Derivation of Bagnold’s velocity profile for free-surface granular flows

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A rheology describes the stress-strain relationship in terms of frictional parameters. In the dense inertial regime, the shear stress and pressure are related as:

\[ \frac{\tau}{P} = \mu(I), \quad (1) \]

with the inertial number \( I \), representing the ratio between the microscopic (rearrangement) and macroscopic (deformation) timescale:

\[ I = \frac{\gamma d}{\sqrt{\frac{\tau}{P}}}, \quad (2) \]

For both the plane shear and the inclined-plane geometry, the stress distribution is constant, hence \( P = \text{constant} \) and \( \frac{\tau}{P} = \text{constant} \). As a result, the inertial number \( I \) needs to be constant and equal to:

\[ I = \mu^{-1}\left(\frac{\tau}{P}\right). \quad (3) \]

In plane shear at low inertia numbers, the shear rate \( \dot{\gamma}(y) \) is uniform and the velocity profile is linear:

\[ V_{\text{wall}} = \dot{\gamma}_{\text{wall}}L, \quad (4) \]

In the case of surface flows, the velocity as a function of position is:

\[ V(y) = \int_0^h \dot{\gamma}(y)dy, \quad (5) \]

and the stress distribution is:

\[ P = \rho g (h - y) \cos(\theta), \quad (6) \]

with

\[ \frac{\tau}{P} = \tan(\theta). \quad (7) \]

Now, the shear rate, from the equation with the inertial number, is:

\[ \dot{\gamma}(y) = \frac{1}{d} \sqrt{\frac{P(y)}{\rho} \mu^{-1}(\tan(\theta))} = \frac{1}{d} \sqrt{g (h - y) \cos(\theta) \mu^{-1}(\tan(\theta))}. \quad (8) \]

Integrating this once over the height of the flow, and scaling with the factor \( \sqrt{gd} \), gives the velocity profile as a function of \( y \):

\[ \frac{V(y)}{\sqrt{gd}} = A(\theta) \frac{3}{2} \frac{h^{3/2} - (h - y)^{3/2}}{d^{3/2}}, \quad (9) \]

with

\[ A(\theta) = \frac{2}{3} \mu^{-1}(\tan(\theta)) \sqrt{\cos(\theta)} = \frac{2}{3} I(\theta) \sqrt{\cos(\theta)}. \quad (10) \]

The velocity profile is in this case not linear, but scales with the depth as \( h^{3/2} \) and is called a “Bagnold velocity profile”. Analyzing the average velocity \( \langle V \rangle \) can be achieved by:

\[ \langle V \rangle = \frac{1}{h} \int_0^h \frac{V(y)}{\sqrt{gd}}dy = \frac{3}{5} A(\theta) \left(\frac{h}{d}\right)^{3/2}. \quad (11) \]
Rewriting this in terms of a Froude number $Fr$ gives:

$$Fr = \frac{<V>}{\sqrt{gh}} = \frac{3}{5} A(\theta) \frac{h}{d},$$  \hspace{1cm} (12)

By comparing this with the equation for the Froude number as derived from experiments and simulations:

$$Fr = \frac{<V>}{\sqrt{gh}} = \alpha + \beta \frac{h}{h_{stop}(\theta)},$$  \hspace{1cm} (13)

some functional similarities and scalings are observed. Assuming $\alpha = 0$, which has been observed for experiments with smooth glass beads and simulations with spherical particles, one can derive an expression for $\beta$:

$$\beta = \frac{3}{5} A(\theta) \frac{h_{stop}(\theta)}{d}.$$  \hspace{1cm} (14)