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Multiple sources of buoyancy in a naturally ventilated enclosure

By P. F. LINDEN[1] AND P. COOPER[2]

[1] Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, UK

[2] Department of Mechanical Engineering, University of Wollongong, Wollongong, NSW 2522, Australia

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This paper presents an approximate model of the flow and stratification within a naturally ventilated enclosure containing multiple sources of buoyancy. The sources are assumed to produce plumes which rise without interaction throughout the enclosure. Buoyant fluid accumulates at the top of the enclosure and flows out through upper level openings, and ambient air flows in at low levels to generate an upward displacement ventilation. It is shown that the sources produce a multiple layered stratification with each plume terminating in a given layer. The weakest plume, with the lowest buoyancy flux, produces the lowest interface, and stronger plumes rise higher up within the space before discharging their buoyant fluid into the environment. The model is approximate in that it ignores the stratification within the space when calculating the properties of each plume, and it is shown that this approximation is satisfactory over a wide range of conditions. As a result it is possible to calculate the stratification and ventilation rate for any number of unequal, and equal, sources of buoyancy within a space.

1. Introduction

The modelling of buoyancy-driven flow within naturally ventilated enclosures raises many complex issues associated with the rise of convective elements and their interaction with the stratification in the space. In an attempt to reduce the problem to its simplest form Linden, Lane-Serff & Smeed (1990) have investigated the case of a single source of buoyancy within an enclosure with openings at the top and bottom. This leads to a steady-state configuration in which the stratification takes the form of two uniform layers separated by a sharp interface, and the transport of air in through the lower openings through the space and out through the upper openings is via the plume. Therefore, the position of the interface is determined by the volume flux within the plume which, in turn, depends on the distance from the source. It was shown that the position of the interface depends only on the openable areas and on the entrainment into the plume, and is not dependent on the strength of the buoyancy flux of the source.

Cooper & Linden (1996) have extended this work to the case of two plumes of unequal strength within an enclosure. They generalized the results for a single plume by assuming that the flow developed a three-layer stratification as illustrated in figure 1.

The two sources of buoyancy are located on the floor of the enclosure and both have positive buoyancy fluxes B_1 and B_2, where $B_1 < B_2$ for definiteness. In this case the

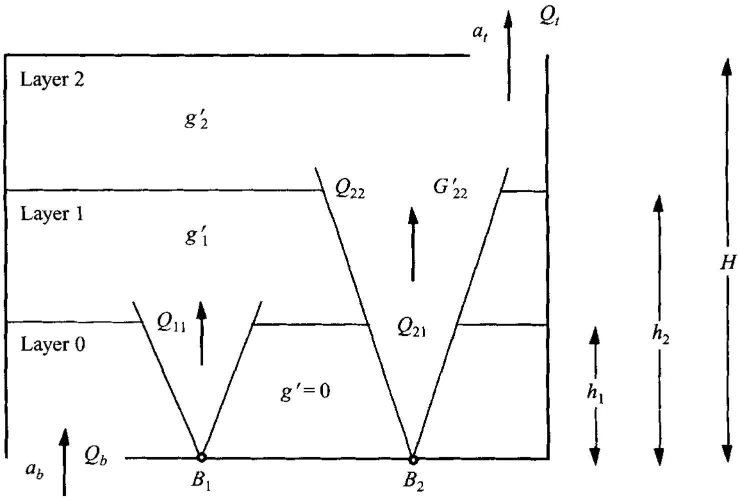

FIGURE 1. Ventilated enclosure with two positive buoyancy sources.

stronger plume rises to the top of the enclosure and produces a buoyant layer underneath the ceiling. The weaker plume cools as it rises and reaches its neutral buoyancy below the upper layer and produces an intermediate layer as shown in figure 1. The lower layer is at ambient temperature $g' = 0$, and the buoyancy of layers 1 and 2 is denoted by g'_1 and g'_2, respectively, with $g'_1 < g'_2$. Thus two interfaces are formed at heights h_1 and h_2 as shown.

The essential features of this model are the following. First, since the interfaces are stable, the only way air can be transported from one layer to another is within the individual plumes. Thus the flowrate through each interface can be directly connected to the volume flux within the two plumes. Secondly, it is assumed that the buoyancy of layer 1 is given by the buoyancy of plume B_1 at the height $z = h_1$, and similarly for the buoyancy g'_2. The buoyancies within layers 1 and 2 are assumed to be uniform with height. Thus ambient air enters through the lower opening and is entrained into the two plumes in layer 0. It is carried across the interface $z = h_1$ by both plumes and the weaker plume discharges into layer 1. This air is entrained into the stronger plume B_2 and carried upwards across the interface $z = h_2$ and discharged into layer 2. Finally, this air flows out through the upper level openings as shown in figure 1.

