A nonlinear journey - from chaos to random media

Varsity event
Emmy Noether Society and Mirzakhani Society

O. Rath Spivack
University of Cambridge

March 1, 2020

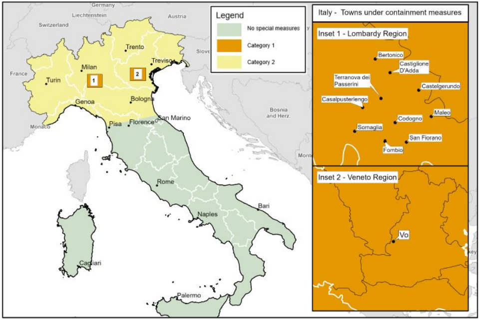
Outline

1. Where it all started
2. Current research
   - Scattering and propagation in random media
   - Inverse problems

A little about me

- Where I come from

Map showing specified areas of Italy, as of 25 February 2020.



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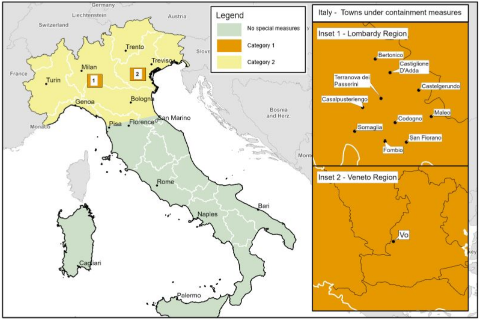

- Where I come from

- First real maths fun stuff

$$p \supset q \iff \sim p \sim q$$

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A little about me

- Where I come from

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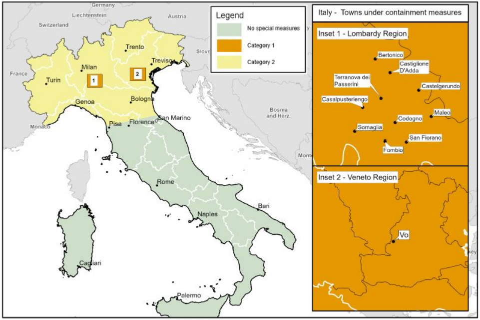

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- First degree: Physics
  (or, as a colleaugue asked: “Where did it all go wrong?”)

Applied Mathematics and other pursuits

- Favourite course during degree in Physics
  - Analisi I
  - Metodi Matematici della Fisica
    (i.e. Functional Analysis / Operator Theory)
- Dissertation: "Some aspects of the theory of polyatomic gases" , Supervisor Carlo Cercignani
  (Boltzmann equation, Waldmann-Snider equation, ...)

> An interesting interlude:
> teaching Maths and Physics in schools in Milan

- Dynamical systems and transition to chaos
- Applied Maths PhD: "The dynamics of excited hydrogen atoms in strong electric and magnetic fields", Supervisor: Derek Richards
  (Classical and quantum chaos)

More Applied Maths and other pursuits

- First postdoc, at the Cavendish Laboratory:
  Diffusion Limited Aggregates (DLAs) and their fractal measure
  - Abstract maths: probability measures, multifractals
  - Applications: gas reservoirs, viscous fingering

> Another interesting interlude:
> looking after 2 small children!

- Daphne Jackson Research Fellowship, in the Maths Faculty!
  (e,2e) processes with hydrogenic ion targets
  - Abstract maths: not a lot, though:
    classical / quantum correspondence,
    operator series approximations
  - Applications: astrophysics, nuclear fusion
- Lu Gwei Djen Research Fellowship

And even more Applied Maths and other pursuits

A long, interesting mixed stretch:

- Graduate Tutor at Lucy Cavendish College
  - College graduate admissions
  - Pastoral tutor
  - University Educational policy and governance
- Another postdoc in the Maths Faculty
  propeller noise, underwater acoustics, wave scattering
  - Abstract maths: operator series approximations
    wave equations, integro-differential equations
  - Applications: ship hull vibrations, signature of submarines
- Director of Studies in Mathematics
  - College undergraduate admissions
  - College teaching
- Affiliated Lecturer
  - Faculty teaching

overlapping at various times.

