Sediment continuity: how to model sedimentary processes?

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1 Sediment transport

The total sediment transport rate per unit width is a combination of bed load \( q_b \), suspended load \( q_s \) and wash-load \( q_w \):

\[
q_t = q_b + q_s + q_w.
\]  

(1)

To determine in which regime the flow lies, one can differentiate the different regions by using the Rouse number \( P = \frac{u_s}{\beta u^*} \), with settling velocity \( u_s \), friction velocity \( u^* \), the von Karman number \( \kappa = 0.4 \) and a constant \( \beta \) depending on the ability of particles to move with the fluid. For small particles, \( \beta = 1 \), but larger particles won’t be able to keep up and therefore \( \beta < 1 \).

In bed load \( q_b \), \( 2.5 < P < 7.5 \), the coarsest particles are in motion along the bed by rolling, quivering and hopping (saltating). Movement is not continuous or uniform, brief gusts and pulses.

In suspended load \( q_s \), \( 0.8 < P < 2.5 \), the medium-sized particles are in suspension and in continuous balance between upwards turbulent mixing and downwards settling. They travel downstream in trajectories eventually returning to bed.

In wash load \( q_w \), \( P < 0.8 \), fine particles are in near-permanent suspension: turbulent mixing is dominant and random motions are Brownian.

2 Shallow water equations in conservative and non-conservative form

The typical x-component of the Navier-Stokes equation states:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + f_x,
\]

(2)

with velocity field \( \vec{u} = (u, v, w) \), time \( t \), pressure \( p \), density \( \rho \), kinematic viscosity \( \nu \) and the body force in x-direction \( f_x \). Now, assume the following: (1) no frictional forces \( \nu = 0 \) or body forces \( f_x = 0 \) exist, (2) due to 1D flow in the x-direction:

\[
v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = 0,
\]

(3)

(3) the pressure distribution is hydrostatic, therefore \( p = \rho gh \) and:

\[-\frac{1}{\rho} \frac{\partial p}{\partial x} = -g \frac{\partial h}{\partial x}.
\]

(4)

with these assumptions, the Navier-Stokes equation simplifies to the shallow-water equation:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x}.
\]

(5)

Another way of looking at the conservation equations is to analyze the equation in conservative form. In the case of no rotational, frictional or viscous forces, the conservation of mass and momentum in conservative form state:

\[
\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0.
\]

(6)
and
\[ \frac{\partial u h}{\partial t} + \frac{\partial u^2 h}{\partial x} = -\frac{\partial g h^2}{\partial x}. \]  
(7)

By expanding the terms out, one can derive the non-conservative form of the momentum equations:
\[ u \frac{\partial h}{\partial t} + h \frac{\partial u}{\partial t} + u h \frac{\partial u}{\partial x} = -g h \frac{\partial h}{\partial x}, \]
(8)

and by applying the mass conservation equation and dividing by \( h \), one obtains:
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x}, \]
(9)

which is the same form as derived above.

3 Exner-Polya equation for sediment continuity in bedload transport

The Exner-Polya equation describes the sediment continuity, or material mass balance equation, by calculating the transient mobile-bed evolution involving the bedload and its evolution. It evaluates the time evolution of the mass within the layer, adds the mass flow rate out and in the bedload and equates it to the net vertical mass flux to the bed, writing:
\[ \frac{\partial}{\partial t} \left[ \eta \Delta x (1 - \lambda_p) \rho_s \right] + \rho_s \left[ q_{bx}(x + \Delta x) - q_{bx}(x) \right] = F, \]
(10)

with the flux \( F \) as the net vertical mass flux from the bed.

Taking the limit as the differentials approach 0 and looking at the 2D case, the Exner equation reduces to:
\[ (1 - \lambda_p) \frac{\partial \eta}{\partial t} + \frac{\partial q_b}{\partial x} = u_s (e_b - E), \]
(11)

with on the left-hand side the sediment material porosity of the bed \( \lambda_p \), the evolution of bed elevation \( \eta \), volume bedload flux vector or transport rate per unit width \( q_b \). Note that the packing fraction \( \phi = 1 - \lambda_p \), with the packing fraction preferred by physicists and the porosity preferred by the geomorphological community. The schematic cross-section of bedload characteristics is given in figure 1.