It is clear, therefore, that the properties of the plume are important in determining the flow and stratification. In order to calculate them we considered plumes arising from point sources of buoyancy and evoked the entrainment assumption of Morton, Taylor & Turner (1956) which states that the entrainment into a plume is proportional to the vertical velocity within the plume at any level. The constant of proportionality is known as the entrainment constant α. Using the conservation of volume and buoyancy fluxes and the plume properties in this manner it is possible to derive a relationship equivalent to that for a single plume. The dimensionless heights of the two interfaces are given by

$$\frac{A^*}{H^2C^{3/2}} = \frac{(1+\psi^{1/3})^{3/2}}{(1+\psi)^{1/2}}\left(\frac{(h_1/H)^5}{1-h_1/H-\dfrac{(1-\psi^{2/3})}{(1+\psi)}\left(\dfrac{h_2-h_1}{H}\right)}\right)^{1/2}, \tag{1}$$

where A^* is the equivalent area of the openings (see Cooper & Linden 1996), $C = \frac{6}{5}\alpha(\frac{9}{10}\alpha)^{1/3}\pi^{2/3}$, and $\psi \equiv B_1/B_2 \leqslant 1$ is the ratio of the buoyancy fluxes. This

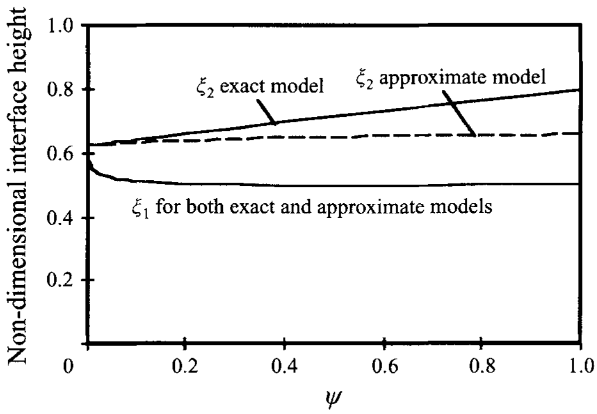

FIGURE 2. Comparison of the exact and approximate theoretical predictions of the non-dimensional interface heights ξ_1 and ξ_2 as functions of the ratio $\psi = B_1/B_2$ of the buoyancy fluxes, for the dimensionless vent area $A^*/H^2 = 0.0167$. The solid curves are the exact solutions given by Cooper & Linden (1996) neglecting entrainment flux Q^* and the dashed curves are the approximate model discussed in §2.

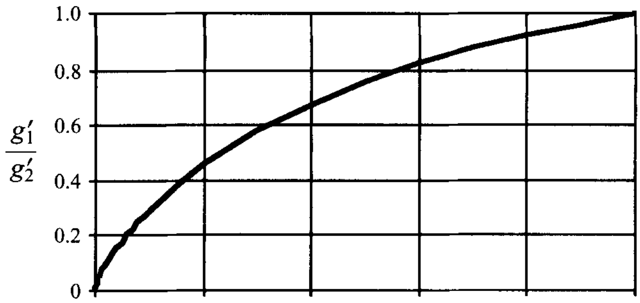

FIGURE 3. Layer buoyancy ratio for any value of A^*/H^2, and total source strengths for two positive plumes.

relationship is of a very similar form to that for a single plume where the interface heights are independent of the total buoyancy fluxes and depend only on the openable areas of the vents, the height of the enclosure and the ratio of the buoyancy fluxes ψ.

In order to complete the solution to this problem it is necessary to find a relationship between h_1 and h_2. The difficulty here is that it is necessary to take account of the fact that once the stronger plume passes the interface at $z = h_1$ its buoyancy flux is changed as a result of the stratification in layer 1. In Cooper & Linden (1996) we showed that it is possible to determine this relationship by calculating the plume B_2 in layer 1 as a forced plume arising from a virtual origin. This computation is somewhat lengthy and is given in detail in Cooper & Linden (1996, §§2.1 and 2.2). The solution for the dimensionless interface heights $\xi_i = h_i/H$, $i = 1, 2$ is shown in figure 2 and the buoyancies of the two layers are shown in figure 3. In order to compare these theoretical results with experiments it was necessary to include the entrainment flux Q^* from the top layer by the weaker plume (see §2.3 of Cooper & Linden 1996). However, for our present purposes since we are concerned with the effects of the internal stratification we will ignore this entrainment, both in the exact and approximate models. Hence, the results shown in figure 2 differ from those shown in figure 4 of Cooper & Linden (1996).

As shown by (1) the interface heights depend only on the dimensionless area A^*/H^2

and the ratio ψ of the buoyancy fluxes. When $\psi = 0$ a single interface forms, and for $\psi > 0$ this interface splits into two with the lower interface ξ_1 descending and the upper interface ξ_2 ascending as ψ increases. As can be seen from figure 3, the buoyancy of the intermediate layer increases, while that of the upper layer decreases as ψ increases, and the two are equal when $\psi = 1$. At this point the interface ξ_2 disappears and the result for two equal plumes given by (3) of Cooper & Linden (1996) with $n = 2$ is obtained.