Basic equations

Conservation of mass and momentum

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \ , \qquad \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p \tag{1}$$

Helmholtz equation

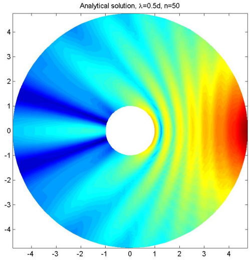
$$\nabla^2 \psi + k^2 \psi = 0 \ . \tag{2}$$

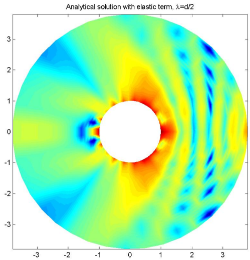
Kirchhoff-Helmholtz equation

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_S \left[\psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial \psi}{\partial n'}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \right] ds' \ . \tag{3}$$

Lighthill acoustic analogy

$$\frac{\partial^2 \rho}{\partial t^2} c_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{4}$$

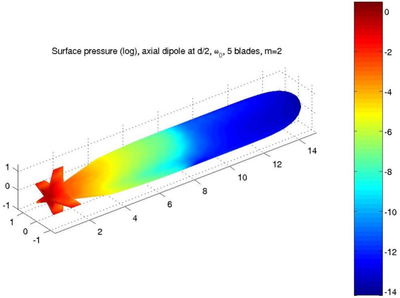




Acoustic pressure on submarine hull

5-bladed propeller

$\omega \sim 2Hz$



What I do now

- Director of Studies in Mathematics
  - College undergraduate admissions
  - College teaching
- Affiliated Lecturer
  - Faculty teaching
- Faculty Admissions Officer
  - University Admissions and Education policy
  - Outreach

i.e. much more Maths!

Acoustic wave scattering by rough surfaces

The unknown field ψ on the surface is expressed as the solution to the Kirchhoff-Helmholtz integral equation. Formally

$$A\psi = \psi_i \ ,$$

where ψ_i is the incident field impinging (say) from the left, so that we require

$$\psi = A^{-1}\psi_i \ .$$

The region of integration is split into two, to the left and right of the point of observation, allowing A to be written as the sum of 'left' and 'right' components:

$$(L + R)\psi = \psi_i$$

The inverse of A can formally be expressed as a series

$$A^{-1} = L^{-1} - L^{-1}RL^{-1} + \dots \tag{5}$$

Application to rough surface with Neumann b.c.

With Neumann boundary condition $\frac{\partial \psi}{\partial n} = 0$ the field at the surface is

$$\psi_i(\mathbf{r}_s) = \psi(\mathbf{r}_s) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} \psi(\mathbf{r}')\, \gamma(\mathbf{r}') dx' dy' \tag{6}$$

so

$$\psi = (L+R)^{-1}\psi_{inc} = \left[L^{-1} - L^{-1}RL^{-1} + \ldots\right]\psi_{inc}. \tag{7}$$

where the right- and left-going operators L and R with respect to the x-direction are (for an L^2 function f):

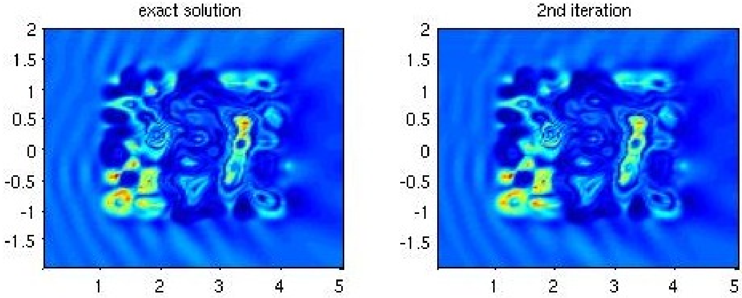
$$Lf(\mathbf{r}) = f - \int_{-\infty}^{\infty}\int_{-\infty}^{x} \frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} f(\mathbf{r}')\, \gamma(\mathbf{r}')\, dx'\, dy', \tag{8}$$

$$Rf(\mathbf{r}) = -\int_{-\infty}^{\infty}\int_{x}^{\infty} \frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} f(\mathbf{r}')\, \gamma(\mathbf{r}')\, dx'\, dy' \tag{9}$$

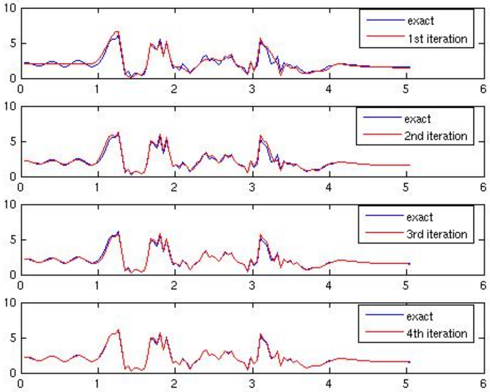
Application to rough surface with Neumann b.c.

