of motion

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx}\,. \tag{4.14}$$

This is the general form of the equation of motion that is derived from the Principle of Least Action. The true motion, $X(t)$, is a solution of this.

Equation (4.14) has the form of Newton's second law of motion. We identify the force F that acts on the body as $-\frac{dV}{dx}$. This force is a function of x and needs to be evaluated at the position where the body is, namely $x(t)$. This, in fact, is the main lesson from the Principle of Least Action. The potential $V(x)$ is the fundamental input, and the force $F(x)$ is derived from it. The force is minus the derivative of the potential.

Newton's first law follows as a special case. When the potential V is a constant V_0, then its derivative vanishes, so there is no force, and the equation of motion is

$$m\frac{d^2x}{dt^2} = 0\,, \tag{4.15}$$

which implies that $\frac{dx}{dt} =$ constant, i.e. the motion is at constant velocity.

Even in a non-constant potential $V(x)$, at the points $\tilde{x}$ where $\frac{dV}{dx}$ vanishes there is no force. These are possible equilibrium points for the body, where the body can remain at rest. Such equilibrium points may or may not be stable.

We argued earlier that the gravitational potential of a body close to the Earth's surface is $V(x) = V_0 + mgx$, but had not confirmed that g is the acceleration due to gravity. For this potential, $-\frac{dV}{dx} = -mg$. This is precisely the gravitational force on a body of mass m, if g is the acceleration due to gravity.

Because equation (4.14) is general, we can now go back and see if the simplifications we made in subsection 4.1 gave the right or wrong answer. In fact, the equation of motion we obtained, (4.12), is correct for the linear potential $V(x) = kx$. This is because the force is the constant $-k$, so the acceleration is constant. The true motion $X(t)$ is therefore quadratic in t and has a parabolic graph, just as we assumed.[4]

4.3 The Calculus of Variations

[This subsection can be skipped.]

We gave above a derivation of the equation of motion (4.14), starting from the Principle of Least Action, but our method, based on calculations involving parabolas, is not the most rigorous one, and does not easily extend to more complicated problems. For completeness, we show here how to minimise the action S by the method of calculus of variations [6, 7]. As above, this method gives the differential equation that the true path $X(t)$ obeys, which is Newton's second law of motion. One must still solve the differential equation to find $X(t)$.

Recall that for a general path $x(t)$, running between the fixed endpoints $x(t_0) = x_0$ and

[4] There is an online, interactive tool to investigate motion in a linear potential, directly using the Principle of Least Action [5]. Using this, one gains intuition into how S varies as the path varies, and one can construct an arbitrarily good approximation to the actual motion by searching for the minimum of S, using step-by-step path adjustments.

$x(t_1) = x_1$,

$$S = \int_{t_0}^{t_1} \left(\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x(t)) \right) dt \,. \tag{4.16}$$

Assume that $x(t) = X(t)$ is the minimising path (we need to assume that such a path *does* exist and has the smoothness mentioned above). Let S_X denote the action of this minimising path, so

$$S_X = \int_{t_0}^{t_1} \left(\frac{1}{2} m \left(\frac{dX}{dt} \right)^2 - V(X(t)) \right) dt \,. \tag{4.17}$$

Now suppose that $x(t) = X(t) + h(t)$ is a path infinitesimally close to $X(t)$. As $h(t)$ is infinitesimal, we shall ignore quantities quadratic in $h(t)$. $h(t)$ is called the path variation, and $X(t) + h(t)$ is called the varied path. For the varied path the velocity is

$$\frac{dx}{dt} = \frac{dX}{dt} + \frac{dh}{dt} \tag{4.18}$$

and the kinetic energy is

$$K = \frac{1}{2} m \left(\frac{dX}{dt} + \frac{dh}{dt} \right)^2 \,. \tag{4.19}$$

Expanding out, and discarding terms quadratic in h, we obtain

$$K = \frac{1}{2} m \left(\frac{dX}{dt} \right)^2 + m \frac{dX}{dt} \frac{dh}{dt} \,. \tag{4.20}$$

Next we do a similar analysis of the potential energy. For the varied path, the potential energy at time t is $V(X(t) + h(t))$. We use the usual approximation of calculus

$$f(X + h) = f(X) + f'(X) \, h \,, \tag{4.21}$$

where f' is the derivative of the function f. Here we find

$$V(X(t) + h(t)) = V(X(t)) + V'(X(t)) \, h(t) \,. \tag{4.22}$$

Note that V is a function of just one variable (originally x), and we differentiate with respect to x to find V'. Then, on the right hand side, both V and V' have to be evaluated at the point where the body is (before the variation), $X(t)$.