Thus this solution gives the full range of behaviour for the two-plume case for all possible values of the ratio of buoyancy fluxes $0 \leqslant \psi \leqslant 1$. In principle, the extension of this result to three or more unequal plumes can be carried out in the same way but the calculation of the plumes as they pass through successive interfaces becomes extremely complicated. It requires a re-definition of a virtual origin for each interface and each plume and, consequently, the calculations become unwieldy. From a practical point of view, the most sensitive design criterion is the height of the lower interface ξ_1 and we see from figure 2 that once $\psi > 0.2$ there is very little variation in ξ_1 with further increase in ψ. This is because the positions of the interface are determined primarily by the plume volume fluxes Q where $Q = CB^{1/3}h^{5/3}$. This formula shows that the volume flux in the plume is very sensitive to the interface height but relatively insensitive to the buoyancy flux. It, therefore, seems worthwhile to investigate the possibility of ignoring the changes in buoyancy flux for the stronger plumes as they pass through the interfaces and concentrate only on the volume fluxes, as though each plume were rising in ambient fluid throughout its full height. This approach is found to provide a very accurate description of the behaviour of the lower interface and reasonably good descriptions of the upper interface in the two-plume case as described in §2. The agreement between the exact and approximate calculations in the two-plume case allows us to generalize to multiple plumes in §3, and the three-plume case is discussed in §4. The results and the implications for building design are discussed in §5, and the conclusions are given in §6.

2. The approximate two-plume solution

In this section we repeat the analysis of Cooper & Linden (1996) for the case of two unequal plumes, but with the approximation that the buoyancy flux in the strong plume B_2 remains unchanged as it rises through layer 1 (see figure 1). We also retain the assumption in the approximate theory that the entrainment flux Q^* into layer 1 by the weaker plume is zero for the sake of this comparison. We first write down a series of conservation relations and also the plume properties based on top hat profiles and the entrainment assumption.

The application of Bernoulli's theorem shows that the volume flux through the top and bottom openings is given by

$$Q_t = Q_b = A^*(g_2'(H-h_2)+g_1'(h_2-h_1))^{1/2}. \tag{2}$$

Given the form of the flow shown in figure 1 a number of volume flux, buoyancy flux and density relations may then be identified. Since the only vertical transfer of fluid across the stable interfaces takes place within the plumes, conservation of volume flux implies

$$Q_t = Q_b = Q_{22} = Q_{11}+Q_{21}. \tag{3}$$

[*Note*: the same terminology as used in Cooper & Linden (1996) is used here where the first term of the double subscripts refers to the plume and the second term to the interface, e.g. Q_{21} is the volume flux in plume 2 passing through interface 1 and between layers 1 and 2, see Cooper & Linden (1996) figure 11.]

In a steady state the buoyancy fluxes into and out of each of the layers are equal and hence

$$B_1 + B_2 = Q_{11} G'_{11} + Q_{21} G'_{21} = Q_{22} G'_{22}. \tag{4}$$

In the lower layer (layer 0), in which the density is constant (at the ambient value), the buoyancy fluxes within each plume are constant. For the weaker plume

$$B_1 = G'_1 Q_1 = \text{constant}, \tag{5}$$

and volume flux and reduced gravity at $z = h_1$ are given by

$$Q_{11} = C(B_1 h_1^5)^{1/3}, \tag{6}$$

$$G'_{11} = \frac{1}{C}(B_1^2 h_1^{-5})^{1/3} = g'_1, \tag{7}$$

respectively. As stated above $C = \frac{6}{5}\alpha(\frac{9}{10}\alpha)^{1/3}\pi^{2/3}$, where α is the entrainment constant for the plume. Throughout this paper we take $\alpha = 0.083$, and so $C = 0.11$.

Equivalent relations hold for the strong plume in layer 0:

$$B_2 = G'_2 Q_2 = \text{constant}, \tag{8}$$

$$Q_{21} = C(B_2 h_1^5)^{1/3}, \tag{9}$$

$$G'_{21} = \frac{1}{C}(B_2^2 h_1^{-5})^{1/3}. \tag{10}$$

The strong plume entrains the buoyancy flux from the weaker plume in layer 1, and hence at interface 2, $z = h_2$,

$$Q_{22} = C(B_1 + B_2)^{1/3} h_2^{5/3}, \tag{11}$$

$$G'_{22} = \frac{1}{C}(B_1 + B_2)^{2/3} h_2^{-5/3} = g'_2. \tag{12}$$

In (11) and (12) we have assumed that the stratification in layer 1, g'_1, does not affect the plume. This is the only change to the model presented in Cooper & Linden (1996) but it represents a major simplification to the problem as it is no longer necessary to re-calculate the behaviour of the strong plume in this layer. Now, (3), (6) and (11) imply that

$$\left(\frac{h_2}{h_1}\right)^{5/3} = \frac{1 + \psi^{1/3}}{(1 + \psi)^{1/3}}. \tag{13}$$

Thus, we have a direct relationship between the two interface heights and this may be substituted into (1), and the problem is now solved in this approximate manner. The approximate solution may be written as

$$\frac{A^*}{H^2 C^{3/2}} = \frac{\xi_2^{5/2}}{\left[1 - \xi_2\left(1 - \frac{g'_1}{g'_2}\right) - \xi_1 \frac{g'_1}{g'_2}\right]^{1/2}}, \tag{14}$$

$$\left(\frac{\xi_2}{\xi_1}\right)^{5/3} = \frac{1 + \psi^{1/3}}{(1 + \psi)^{1/3}}, \tag{15}$$

$$\frac{g'_1}{g'_2} = \frac{\psi^{2/3}(1 + \psi^{1/3})}{1 + \psi}, \tag{16}$$

where $\xi_i = hi/H$, $i = 1, 2$ are the dimensionless interface heights.