![Figure 1: Cross-section of mobile bed characteristics, derived from Sotiropoulos and Khosronejad, 2015](image-url)
The bedload flux vector is a critical parameter and is defined as:

\[ q_b = \int_0^{\delta_b} u(\eta)\phi_b d\eta \approx \bar{\phi}_b \int_0^{\delta_b} u(\eta) d\eta, \]

(12)

with bedload layer thickness \( \delta_b \), settling velocity \( u_s \), the bedload concentration across the bedload layer \( \phi_b \). For the last simplification, we assume that the bedload concentration \( \phi_b \) in vertical direction in the thin bedload layer is constant and can be approximated by its depth-averaged equivalent \( \bar{\phi}_b \).

The right-hand side of the Exner-Polya equation includes the net rate (= velocity times concentration) of material in the control volume. In other words, this is the rate of entrainment of bed material into the suspended load \( u_s \delta_b \) minus the rate at which suspended load is deposited on the mobile bed \( u_s E \).

The bedload concentration across the bedload layer has a functional dependance as:

\[ \phi_b = f(F_f, \tau_b, \rho_s, \rho_f, D), \]

(13)

with the forces exerted by turbulent flow on the sediment \( F_f [N = \text{kg/m/s}^2] \), the bed shear stress \( \tau_b [N/m^2 = \text{kg/(m/s)^2}] \), the fluid \( \rho_f [\text{kg/m}^3] \) and sediment \( \rho_s [\text{kg/m}^3] \) material density and fluid kinematic viscosity \( \nu [\text{m}^2/\text{s}] \) and the median size of bed material \( D [\text{m}] \).

The bed shear stress \( \tau_b \) can be subdivided into skin friction \( \tau_{sf} \) due to friction forces on particles and form drag \( \tau_{fd} \) due to energy loss in the lee of bedforms. The stability of individual grains is not affected by form drag, as this is generated by the difference in pressures between the upstream and downstream sides of bedforms. The main influence to the motion of surface grains comes from the skin friction, forming the dominant hydrodynamic parameter influencing both bedload and suspension processes. The common approach is to use the skin friction in sediment transport models, and the total friction in hydrodynamical models.

4 Non-dimensionalizing for the bedload flux

There are just a few dimensionless groups influencing the bedload sediment transport rate that can be formed from these parameters: the dimensionless boundary shear stress \( \tau^*_b \), the relative roughness of the flow \( L_b \) and the wall or boundary Reynolds number \( Re_{wall} \) and the density ratio \( \rho_b/\rho_f \). We will discuss these one by one.

4.1 Boundary shear stress

The Shield’s parameter \( \tau^*_b \) describes the influence of the strength of the flow on the particle movement. The boundary shear stress is more appropriate than a velocity, as the grains are moved by a force acting on the bed. Furthermore, a mean velocity in the bulk of the fluid will not be very relevant for the near-bed processes. The dimensionless shear rate is:

\[ \tau^*_b = \frac{\tau_b}{(\rho_s - \rho_f) gD}. \]

(14)

From the bed shear stress, the friction velocity—a velocity scale—can be defined:

\[ u^* = \sqrt{\tau_b/\rho_f}. \]

(15)

The skin friction Shields parameter:

\[ \tau^* = \frac{\tau_{sf}}{(\rho_s - \rho_f) gD}. \]

(16)

is used to predict initiation of motion and the to estimate sediment concentrations. A critical Shield’s shear stress \( \tau^*_{cr} \), indicating the onset of motion, is determined empirically and is a function of the sediment size and density, of the fluid density and the flow structure. Common values are established as \( \tau^*_{cr} = 0.047 \) (Meyer-Peter and Muller, 1948), \( \tau^*_{cr} = 0.03 \) (Parker, 1982) or \( \tau^*_{cr} = 0.06 \) (Shields, 1936).
4.2 Wall Reynolds number

Secondly, the wall or boundary Reynolds number, also sometimes called the Galileo number in certain scientific communities, is:

\[ Re_{\text{wall}} = Ga = \frac{u^* k_s}{\nu}, \]  

(17)

with friction velocity \( u^* = \frac{\tau_w}{\rho u} \) and roughness of the wall \( k_s = D \). The wall Reynolds number is a measure of the turbulent structure of the flow. If the roughness at the wall \( k_s \) is of the order of the particle diameter \( D \), the wall Reynolds number is equal to the particle Reynolds number:

\[ Re_p = \frac{u^* D}{\nu}. \]  

(18)

4.3 Roughness of the bed, including law of the wall

The third parameter influencing the sediment transport rate is the relative roughness of the bed with respect to the flow \( \delta_\text{B} \), defined as:

\[ \delta_\text{B} = \frac{k_s}{\delta_\nu}, \]  

(19)

with roughness \( k_s \) and thickness of the viscous sublayer \( \delta_\nu \). To understand the physical meaning of a viscous sublayer, we need to refresh our minds on turbulence and how the resulting velocity profile influences the shear stress. The law of the wall defines the inner layer \( z < \delta \) —including the viscous sublayer and the transition zone— and the outer layer \( z > \delta \) —including the log-layer and the defect layer (see figure 2).