Combining these results for K and V, we find that for the varied path, the action, which we denote by S_{X+h}, is

$$S_{X+h} = \int_{t_0}^{t_1} \left(\frac{1}{2} m \left(\frac{dX}{dt} \right)^2 + m \frac{dX}{dt} \frac{dh}{dt} - V(X(t)) - V'(X(t)) \, h(t) \right) dt \,. \tag{4.23}$$

The terms containing neither h nor $\frac{dh}{dt}$ are just the terms that appeared in S_X, so

$$S_{X+h} = S_X + \int_{t_0}^{t_1} \left(m \frac{dX}{dt} \frac{dh}{dt} - V'(X(t)) \, h(t) \right) dt \,. \tag{4.24}$$

S_{X+h} differs from S_X by an integral that is of first order in h.

To relate the two terms in this remaining integral, we do an integration by parts on the first term, integrating $\frac{dh}{dt}$, and differentiating $m\frac{dX}{dt}$. Using the standard formula we find

$$S_{X+h} = S_X + \left[m\frac{dX}{dt}h(t) \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(m\frac{d^2X}{dt^2} + V'(X(t)) \right) h(t)\, dt\,. \tag{4.25}$$

$h(t)$ is a rather general (infinitesimal) function, but it is required to vanish at both t_0 and t_1, because the Principle of Least Action is stated for paths that have fixed endpoints at t_0 and t_1. Consequently, the endpoint contributions to S_{X+h} vanish, so

$$S_{X+h} - S_X = -\int_{t_0}^{t_1} \left(m\frac{d^2X}{dt^2} + V'(X(t)) \right) h(t)\, dt\,. \tag{4.26}$$

Between the endpoints, $h(t)$ is unconstrained, and its sign can be flipped if we want. It follows that if the bracketed expression multiplying $h(t)$ in the integral is non-zero, we could find some $h(t)$ for which $S_{X+h} - S_X$ is negative. (This last claim is not completely obvious, but can be proved rigorously if it is assumed that the bracketed expression is continuous.) S_X is therefore the minimum of the action only if the bracketed expression vanishes at all times t between t_0 and t_1. In other words, the Principle of Least Action requires that

$$m\frac{d^2X}{dt^2} + V'(X(t)) = 0\,. \tag{4.27}$$

This is the differential equation that the actual path $X(t)$ must satisfy, and it is essentially the same as equation (4.14). In the context of the calculus of variations it is called the Euler-Lagrange equation associated with the action S.

As we said before, (4.27) is a version of Newton's second law, in which the force is

$$F(X) = -V'(X)\,. \tag{4.28}$$

We have derived Newton's law from the Principle of Least Action. However, it is no longer the force that is the fundamental quantity – instead it is the potential energy V.

5 Motion of several bodies – Newton's third law

The simplest derivation of Newton's third law is for a system with just two bodies moving in one dimension, and interacting through a potential. The action is a single quantity but it involves both bodies. Let the possible paths of the bodies be $x_1(t)$ and $x_2(t)$, and their masses m_1 and m_2. The kinetic energy is

$$K = \frac{1}{2}m_1\left(\frac{dx_1}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{dx_2}{dt}\right)^2\,. \tag{5.1}$$

For the potential energy V we suppose that there is no background environment, but that the bodies interact with each other. V is some function of the separation of the bodies $l = x_2 - x_1$. So the potential energy is $V(l) = V(x_2 - x_1)$. This is the key assumption. The potential does not depend independently on the positions of the bodies, but only on their separation. The assumption is equivalent to translation invariance – if both bodies are translated by the same distance s along the x-axis then $x_2 - x_1$ is unchanged, so V is unchanged. (Usually, V only depends on the magnitude $|x_2 - x_1|$, but this is not essential.)