The results of this approximate solution given by (1) and (14) are shown in figure 2 for comparison with the exact solution. We see that the height of the lower interface

ξ_1 is very accurately determined by this approximate solution over the full range of ψ. The height of the upper interface on the other hand is under-estimated at large values of ψ, and for $\psi = 1$ when $\xi_2/\xi_1 = 2^{2/5}$ the value is in error by approximately 20%. The lower interface is accurately determined because, as mentioned above, it is controlled by the volume fluxes in the plumes and these remain unchanged as they flow through the ambient layer 0. Since the buoyancy flux of the strong plume is reduced as it flows through the buoyant layer 1 its volume flux is also reduced (see (9)), and consequently it rises further before the required volume flux is achieved compared with that which is calculated assuming there is no change in the buoyancy flux as is done in this approximate solution. Therefore, the underestimate in the position of the upper interface is in accord with the approximations used here. The relative buoyancies of the two layers given by (16) is the same as the exact result (see (33) of Cooper & Linden 1996).

The accuracy of this approximate solution, which depends on the volume flux of a plume being relatively sensitive to the height but insensitive to the buoyancy flux, gives confidence in applying this approach to multiple plumes.

3. Multiple plumes

In this section we extend this approximate solution and treat the case of n unequal plumes of buoyancy fluxes $B_1 < B_2 < \ldots < B_n$, and we denote their relative strengths by $\psi_i = B_i/B_n$, $i = 1, \ldots, n-1$. We assume that all the sources of buoyancy are located on the floor of the enclosure and the flow develops into a series of $n+1$ layers separated by interfaces at $z = h_i$. As before we denote the dimensionless interface heights by $\xi_i = z_i/H$, $i = 1, \ldots, n$. This flow is drawn schematically in figure 4.

The pressure balance for the whole enclosure which relates the total flowrate to the internal stratification equivalent to (2) is given by

$$Q_t = Q_b = A^*(g'_n(H-h_n)+g'_{n-1}(h_n-h_{n-1})+\ldots+g'_1(h_2-h)_1)^{1/2}, \tag{17}$$

where the notation is a simple generalization of that in §2 and is illustrated in figure 4.

Extending the notation described in §2 we may now write down the properties of the plumes at each interface within the enclosure.

Interface 1

All n plumes pass through this interface and so the (exact) relationships for this level are with unchanged buoyancy fluxes

$$\left.\begin{aligned} &B_{i1} = B_i, \\ &Q_{i1} = CB_{i1}^{1/3}h_1^{5/3}, \quad G'_{i1} = \frac{1}{C}B_{i1}^{2/3}h_1^{-5/3}, \quad G'_{11} = g'_1, \end{aligned}\right\} \tag{18}$$

for $i = 1, \ldots, n$.

Interface 2

The weakest plume does not reach this interface and we assume that the buoyancy flux B_1 carried by this plume is entrained equally into the remaining $n-1$ plumes so that

$$\left.\begin{aligned} &B_{i2} = B_i + \frac{1}{(n-1)}B_1, \\ &Q_{i2} = CB_{i2}^{1/3}h_2^{5/3}, \quad G'_{i2} = \frac{1}{C}B_{i2}^{2/3}h_2^{-5/3}, \quad G'_{22} = g'_2, \end{aligned}\right\} \tag{19}$$

for $i = 2, \ldots, n$.

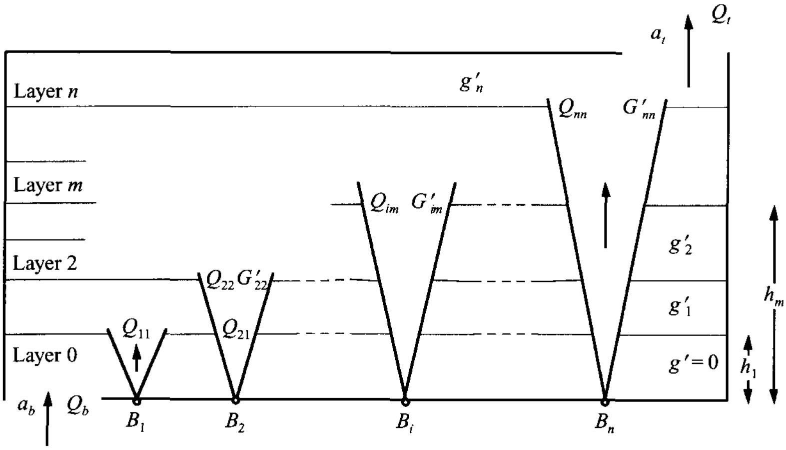

FIGURE 4. Ventilated enclosure with n positive buoyancy sources.

Interface $m < n$

Continuing this procedure for a general interface we have

$$\left.\begin{aligned} B_{im} &= B_i + \frac{1}{(n+1-m)}\sum_{s=1}^{m-1} B_s, \\ Q_{im} &= CB_{im}^{1/3}h_m^{5/3}, \quad G'_{im} = \frac{1}{C}B_{im}^{2/3}h_m^{-5/3}, \quad G'_{mm} = g'_m, \end{aligned}\right\} \tag{20}$$

for $i = m, \ldots, n$.