![Figure 2: Law of the wall, with a viscous sublayer, log-layer and defect layer.](image)

Close to the wall in the viscous sublayer, for \( z < z_f \), there is no turbulence \( (u'v' = 0) \) and laminar flow governs the linear velocity profile satisfying the no-slip velocity condition \( (u(0) = 0) \):

\[ u(z) = \frac{\tau_b z}{\rho \nu} = \frac{u^* z}{\nu} \]

(20)
and the constant viscous shear stress \( \tau = \tau_b \).

The transition between the viscous and inertial sublayer (sometimes also called the transition or buffer layer), is determined by the frictional lengthscale \( z_f = \nu/u^* \) where the viscous and turbulent stresses are of comparable magnitude. Both the law of the wall and the velocity defect law for the turbulent logarithmic layer need to be matched in this transition layer, and the flow is unaware both of the viscosity and of the size of the boundary layer.

In the turbulent logarithmic layer, viscosity can be neglected and the Prandtl mixing length concept prescribes a logarithmic velocity profile:

\[
 u(z) = \frac{u^*}{\kappa} \ln \left( \frac{zu^*}{\nu} \right).
\]

The solution is only valid outside of the viscous boundary layer and the no-slip condition on the wall is not satisfied in this solution.

Lastly, in the turbulent outer layer, strong mixing occurs due to large eddies, resulting in an almost constant velocity \( u(z) = U \) and a varying turbulent shear stress \( \tau_{\text{turb}} \).

Turning our attention back to the effect of the roughness of the bed, the influence of structures (e.g. particles or bedforms) on the flow greatly depends whether they are within (= hidden) or outside (= sticking out) the viscous sublayer with thickness \( \delta_v \). For hydraulically smooth flow, resulting in a lubricated laminar regime which is valid for \( Re_{\text{wall}} < 5 \), we assume \( k_s < \delta_v \). In transitional flow for \( 5 < Re_{\text{wall}} < 70 \), viscosity and roughness have a limited effect as \( k_s \approx \delta_v \), while hydraulically rough flow for \( Re_{\text{wall}} > 70 \) and \( k_s > \delta_v \) features flow separation and form drag. Assuming that the roughness of the flow is of the order of the particle diameter and the viscous sublayer is of the order of the bedload layer thickness \( \delta_b \), we can assume that the relative roughness \( L_b \) is not a critical parameter in determining the sediment transport rate.

### 4.4 Density ratio

The last parameter potentially influencing the sediment transport is the density ratio \( \rho_p/\rho_f \). For water/sand systems, the density ratio is close to unity and is therefore usually not included in the analysis.

### 4.5 Bedload sediment transport rate

The bedload sediment transport rate \( q_b \) can be non-dimensionalized by the specific weight \( \gamma = g(\rho_s - \rho_f) \), the density of the fluid \( \rho_f \) and the particle diameter \( D \), resulting in the Einstein number:

\[
 q^*_b = \frac{q_b}{D \left( \frac{\rho_s - \rho_f}{\rho_f} gD \right)^{1/2}}.
\]

Based on the arguments above, we determine that the boundary shear stress and the particle size are the most important parameters in determining the sediment transport rate:

\[
 q^*_b = f(\tau^*, Re_p) = f\left( \frac{\tau_{sf}}{(\rho_s - \rho_f) gD}, \frac{u^* D}{\nu} \right).
\]

### 4.6 Empirical data and diagrams

Empirical experiments indicate that the sediment transport rate scales as a power of \( 3/2 \) to the dimensionless shear stress, for example the Meyer Peter and Muller (1948) relation: \( q^*_b \sim 8 (\tau^* - \tau^*_c)^{3/2} \), although slightly different relations have been proposed:

- **Einstein-Brown (1950):** \( q_b \sim \tau_{sf} \)
- **Yalin (1963):** \( q_b \sim \tau_{sf}^{3/2} \)
- **Bagnold, empirical (1966) and Andreotti, theoretical (2012):** \( q_b \sim \tau_b^{3/2} \)
The Shields’ parameter is critical in determining when sediment starts to move, which can be visualized by a Shields diagram. Shields (1936) noted that the drag $F_D$ and life force $F_L$ exerted on a bed sediment particle are proportional to $u^*^2$ and to functions of the grain Reynolds number $Re_p$. The curve indicating the threshold between movement and non-movement is called the Shields curve, as illustrated in figure 4.