The action for the pair of bodies is then

$$S = \int_{t_0}^{t_1} \left(\frac{1}{2} m_1 \left(\frac{dx_1}{dt} \right)^2 + \frac{1}{2} m_2 \left(\frac{dx_2}{dt} \right)^2 - V\big(x_2(t) - x_1(t)\big) \right) dt \,. \tag{5.2}$$

The possible paths of the two bodies are independent, but the endpoints $x_1(t_0), x_2(t_0)$ and $x_1(t_1), x_2(t_1)$ are specified in advance. The Principle of Least Action now says that the true paths of both bodies, which we denote by $X_1(t)$ and $X_2(t)$, are such that the action S is minimised. This principle leads to the equations of motion. The equations are found by requiring that the minimised action has no first order variation under independent path variations $X_1(t) \to X_1(t) + h_1(t)$ and $X_2(t) \to X_2(t) + h_2(t)$. Analogously to what we found for a single body, the equations have the form

$$\begin{aligned} m_1 \frac{d^2 X_1}{dt^2} + \frac{dV}{dX_1} &= 0 \,, \\ m_2 \frac{d^2 X_2}{dt^2} + \frac{dV}{dX_2} &= 0 \,. \end{aligned} \tag{5.3}$$

Here we have not yet made use of the assumption that V is a function of just the single quantity $l = X_2 - X_1$, the separation of the bodies. Let us denote by V' the derivative $\frac{dV}{dl}$. Then, by the chain rule, $\frac{dV}{dX_2} = V'$ and $\frac{dV}{dX_1} = -V'$. The equations of motion for the two bodies (5.3) therefore simplify to

$$\begin{aligned} m_1 \frac{d^2 X_1}{dt^2} - V'(X_2 - X_1) &= 0 \,, \\ m_2 \frac{d^2 X_2}{dt^2} + V'(X_2 - X_1) &= 0 \,. \end{aligned} \tag{5.4}$$

Each is in the form of Newton's second law. For body 1 the force is $V'(X_2 - X_1)$ whereas for body 2 the force is $-V'(X_2 - X_1)$. The forces are equal and opposite. So we have derived Newton's third law, and seen that the reason for it is the translation invariance of the potential.

Now is a good time to discuss momentum, the product of mass and velocity. The form of Newton's second law for one body suggests that it is worthwhile to define the momentum P as

$$P = m \frac{dX}{dt} \,. \tag{5.5}$$

The equation of motion (4.27) then takes the form

$$\frac{dP}{dt} + V'(X(t)) = 0\,, \tag{5.6}$$

which is hardly more than a change of notation. If V' is zero, and there is no force, then $\frac{dP}{dt} = 0$, so P is constant. In this case we say that P is conserved.

Momentum is more useful when there are two or more bodies. Suppose we add the two equations (5.4). The force terms cancel, leaving

$$m_1\frac{d^2X_1}{dt^2} + m_2\frac{d^2X_2}{dt^2} = 0\,. \tag{5.7}$$

We can integrate this once, finding

$$m_1\frac{dX_1}{dt} + m_2\frac{dX_2}{dt} = \text{constant}\,. \tag{5.8}$$

In terms of the momenta P_1 and P_2 of the two bodies we see that

$$P_1 + P_2 = \text{constant}\,. \tag{5.9}$$

This is an important result. Even though there is a complicated relative motion between the two bodies, the total momentum $P_{\rm tot} = P_1 + P_2$ is unchanging in time. $P_{\rm tot}$ is conserved. This is a consequence, via Newton's third law, of the bodies not interacting with the environment but only with each other.

One interpretation is that the two bodies act as a composite single body, with a total momentum that is the sum of the momenta of its constituents. The total momentum of the composite body is conserved, which is what is expected for a single body not exposed to an external force. One can go further and identify for a composite body the preferred, central position that it would have as a single body. First we note that

$$P_{\rm tot} = m_1\frac{dX_1}{dt} + m_2\frac{dX_2}{dt} = \frac{d}{dt}\big(m_1X_1 + m_2X_2\big)\,, \tag{5.10}$$

and that the mass of the composite body is $m_1 + m_2$. Then we write

$$P_{\rm tot} = (m_1 + m_2)\frac{d}{dt}\left(\frac{m_1}{m_1+m_2}X_1 + \frac{m_2}{m_1+m_2}X_2\right)\,. \tag{5.11}$$

This expresses the total momentum in the form expected for a single body, as the product of the total mass $M = m_1 + m_2$ and a velocity, the time-derivative of the central position

$$X_{\rm C.M.} = \frac{m_1}{m_1+m_2}X_1 + \frac{m_2}{m_1+m_2}X_2\,. \tag{5.12}$$

$X_{\rm C.M.}$ is called the Centre of Mass. Because of the conservation of total momentum, $X_{\rm C.M.}$ moves at a constant velocity. The composite body obeys, essentially, Newton's first law, despite the internal relative motion of its constituents. The Centre of Mass is an average of the positions of the constituents, weighted by their masses. It is the ordinary average if the masses are equal.