Interface n

At the top of the enclosure all the buoyancy flux is carried by the strongest plume so that

$$\left.\begin{aligned} B_{nn} &= \sum_{i=1}^{n} B_i, \\ Q_t = Q_{nn} &= CB_{nn}^{1/3}h_n^{5/3}, \quad G'_{nn} = \frac{1}{C}B_{nn}^{2/3}h_n^{-5/3} = g'_n. \end{aligned}\right\} \tag{21}$$

Again, assuming that the transport across each interface takes place only in the plumes, we can write down the following relationships for the volume flowrates across each interface

$$Q_t = Q_b = Q_{nn} = \sum_{i=m}^{n} Q_{im} \quad (m = 1, \ldots, n). \tag{22}$$

The pressure balance (17), the plume relations (20) and the volume conservation relation (22) now give

$$\frac{A^*}{H^2C^{3/2}} = \frac{\xi_n^{5/2}}{\left[1 - \sum_{m=1}^{n} \Gamma_m \xi_m\right]^{1/2}}, \tag{23}$$

where
$$\Gamma_m = \frac{g'_m - g'_{m-1}}{g'_n} \quad (m = 1, \ldots, n, \quad g'_0 = 0), \tag{24}$$

$$\frac{g'_m}{g'_n} = \left(\frac{B_{mm}}{B_{nn}}\right)^{2/3}\left(\frac{\xi_n}{\xi_m}\right)^{5/3}. \tag{25}$$

The interface heights are related by

$$\left(\frac{\xi_m}{\xi_1}\right)^{5/3}=\frac{\sum_{i=1}^{n} B_{i1}^{1/3}}{\sum_{i=m}^{n} B_{im}^{1/3}}. \tag{26}$$

Equations (23)–(26) give the full solution to the approximate problem. We see that the interface heights are again dependent only on the dimensionless vent area A^*/H^2 and the ratios of the buoyancy fluxes. Consequently, the general result obtained for the special cases of one and two plumes, that the positions of the interfaces are independent of the total buoyancy from the sources, continues to apply. The buoyancies of the layers, however, do depend on the total buoyancy flux. We now explore some special cases of these solutions in §4.

4. Three plumes

For the case of 3 plumes $B_1 < B_2 < B_3$, (20) implies that

$$B_{11}=B_1,\quad B_{21}=B_2,\quad B_{31}=B_3,$$
$$B_{22}=B_2+\tfrac{1}{2}B_1,\quad B_{32}=B_3+\tfrac{1}{2}B_1,$$

and

$$B_{33}=B_1+B_2+B_3. \tag{27}$$

Thus (26) gives the relative interface heights as

$$\left(\frac{\xi_2}{\xi_1}\right)^{5/3}=\frac{1+\psi_1^{1/3}+\psi_2^{1/3}}{(1+\frac{1}{2}\psi_1)^{1/3}+(\frac{1}{2}\psi_1+\psi_2)^{1/3}}, \tag{28}$$

$$\left(\frac{\xi_3}{\xi_1}\right)^{5/3}=\frac{1+\psi_1^{1/3}+\psi_2^{1/3}}{(1+\psi_1+\psi_2)^{1/3}}, \tag{29}$$

where $\psi_i = B_i/B_3$, $i = 1,2$. The relative buoyancies of the layers (25) are

$$\frac{g_1'}{g_3'}=\frac{\psi_1^{2/3}(1+\psi_1^{1/3}+\psi_2^{1/3})}{1+\psi_1+\psi_2}, \tag{30}$$

$$\frac{g_2'}{g_3'}=\frac{(\psi_2+\frac{1}{2}\psi_1)^{2/3}[(1+\frac{1}{2}\psi_1)^{1/3}+(\frac{1}{2}\psi_1+\psi_2)^{1/3}]}{(1+\psi_1+\psi_2)}. \tag{31}$$

The interface height is then given by (23) and (24) which in this case reduce to

$$\frac{A^*}{H^2C^{3/2}}=\frac{\xi_3^{5/2}}{[1-\Gamma_1\xi_1-\Gamma_2\xi_2-\Gamma_3\xi_3]^{1/2}}, \tag{32}$$

where

$$\Gamma_1=\frac{g_1'}{g_3'},\quad \Gamma_2=\frac{g_2'-g_1'}{g_3'},\quad \Gamma_3=1-\frac{g_2'}{g_3'}. \tag{33}$$

We now examine a number of special cases.

4.1. $B_1 = B_2 = B_3$: *three equal plumes*

When the strengths of the plumes are equal $\psi_1 = 1$, $\psi_2 = 1$ and from (28)–(31) we have

$$\frac{\xi_2}{\xi_1}=\left(\frac{3}{2}\right)^{2/5},\quad \frac{\xi_3}{\xi_1}=3^{2/5}, \tag{34}$$

$$\frac{g_1'}{g_3'}=\frac{g_2'}{g_3'}=1. \tag{35}$$

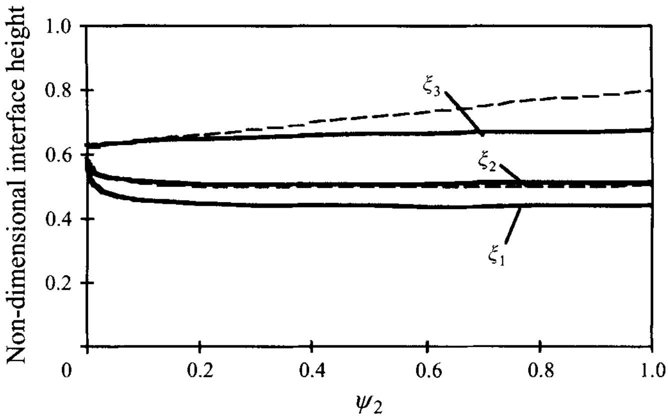

FIGURE 5. Dimensionless interface heights ξ_1, ξ_2 and ξ_3 (solid curves), for the approximate solution for three plumes with $B_1 = \frac{1}{2}B_2$, plotted against the buoyancy flux ratio B_2/B_3. The dashed curves are the two-plume solution for the same dimensionless vent area $A^*/H^2 = 0.0167$.