Defining the n-th order approximation as

$$\psi_n = \sum_{1}^{n} L^{-1} \left(RL^{-1}\right)^{n-1} \psi_i. \tag{10}$$



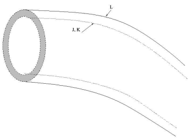
Application to rough surface with Neumann b.c.

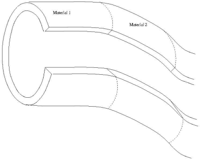


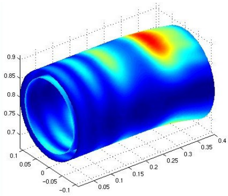
With surface discretized using a rectangular M by N grid, $(L+R)$ becomes an $(MN \times MN)$ matrix, and exact inversion takes $O((MN)^3)$ operations. Evaluation of each term of eq. (7) involves inversion of an $M \times M$ matrix at each of N range steps, $O(NM^3)$ operations and far less memory.

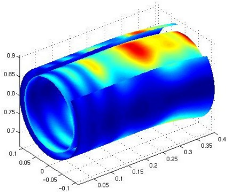
Lined duct

Lined duct as a waveguide of varying cross-section with a layer of dielectric material.









Similar problems arise in acoustics, for example in modelling acoustically lined aeroengine ducts.

Ill-posed problems

- Jacques Hadamard (1865 - 1963)
  a mathematical problem is well-posed if:
  1. a solution exists
  2. the solution is unique
  3. the solution's behaviour changes continuously with the initial conditions
- 'sensible' problem:
  "given the shape of an object, how does it vibrate?"
- 'improper' problem:
  "can we hear the shape of a drum?"

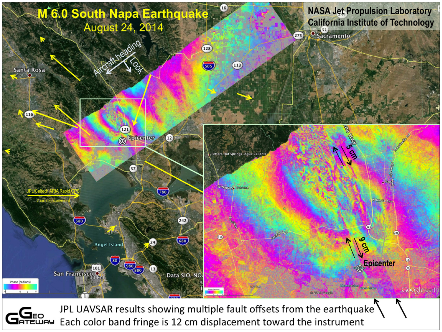
Most problems in real life are ill-posed, and do not satisfy (1), (2), (3). There are rigourous mathematical techniques for dealing with this.

- Moore (1862-1932) and Penrose (1831-)
- Tikhonov (1906-1993)

Ill-posed problems

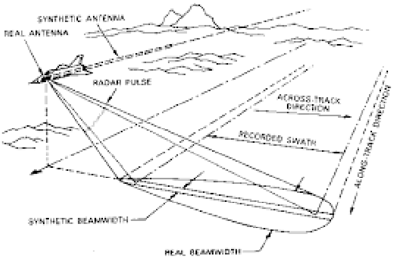
Lined duct as a waveguide of varying cross-section with a layer of dielectric material.

A practical application



SAR (Synthetic Aperture Radar) images of active earthquake faults

- Radar Remote Sensing



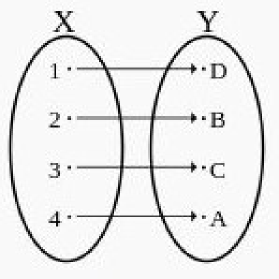
from M J Underhill, IEEE Conference 2015

- radiation sent from antenna on aircraft
- scattered radiation measured by receiving antenna
- shape of earth surface retrieved using mathematical model

Inverse problems - abstract formulation

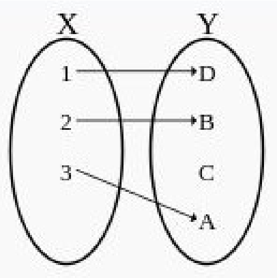
$$Ax = y$$

A = 'operator' = a mapping
that takes an object x from a set X
and maps it to an object y in a set Y