Further extensions of these results can be found for any collection of bodies, and for motion in 2 or 3 dimensions, but a proper discussion requires the use of vectors and partial derivatives. The potential energy is a function of the positions $\mathbf{x}$ of all the bodies, that is, a function of $3N$ variables for N bodies moving in 3 dimensions. One needs to be able to differentiate with respect to any of these variables. Here are some of the results one can obtain.

For one body, moving in 3 dimensions, the action S has kinetic energy and potential energy contributions. The equation of motion, Newton's second law, involves a vector force $F(\mathbf{x})$ that is not fundamental, but comes from the (partial) derivatives of the potential $V(\mathbf{x})$. That means that $F(\mathbf{x})$ is not an arbitrary vector function, but is subject to some restrictions. Such a force is called conservative, and is the only type that arises from a Principle of Least Action. The forces in nature, like the gravitational force on a body due to other bodies, or the force produced by a static electric field on a charged particle, are of this conservative type.

If N bodies interact with each other, through a potential energy V that depends on the positions of all of them, then one can derive from a single Principle of Least Action the equations of motion of all of them. The equation of each body has the form of Newton's second law of motion for that body. If the whole system is isolated from the environment, then the system is translation invariant, and V only depends on the relative positions of the bodies. In this case, the sum of the forces acting on the N bodies vanishes, i.e. $F_1 + F_2 + \cdots + F_N = 0$. This is a more general version of Newton's third law, but it implies the usual third law. For example, the force on the first body, F_1, which is produced by all the other bodies, is equal and opposite to the total force on the other bodies combined, $F_2 + \cdots + F_N$.

One can define momenta for each of the bodies, $P_1 = m_1 \frac{dX_1}{dt}$, $P_2 = m_2 \frac{dX_2}{dt}$, etc., and a total momentum $P_{\rm tot} = P_1 + P_2 + \cdots + P_N$. For an isolated system, where $F_1 + F_2 + \cdots + F_N = 0$, $P_{\rm tot}$ is conserved. It follows that if we define, for N bodies, the total mass as $M = m_1 + m_2 + \cdots + m_N$, and the Centre of Mass as

$$X_{\rm C.M.} = \frac{m_1}{M} X_1 + \frac{m_2}{M} X_2 + \cdots + \frac{m_N}{M} X_N \,, \qquad (5.13)$$

then the Centre of Mass has a constant velocity. We can think of the N bodies as forming a single composite body, characterised by its total mass and a simple Centre of Mass motion. If this composite body is not isolated from the environment, then the sum of the forces $F_{\rm tot}$ will not vanish. The equation of motion for the Centre of Mass is then

$$M \frac{d^2 X_{\rm C.M.}}{dt^2} = F_{\rm tot} \,. \qquad (5.14)$$

This simple result helps to explain the motion of a composite system, like the Earth and Moon together, around the Sun.

6 Conservation of Energy

An important topic we have ignored so far is that of total energy, and its conservation. This is because it is a little tricky to get insight into energy conservation directly from the Principle of Least Action. Instead it is easier to start with the equations of motion. Let us return to the situation of a single body moving in one dimension, with equation of motion

$$m\frac{d^2x}{dt^2} + \frac{dV}{dx} = 0\,. \tag{6.1}$$

Multiply by $\frac{dx}{dt}$ to obtain

$$m\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{dV}{dx}\frac{dx}{dt} = 0 \tag{6.2}$$

and notice that this can be expressed as a total derivative,

$$\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x(t))\right) = 0\,. \tag{6.3}$$

Integrating, we see that

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x(t)) = \text{constant}\,. \tag{6.4}$$

The constant is called the total energy of the body, and denoted by E. Since E is constant, one says that energy is conserved. Notice that $E = K + V$, but K and V are not separately conserved. Note also the plus sign here, not the minus sign that occurs in the Lagrangian $L = K - V$.