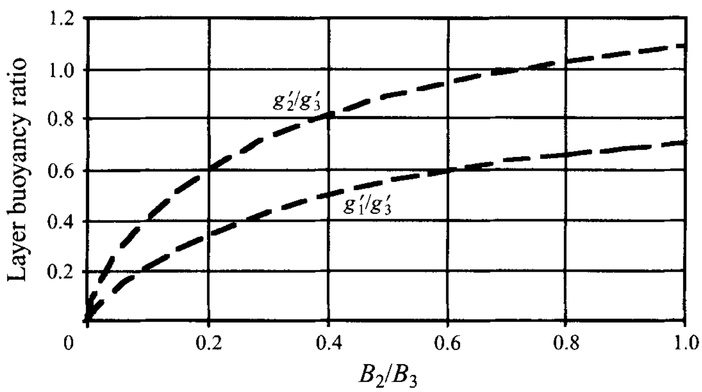

FIGURE 6. Ratio of layer buoyancies for the three-plume case shown in figure 5.

Thus the densities of layers 1, 2 and 3 are equal and the stratification reverts to a simple two-layer form, with the upper layer at a uniform density. The height of the remaining interface ξ_1 is given by (32) and (34) with $\Gamma_1 = 1$ and $\Gamma_2 = \Gamma_3 = 0$. In this limit (32) reduces to

$$\frac{A^*}{H^2 C^{3/2}} = \frac{3\xi_1^{5/3}}{[1-\xi_1]^{1/2}},$$

which recovers the required result (3) of Cooper & Linden (1996) with $n = 3$.

4.2. $B_1 = \frac{1}{2}B_2$

The properties of the system may be determined for any values of the buoyancy fluxes and so, by way of an example, we choose the case $B_1 = \frac{1}{2}B_2 < B_3$. The results for the heights of the three interfaces are plotted against $\psi_2 = B_2/B_3$ in figure 5, and the corresponding buoyancies are shown in figure 6. The value of the dimensionless area $A^*/H^2 = 0.0167$ as in figures 2 and 3 for the two-plume case, and the results for the interface heights in the latter case are shown as solid curves in figure 5.

We see that the presence of a third plume is to lower the height of the ambient layer 0. This reduction is caused by the increased volume flux carried by three plumes and,

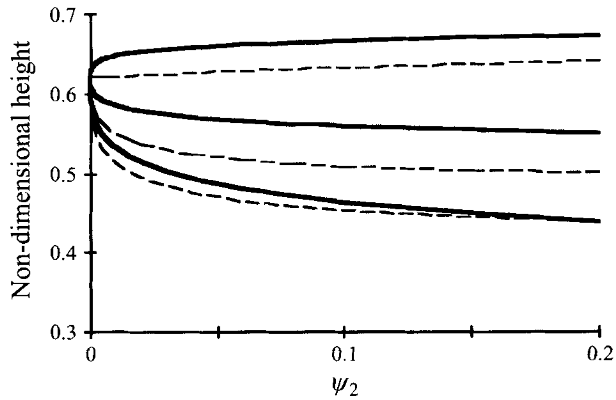

FIGURE 7. Comparison of the approximate solutions (solid curves) with the asymptotic solution (dashed curves) for the case $B_1 < B_2 \ll B_3$, plotted against ψ_2.

hence, the interface is lowered in order to compensate for this flux increase. As ψ_2 increases interfaces 1 and 2 decrease in height while interface 3 rises within the enclosure. The total width of the stratified zone as measured by the distances between the top and bottom interfaces is larger at small ψ_2 in the three-plume case, but is less as $\psi_2 \rightarrow 1$. However, the comparison between the exact and the approximate solutions in the two-plume case (§2) suggests that the interface heights for interfaces 2 and 3 will be underpredicted in this limit. The corresponding ratios of the layer buoyancies are shown in figure 6. The strength of the stratification $(g_2'-g_1')$ across the lowest interface increases with B_2/B_3, while the buoyancies of layers 2 and 3 approach the same value as $B_2/B_3 \approx 0.7$. For values of the flux ratio larger than this value, the approximate model predicts $g_2' > g_3'$ which is an unstable stratification. However, this is not a physical effect but a result of inaccuracies caused by ignoring the detailed dynamics of the plumes.