Note that there is considerable scatter in the data and that there is only data in the range of $2 < Re_p < 600$. However, this diagram has been widely used to indicate movement of particles. An updated version of the Shields diagram, presented in work by Miller, McCabe and Komar in 1977, included a larger range of $Re_p$ and more data. Parker et al, 2003 provide a regression for the

Figure 3: Dimensionless sediment transport rate (expressed as volume, not mass) against dimensionless boundary shear stress (in the form of the Shields parameter) for various sets of measurement data from OCW MIT course notes, modified from Vanoni, 1975).
Shields curve:

\[ \tau^*_{cr} = 0.5 \left[ 0.22Re_p^{-0.6} + 0.0610^{-7.7Re_p^{-0.6}} \right]. \tag{24} \]

The complication with the Shields diagram is that it entangles the critical shear stress \( \tau^* \) and the particle diameter \( D \), as they both appear in the particle Reynolds number \( Re_p \) and the Shields number \( \tau^* \): it is not easy to solve for either parameter. Shields diagrams with alternative non-dimensional axes have been proposed to reconcile this complication, for example Yalin’s curve with a dimensionless grain diameter \( D^{*1/2} = Re_p = \sqrt{\frac{(\rho_s - \rho_f) g D^3}{\rho_f \nu^2}} \) and the Shields number \( \tau^* \).

Again another approach is to use the Hjulstrom diagram, as illustrated in figure 5, plotting particle size against velocity. Obviously, the use of a mean velocity to describe the flow is limited and the boundaries were on purpose drawn widely to represent uncertainty, but it shows the effect of cohesion and sedimentation on the bedload transport very nicely.

5 Suspension

The advection-diffusion equation for suspended sediment concentration is given as:

\[ \frac{\partial \phi}{\partial t} + \nabla [u \phi] = \nabla [D \nabla \phi], \tag{25} \]

with volume concentration of particles \( \phi \), advection velocity \( u \) exerted by the flow and molecular diffusion coefficient \( D = D_x, D_y, D_z \).
Let’s now assume a flow in two dimensions, the x- (downstream) and z-direction (depth). The advection-diffusion equation reduces to:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial [u\phi]}{\partial x} + \frac{\partial [w\phi]}{\partial z} = \frac{\partial}{\partial x} \left[ D_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial z} \left[ D_z \frac{\partial \phi}{\partial z} \right].
\] (26)

Introducing the mean and fluctuating components of the concentration \( \phi = \bar{\phi} + \phi' \) and velocity \( u = \bar{u} + u' \) and \( w = \bar{w} + w' \) (= Reynolds’ decomposition), reduces the advection diffusion equation to:

\[
\frac{\partial (\bar{\phi} + \phi')}{\partial t} + \frac{\partial [(\bar{u} + u')(\bar{\phi} + \phi')]}{\partial x} + \frac{\partial [(\bar{w} + w')(\bar{\phi} + \phi')]}{\partial z} = \frac{\partial}{\partial x} \left[ D_x \frac{\partial (\bar{\phi} + \phi')}{\partial x} \right] + \frac{\partial}{\partial z} \left[ D_z \frac{\partial (\bar{\phi} + \phi')}{\partial z} \right].
\] (27)

By taking a time average, we can simplify this equation by realizing that \( \phi = \int \phi dt, \bar{\phi} = \int \bar{\phi} dt, \int \phi' dt = 0, \int \phi' u' dt = 0, \int \phi' w' dt = 0 \) and \( \int \phi' u' dt = \bar{\phi'} u' \) (and equivalently for \( w \) instead of \( u \)):

\[
\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial [\bar{u}\bar{\phi} + \bar{u}'\phi']}{\partial x} + \frac{\partial [\bar{w}\bar{\phi} + \bar{w}'\phi']}{\partial z} = \frac{\partial}{\partial x} \left[ D_x \frac{\partial \bar{\phi}}{\partial x} \right] + \frac{\partial}{\partial z} \left[ D_z \frac{\partial \bar{\phi}}{\partial z} \right].
\] (28)