Total energy is also important for a body moving in more than one dimension. For a body subject to a general force $F(\mathbf{x})$ in 3 dimensions, there is no conserved energy. But if there is a potential $V(\mathbf{x})$, and $F(\mathbf{x})$ is derived from this by differentiation, which is what follows from the Principle of Least Action, then there is a conserved total energy $E = K + V$. This is why the force obtained by differentiating V is called conservative. The proof of energy conservation is slightly more difficult than in one dimension, as it requires partial derivative notation. Total energy is also conserved for a collection of interacting bodies, provided the forces are obtained from a single potential energy function – again, precisely the condition for the equations of motion to be derivable from a Principle of Least Action.

6.1 Energy conservation from the Principle of Least Action

[This subsection can be skipped.]

Conservation of energy *can* be directly derived from the Principle of Least Action. This is done by considering how the action S changes if the bodies move along their true paths but at an infinitesimally modified speed. This is a special kind of path variation which, like all others, must leave S unchanged to first order.

Let us simplify to the case of a single body moving in one dimension. Let $X(t)$ be the true motion between given endpoints. Changing the speed is equivalent to changing the time at which the body reaches each point along its path. The varied path can be expressed as

$$\widetilde{X}(t) = X(t + \varepsilon(t)) \,, \tag{6.5}$$

where ε is an infinitesimal function of t, the time shift. We do not want the endpoints of the path to change, so $\varepsilon(t_0) = \varepsilon(t_1) = 0$.

Using the usual formula for infinitesimal changes, we see that

$$\widetilde{X}(t) = X(t) + \frac{dX}{dt}\,\varepsilon(t) \,, \tag{6.6}$$

so the velocity along the varied path is

$$\frac{d\widetilde{X}}{dt} = \frac{dX}{dt} + \frac{d}{dt}\left(\frac{dX}{dt}\,\varepsilon(t)\right) \,, \tag{6.7}$$

and the potential energy is

$$V(\widetilde{X}(t)) = V(X(t)) + V'(X(t))\frac{dX}{dt}\,\varepsilon(t) \,. \tag{6.8}$$

Now recall that

$$S_X = \int_{t_0}^{t_1} \left(\frac{1}{2}m\left(\frac{dX}{dt}\right)^2 - V(X(t))\right) dt \,, \tag{6.9}$$

and $S_{\widetilde{X}}$ is the same integral with $\widetilde{X}$ replacing X. Using the expressions above we see that to first order in ε

$$S_{\widetilde{X}} - S_X = \int_{t_0}^{t_1} \left(m\frac{dX}{dt}\frac{d}{dt}\left(\frac{dX}{dt}\,\varepsilon(t)\right) - V'(X(t))\frac{dX}{dt}\,\varepsilon(t)\right) dt \,. \tag{6.10}$$

Integrating the first term by parts, we find

$$S_{\widetilde{X}} - S_X = -\int_{t_0}^{t_1} \left(m\frac{d^2X}{dt^2}\frac{dX}{dt} + V'(X(t))\frac{dX}{dt}\right) \varepsilon(t)\, dt \,. \tag{6.11}$$

There is no contribution from the endpoints because ε vanishes there.

By the Principle of Least Action, $S_{\widetilde{X}} - S_X$ must vanish. Between the endpoints, ε is arbitrary, so the expression multiplying ε in the integral (6.11) must vanish. Therefore, at all times,

$$m\frac{d^2X}{dt^2}\frac{dX}{dt} + V'(X(t))\frac{dX}{dt} = 0 \,. \tag{6.12}$$

Apart from slightly different notation, this is the equation (6.2) that we used before when showing energy conservation. The final step is the same. Eq.(6.12) can be rewritten as

$$\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dX}{dt}\right)^2 + V(X(t))\right) = 0 \tag{6.13}$$

so the total energy,

$$E = \frac{1}{2}m\left(\frac{dX}{dt}\right)^2 + V(X(t))\,, \tag{6.14}$$

is conserved.