4.3. $B_1 < B_2 \ll B_3$

The results of §4.2 show that most of the variation in the height ξ_1 of the lowest interface occurs for $\psi_2 < 0.2$. A similar result was found in the two-plume case. Hence, we can further approximate the flow by considering the solution in the limit $\psi_1 < \psi_2 \ll 1$. In this case there is one strong plume and two weak ones, and (29)–(31) reduce to

$$\left(\frac{\xi_2}{\xi_1}\right)^{5/3} \approx 1+\psi_1^{1/3}, \quad \left(\frac{\xi_3}{\xi_1}\right)^{5/3} \approx 1+\psi_1^{1/3}+\psi_2^{1/3},$$

$$\frac{g_1'}{g_3'} \approx \psi_1^{2/3}, \quad \frac{g_2'}{g_3'} \approx \psi_2^{2/3}. \tag{36}$$

The height of the lowest interface is given by

$$\frac{\xi_1^{5/2}}{(1-\xi_1)^{1/2}} = (1+\psi_1^{1/3}+\psi_2^{1/3})^{-3/2}\frac{A^*}{H^2C^{3/2}}. \tag{37}$$

This asymptotic solution is plotted against the full solution for $\psi_1 = \frac{1}{2}\psi_2$, $A^*/H^2 = 0.0167$ over the range $0 \leqslant \psi_2 \leqslant 0.2$ in figure 7. We see that the agreement is quite good over this range of buoyancy fluxes, and hence that (36) and (37) give useful approximate forms for the interface heights and the stratification. From the results shown in figure 4, we conclude that the height ξ_1 of the lowest interface remains fairly constant with further increase in ψ_2, and hence (37) provides a good approximation to the solution. In particular, we note that adding two plumes of strengths both equal to 10% of the strongest plume (i.e. $\psi_1 = \psi_2 = 0.1$) is equivalent to reducing the dimensionless area by a factor of 2.7. Thus the distribution of the buoyancy flux among a number of plumes has a large influence on the flow.

5. Discussion

We return now to the case of multiple plumes and, in the light of the results obtained for three plumes in §4, we consider some general implications for the flow and stratification within the enclosure.

Consider first the case where the n plumes have equal strength $B_i = B$, $i = 1, \ldots, n$. Then from (20) and (26) we have that

$$\frac{\xi_m}{\xi_1} = \left(\frac{n}{n+1-m}\right)^{2/5} \quad (m = 1, \ldots, n). \tag{38}$$

Substituting this result into (25) we find

$$\frac{g'_m}{g'_n} = 1 \quad (m = 1, \ldots, n) \tag{39}$$

and hence a two-layer stratification occurs in this case also. From (24) we see that

$$\Gamma_1 = 0, \quad \Gamma_m = 0 \quad (m = 2, \ldots, n),$$

and so, using (38) in (23), we recover the result (3) from Cooper & Linden (1996) for n equal plumes. Thus, although the solution given in §3 is approximate we expect the height of the lowest interface to be predicted accurately over the whole range of $\psi_i \leqslant 1$. The heights of the higher interfaces are likely to be underpredicted by the model.

The solutions for the two- and three-plume cases showed that accurate estimates of the lower interface were predicted by considering the asymptotic forms of the flow when $\psi_i \ll 1$, $i = 1, \ldots, n-1$. The results given in §4.3 may be extended to the multiple plume case in this limit and we find

$$\left.\begin{aligned} \left(\frac{\xi_m}{\xi_1}\right)^{5/3} &\approx 1 + \psi_1^{1/3} + \ldots + \psi_{m-1}^{1/3}, \\ \frac{g'_m}{g'_n} &\approx \psi_m^{2/3} \quad (m = 1, \ldots, n-1). \end{aligned}\right\} \tag{40}$$

The case of $n-1$ equal plumes with buoyancy flux $B_i = B$, $i = 1, \ldots, n-1$ and one strong plume B_n such that $\psi_i = B/B_n \ll 1$, $i = 1, \ldots, n-1$ may be calculated explicitly. In this case (20) gives

$$\left.\begin{aligned} B_{im} &= \frac{n}{n+1-m} B \quad (i-1, \ldots, n-1), \\ B_{nm} &= B_n + \frac{m-1}{n+1-m} B. \end{aligned}\right\} \tag{41}$$

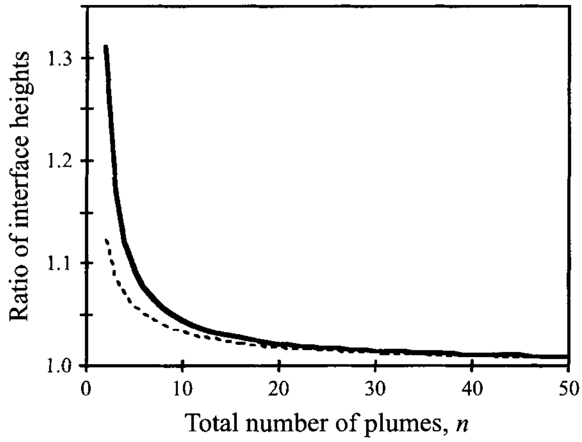

FIGURE 8. The ratio of interface heights ξ_2/ξ_1 plotted against n for the case of $(n-1)$ equal plumes of strength $B_i = B$, and one strong plume of strength B_n, such that $\psi_i = B/B_n \ll 1$, $i = 1, \ldots, n-1$. The dashed curve is for $\psi = 0.01$, and the solid curve is for $\psi = 0.5$.