Inverse problems - complications

Where it can all go wrong



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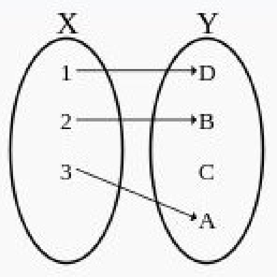

$$x + 3 = y$$
$$X = \mathbb{N}\ ,\ \ Y = \mathbb{N}$$

Inverse problems - complications

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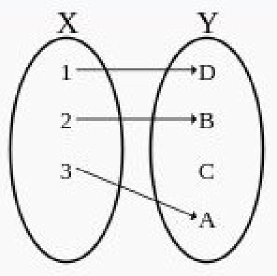

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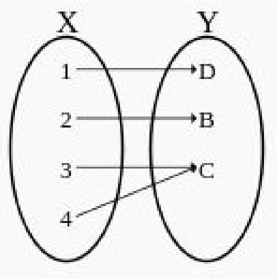

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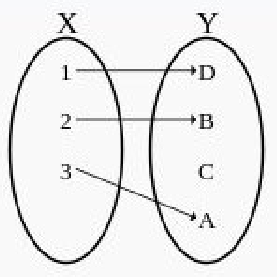
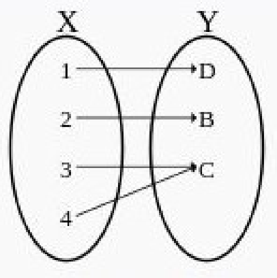




$x + 3 = y$

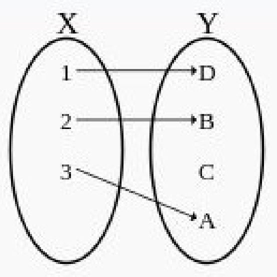
$X = \mathbb{N}\ ,\ \ Y = \mathbb{N}$

$x^2 = y$

$X = \mathbb{R}\ ,\ \ Y = \mathbb{R}^+$

Inverse problems - complications

Where it can all go wrong



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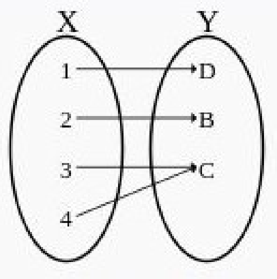
Non-Existence



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Inverse problems - complications

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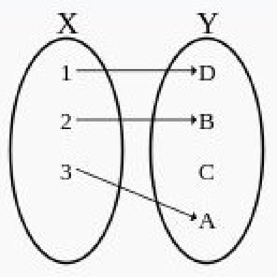

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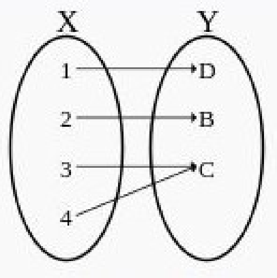

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Non-Uniqueness

Inverse problems - complications

Consider

$$\begin{aligned} ax_1 + bx_2 &= y_1 \\ cx_1 + dx_2 &= y_2 \end{aligned}$$

Define vectors

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Then we can write as matrix equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$Ax = y$$

and the solution is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$x = A^{-1} y$$

Inverse problems - complications

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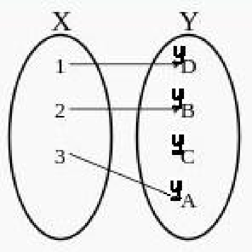
Inverse problems - complications

Measured data always has some error $\longrightarrow$

- inverse problems usually unstable w.r.t. data
- error can also lead to Non-Existence:



if error ε in data $\underline{y}$ is such that

$$\underline{y}_A + \varepsilon = \underline{y}_C$$

The CT scan inverse problem

- for each measurement g_i given by attenuation along the line i :
$$g_i = \sum_{j=1}^{N} \Delta x_{ij} \mu_j,$$
where Δx_{ij} = distance along line i in j^{th} pixel
- matrix equation:

$$\begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_M \end{pmatrix} = A \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \mu_N \end{pmatrix}$$

The CT scan inverse problem

Note that we have $M \neq N$!

So solution may not exist, and we have also many other issues to deal with, related to error in measurements

Problem is ill-posed, like most inverse problems

Not enough time today to see how we deal with ill-posedness!

Thanks!