Very similar calculations work for N bodies, and for bodies moving in more than one dimension. If one replaces the actual motion of each body $X_1(t), X_2(t), \cdots, X_N(t)$ by a motion along the same geometrical path, but shifted in time to $X_1(t+\varepsilon(t)), X_2(t+\varepsilon(t)), \cdots, X_N(t+\varepsilon(t))$, and requires the action S to be unchanged to first order in ε, then one finds the equation

$$\frac{d}{dt}(K+V) = 0 \tag{6.15}$$

where K and V are the kinetic and potential energies of the N bodies combined. Therefore the total energy $E = K + V$ is conserved. The calculation requires some vector notation, and partial derivatives, but is essentially the same as for one body moving in one dimension.

In summary, we have shown that energy conservation follows from the Principle of Least Action. It is a consequence of the action being stationary under path variations that come from modifying the time parameter along the true path. There is a more sophisticated understanding of energy conservation, related to time-translation invariance, but requiring a re-think of the fixed endpoint conditions we have been imposing. This illustrates a very general relationship between symmetries of a dynamical system and the conserved quantities that the motion exhibits. This general relationship is known as Noether's Theorem.

7 Summary

Optimisation principles play a fundamental role in many areas of physical science. We have discussed the Fermat Principle of minimal time in ray optics, and derived the laws of reflection and refraction.

A profound understanding of the dynamics of bodies comes from the Principle of Least Action. From this one derives all three of Newton's laws of motion. We have found these laws without using the calculus of variations, but a more elementary method that relies on ordinary calculus. For completeness we also gave the calculus of variations derivation. The action is constructed from the kinetic energies of the bodies participating, and a single potential energy function V which depends on the positions of all the bodies. The kinetic energy of each body is of a standard form, being proportional to mass and quadratic in velocity, but the potential energy has a variety of forms depending on the physical situation. The force on a body was shown to be minus the derivative of V with respect to the body's position.

From the equations of motion one can prove that for a body or collection of bodies not subject to external forces, the total momentum is conserved. In addition, there is a conserved total energy. By considering the total momentum of a collection of bodies, we saw how to define the Centre of Mass, and study its motion.

The Principle of Least Action can be developed further, to deal with rotating bodies, for example. One good feature, not discussed here, is that the action can be written down using any coordinates, which makes it easier to understand certain kinds of motion. Converting Newton's laws of motion to polar coordinates, for example, is a little tricky, but much easier if one starts with the Principle of Least Action.

Acknowledgement

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Appendix: Motion under gravity – another problem

Here is a problem that can be studied by direct application of the Principle of Least Action. Consider a body of mass m moving vertically in the Earth's gravitational potential. Suppose the body is thrown upwards from height zero at time $-T$, and returns to height zero at time T. Find the height H that the body reaches?

The potential is $V(x) = mgx$, so the action is

$$S = \int_{-T}^{T} \left(\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - mgx(t) \right) dt \,. \tag{7.1}$$

We assume the graph $x(t)$ is a parabola, reaching its maximal height H at time $t = 0$ and satisfying the endpoint conditions, $x(\pm T) = 0$, so

$$x(t) = H \left(1 - \frac{t^2}{T^2} \right) \,. \tag{7.2}$$

H is the one free parameter here. As the velocity is

$$\frac{dx}{dt} = -\frac{2Ht}{T^2} \,, \tag{7.3}$$

the action S simplifies to the integral

$$S = m \int_{-T}^{T} \left(\frac{2H^2}{T^4} t^2 - gH + \frac{gH}{T^2} t^2 \right) dt \,, \tag{7.4}$$

whose value is

$$S = \frac{4}{3}m\left(\frac{H^2}{T} - gHT\right). \tag{7.5}$$

Note that if H is zero then S is zero, because in this case the body does not move, and both kinetic and potential energy vanish. The minimised action S will be less than this, i.e. negative.

We find the minimum of S by setting $\frac{dS}{dH} = 0$. This gives the equation $\frac{2H}{T} - gT = 0$, so the height reached is

$$H = \frac{1}{2}gT^2. \tag{7.6}$$

The action S, for this H, is

$$S = -\frac{1}{3}mg^2T^3. \tag{7.7}$$

The calculation correctly gives the height reached, because the true path that minimises the action and solves the equation of motion has a parabolic graph, as we assumed.

It is in fact easier to solve this problem using the equation of motion. We know that motion at constant acceleration $-g$ produces a term $-\frac{1}{2}gt^2$ in the solution $x(t)$. Comparing with (7.2), we see that H has to be $\frac{1}{2}gT^2$.

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