Now, let’s move the “Reynolds stresses” \( u'\phi' \) and \( w'\phi' \), also sometimes called the statistical correlations between the concentration and the velocity fluctuations, to the right-hand side of the equation. Often, transport due to vertical diffusion \( \partial / \partial z (D_z \partial \phi / \partial z) \) often exceeds transport due to streamwise diffusion \( \partial / \partial x (D_x \partial \phi / \partial x) \), therefore neglecting the latter. Using mass conservation for an incompressible flow \( (u_x + w_z = 0) \), reduces this equation to:

\[
\frac{\partial \bar{\phi}}{\partial t} + \bar{u} \frac{\partial \bar{\phi}}{\partial x} + \bar{w} \frac{\partial \bar{\phi}}{\partial z} = \frac{\partial}{\partial z} \left[ D_z \frac{\partial \bar{\phi}}{\partial z} - w'\phi' \right].
\] (29)

Now assume equilibrium conditions in a rectangular 2D channel, resulting in the result that quantities do not vary in x-direction \( \frac{\partial}{\partial x} \). Furthermore, assume that the mean vertical velocity is equal to the settling velocity \( \bar{w} = -u_s \) and a steady flow and therefore \( \frac{\partial}{\partial t} = 0 \). The advection-diffusion
The equation for suspended sediment concentration now reduces to:

$$\frac{\partial}{\partial z} \left[ D_z \frac{\partial \phi}{\partial z} - w' \phi' \right] + \bar{u}_s \frac{\partial \phi}{\partial z} = 0.$$  
(30)

The first term in this equation is the molecular diffusion, the second term is the Reynolds stresses (often linked to the turbulent diffusion, as we will see momentarily) and the last term represents a flux in z-direction due to settling. For the case of sand particles in water, molecular diffusion will be very small and therefore this diffusion is often neglected $D_z = 0$:

$$- \frac{\partial w' \phi'}{\partial z} + \bar{u}_s \frac{\partial \phi}{\partial z} = 0.$$  
(31)

Unfortunately, resolving the the concentration and velocity fields and their fluctuations (e.g. $w' \phi'$) is rather complicated and in order to make progress, we need to model the fluctuations in terms of the mean values of the velocity and/or concentration using a mixing-length model, in which turbulent motions are characterized by the eddy length-scale.

Let’s first refresh our memory on turbulence. The turbulent stress can be written as:

$$\tau = -\rho K \frac{du}{dz} = -\rho u'^2 = \tau_0,$$  
(32)

with the eddy diffusivity $K_e = \epsilon/\rho$ and friction velocity $u^*$. As a reminder, the friction, or shear, velocity is a shear stress written in terms of velocity units. Research showed that the sediment mass diffusivity $K_s$ is scaling with the water eddy momentum diffusivity $K_e$ as:

$$K_s \approx \beta K_e,$$  
(33)

with constant $\beta$ depending on the ability of particles to move with the fluid. For small particles, $\beta = 1$, but larger particles won’t be able to keep up and therefore $\beta < 1$. The law of the wall indicates a logarithmic profile for the velocity as a function of height close to the boundary:

$$u(z) = \frac{u^*}{\kappa} \ln \left( \frac{z}{z_0} \right),$$  
(34)

with $\kappa = 0.4$ the Von Karman’s constant, resulting in the derivative of the velocity:

$$\frac{du}{dz} = \frac{u^*}{\kappa z}.$$  
(35)

Substituting equation 35 into our expression for the shear stress provides us with an expression for the sediment mass diffusivity $K_s$:

$$K_s = \beta K_e = \beta u^* \kappa z,$$  
(36)

Now, looking back at the advection-diffusion equation for suspended sediment load, let’s model the fluctuations ($w' \phi'$) using the eddy diffusivity concept:

$$w' \phi' = -K_s \frac{\partial \phi}{\partial z}.$$  
(37)

The minus-sign is taken as positive fluctuations in velocity and concentration make the RHS positive, while the gradient of $\phi$ will be negative. The advection-diffusion equation (dropping the top-bars) simplifies to:

$$\frac{\partial}{\partial z} \left[ K_s \frac{\partial \phi}{\partial z} \right] + u_s \frac{\partial \phi}{\partial z} = 0.$$  
(38)

If the settling velocity $u_s$ is not a function of depth, but a constant, then the equation simplifies to:

$$u_s \phi = -K_s \frac{\partial \phi}{\partial z} = -\beta u^* \kappa z \frac{\partial \phi}{\partial z}.$$  
(39)

This is essentially identical to balancing the flux downwards due to particle settling with the flux upwards due to turbulent diffusion. Rewrite this in terms of the Rouse parameter $P = \frac{u_s}{\beta u^* \kappa}$ and group terms:

$$\frac{dz}{z} = -\frac{\beta u^* \kappa}{u_s} \frac{d\phi}{\phi} = -\frac{1}{P} \frac{d\phi}{\phi}.$$  
(40)
A problem appears if we would integrate this all the way down to the bed surface, as the eddy momentum diffusivity is zero at that point. As a solution, we introduce a reference level above the bed $\phi = \phi_0$ at $z = z_0$ and integrate between $z_0$ and $z$:

$$\ln \left[ \frac{\phi}{\phi_0} \right] = -P \ln \left[ \frac{z}{z_0} \right], \quad (41)$$

which can be reduced to a power-law expression:

$$\left[ \frac{\phi}{\phi_0} \right] = \left( \frac{z_0}{z} \right)^P = \left( \frac{z_0}{z} \right)^{\frac{1}{\tau^*}}. \quad (42)$$

The suspended load is created by a balance between particles settling due to gravity and uplifted by turbulent diffusion. The suspended load above the bed can be determined by integrating the concentration multiplied by the mean velocity:

$$q_s = \int_{z=z_0}^{z=H} u(z) \phi(z) \, dz; \quad (43)$$

A typical suspended load is given in the sketch in figure 6, with a maximum away from the bed.

Figure 6: A typical suspended load, derived from figure 3 at leovanrijn-sediment.com.

6 Resuspension

In resuspension, particles are lifted off the bed and are resuspended again in the flow. For a single particle, it is difficult to suspend, but easy to stay suspended. The random motion of the fluid gives a statistical distribution on when a turbulent eddy passes by, which could transfer enough turbulent energy to resuspend a particle.

Resuspension of sediment is an important process in sediment transport. In order to resuspend a particle from the boundary, the hydrodynamic lift force must exceed the particle buoyancy force, giving rise to a critical condition for resuspension in terms of the critical Shields number. The buoyancy force can be quantified as $F_g = (\rho_s - \rho_f) g V$ and the hydrodynamic lift force due to pressure difference around the particle is $F_L = \frac{1}{2} \rho_f C_L (u_T^2 - u_B^2) A$. The latter will be significant close to the bed where there are large velocity gradients due to shear.

For a particle to move along a boundary, the hydrodynamic drag needs to exceed the tangential friction force resulting in incipient particle motion. The drag force $F_D = \frac{1}{2} \rho_s C_D U^2 A$ results from two components: the skin friction drag $\tau_{sf}$ which acts on the particle and influences mobility and the form drag $\tau_{fd}$ which acts on bedforms and increases the overall resistance.
bedload processes, the skin friction will be dominant and form drag is often neglected. The frictional forces are described as the resistance of the bed to motion and can be rolling or Coulomb friction $F_C = \mu_c F_D$ or cohesion friction for smaller $D < 30\mu m$ particles $F_a = \frac{1}{2} \pi \delta \gamma D$.

Let’s analyse particles that are deposited in the viscous sublayer, which has thickness $\delta = \nu/U$. Also, let’s define the critical condition for resuspension in terms of the Shields parameter or Shield’s number, as the ratio between the tangential force and the resisting grain movement:

$$\tau^* = \frac{\tau_{sf}}{(\rho_s - \rho_f)gD},$$

with the skin friction shear stress $\tau_{sf}$, which is the component of the friction that acts on the particle (as oppose to the form drag exerted on bedforms). Another way to define the Shields parameter is in terms of dimensionless groups:

$$\tau^* = \frac{\tau_0}{(\rho_s - \rho_f)gD} = f \left( \frac{\rho_f u^*_D}{\mu}, \frac{\rho_s}{\rho_f} \right) = f \left( Re_{ps}, \frac{\rho_s}{\rho_f} \right),$$

where the dependency of $\rho_s/\rho_f$ is often neglected.

Let’s assume that the gravity force due to weight and buoyancy is:

$$F_g = (\rho_s - \rho_f)gV \approx (\rho_s - \rho_f)gD^3.$$  \hspace{1cm} (46)

Also, assume the lift force scales as:

$$F_L = (1/2) \rho_f C_L (u_T^2 - u_B^2) A \approx \rho_f u^2 D^2.$$ \hspace{1cm} (47)

### 6.1 Resuspension of large particles

Now, for large particles: $k_s > \delta$ and $u \sim U$. Therefore the force balance $F_L = F_g$ reduces to:

$$\rho_f U^2 D^2 = (\rho_s - \rho_f)gD^3.$$ \hspace{1cm} (48)

The critical velocity is now:

$$U > U_c = \left[ \frac{(\rho_s - \rho_f)}{\rho_f} gD \right]^{1/2}.$$ \hspace{1cm} (49)

Rewriting this in terms of the Shield number gives:

$$\tau^* = \frac{\tau_{sf}}{(\rho_s - \rho_f)gD} = \frac{\rho_f U_c^2}{(\rho_s - \rho_f)gD} = 1.$$ \hspace{1cm} (50)

### 6.2 Resuspension of small particles

Now, for small particles: $k_s < \delta$ and $u \sim U k_s \delta$. Therefore the force balance $F_L = F_g$ reduces to:

$$\rho_f \left[ \frac{U^2 D}{\nu} \right]^2 D^2 = (\rho_s - \rho_f)gD^3.$$ \hspace{1cm} (51)

The critical velocity is:

$$U > U_c = \left[ \frac{(\rho_s - \rho_f) g \nu^2}{D} \right]^{1/4}.$$ \hspace{1cm} (52)

Rewriting this in terms of the Shield’s number gives:

$$\tau^* = \frac{\tau_{sf}}{(\rho_s - \rho_f)gD} = \frac{\rho_f U_c^2}{(\rho_s - \rho_f)gD}.$$ \hspace{1cm} (53)

Now substitute $U_c$ to get:

$$\tau^* = \frac{\rho_f \left[ \frac{(\rho_s - \rho_f) g \nu^2}{D} \right]^{1/2}}{(\rho_s - \rho_f)gD} = \frac{\rho_f^{1/2} \nu}{(\rho_s - \rho_f)^{1/2}g^{1/2}D^{3/2}} = \left[ \frac{\rho_f}{(\rho_s - \rho_f)gD} \right]^{1/2} \nu.$$ \hspace{1cm} (54)
As we are looking in this limit for smaller particles and low Reynolds numbers, we should assume that the particles are in the laminar regime. The settling velocity for the laminar regime is:

\[ u_s = \frac{(\rho_s - \rho_f) g D^2}{\mu f} \]  

and substitute that in our expression for the Shield’s number:

\[ \tau^* = \left[ \frac{\rho_f}{(\rho_s - \rho_f) g D D} \right]^{1/2} \left[ \frac{\nu}{\mu} \right]^{1/2} = \left[ \frac{\nu}{u_s D} \right]^{1/2} = \frac{1}{Re_p^{1/2}}, \]

with particle Reynolds number:

\[ Re_p = \frac{u_s D}{\nu}. \]

### 6.3 Summary on resuspension

Large particles are sticking out of the viscous sublayer and as such feel the constant velocity \( U \). The force balance reduces such that the Shield’s number is \( \tau^* = 1 \). Small particles are captured within the viscous boundary layer, encounter a much lower velocity \( U_k s / \delta_v \), resulting in a Shield’s number that depends on the particle Reynolds number \( \tau^* = \frac{1}{Re_p^{1/2}} \). Empirical results, supporting this analysis, are summarized in figure 7 and were derived from Eames and Dalziel, 2000.

![Figure 7: Resuspension depending on particle Reynolds number, reproduced from Eames and Dalziel, 2000 for dust resuspension.](image)
7 Summary of sediment transport

A summary of the information presented in this lecture is given in the Shields-Parker River Sedimentation Diagram, reproduced in figure 8. The Shields’ curve represents the threshold for movement. The onset of significant suspension occurs when the friction velocity is equal to the fall velocity $u^{*\text{far}} = u_*$, or the balance between gravity in vertical direction and the turbulent action to keep particles in suspension.

![Shields-Parker River Sedimentation Diagram](image)

Figure 8: Shields-Parker River Sedimentation Diagram, reproduced from S.A. Miedema, Journal of Dredging Engineering, 2012.