A three-layer stratification will form and from (26) the ratio of the interface heights is given by

$$\left(\frac{\xi_2}{\xi_1}\right)^{5/3} = \frac{1+(n-1)\,\psi^{1/3}}{\left[1+\left(\frac{1}{n-1}\right)\psi\right]^{1/3}+(n-2)\left(\frac{n}{n-1}\right)^{1/3}\psi^{1/3}}. \tag{42}$$

Figure 8 shows plots of ξ_2/ξ_1 against the number n of weak plumes for two values $\psi = 0.01$ and 0.5. When $\xi_2/\xi_1 \to 1$ the effect of the strong plume is lost and the ventilation behaves as though it is driven by the $n-1$ weak plumes alone, producing a two-layer stratification. We see that the heights of the two interfaces approach each other as the number of plumes increases, consistent with the limit of (42) that $\xi_2/\xi_1 \to 1$ as $n \to \infty$. For the case of $\psi = 0.01$, the total strength of the weak plumes is half the strong plume, and the two interfaces are very close. Indeed when $n = 10$, and the weak plumes only provide 10% of the buoyancy into the enclosure, $\xi_2/\xi_1 = 1.03$, and a two-layer structure is a good approximation to the stratification within the space. Even when $\psi = 0.5$ and $n = 10$ and the weak plumes in total provide 5 times the buoyancy of the strong plume, $\xi_2/\xi_1 = 1.04$. This figure emphasizes that when there are multiple sources, the height of the ambient zone is well represented by n equal sources independent of the distribution of buoyancy among the sources. However, the strength of the stratification above this level is determined by the strength of the sources.

The strength of the stratification above the ambient zone may be calculated as follows. Consider the case of n plumes where

$$\left.\begin{aligned} B_i &= \frac{i\beta}{n}B \quad (i = 1, \ldots, n-1, \quad \beta = \text{constant} \ll 1), \\ B_n &= B. \end{aligned}\right\} \tag{43}$$

Then $\psi_i = i\beta/n$, $i = 1, \ldots, n-1$, $\psi_n = 1$, and the buoyancy flux carried by plume i across interface m is

$$\left.\begin{aligned} B_{im} &= \frac{\beta}{n}B\left[i+\frac{m(m-1)}{2(n+1-m)}\right] \quad (i = 1, \ldots, n-1), \\ B_{nm} &= B\left[1+\frac{\beta}{n}\frac{m(m-1)}{2(n+1-m)}\right], \end{aligned}\right\} \tag{44}$$

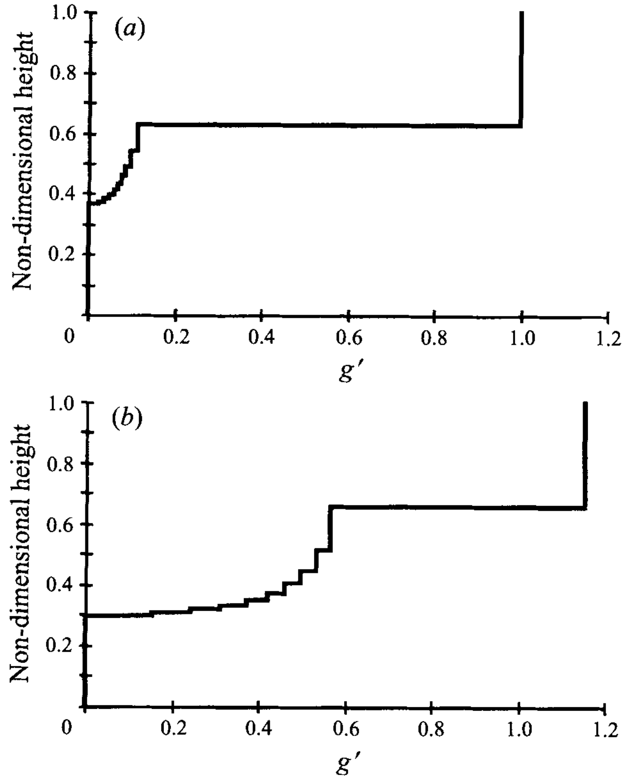

FIGURE 9. The stratification produced by $n = 10$ plumes, with strengths given by the arithmetic progression, plotted against the dimensionless height Z within the enclosure. The strength B of the strongest plume is 20 kW, the dimensionless vent area $A^*/H^2 = 0.0167$ and $H = 5$ m. In (*a*) $\beta = 0.01$, and in (*b*) $\beta = 0.1$. Note that $g' = 1\ \mathrm{m\,s^{-2}}$ corresponds to a temperature difference of about 30 °C.

and the total buoyancy flux through the enclosure is

$$B_{nn} = B\left(1 + \frac{\beta(n-1)}{2}\right). \tag{45}$$

A measure of the stratification at interface m is the buoyancy frequency

$$N_m^2 = \frac{g'_m - g'_{m-1}}{h_m - h_{m-1}}. \tag{46}$$

Substitution of (23)–(26) into (46) and use of (21) gives, after considerable algebra,

$$N_m^2 = \frac{C^{8/15}}{H^{8/3}}\left(\frac{H^2}{A^*}\right)^{16/15}\left\{\frac{B_{mm}^{2/3}}{\left(\dfrac{\xi_m}{\xi_1}\right)^{5/3}} - \frac{B_{m-1\,m-1}^{2/3}}{\left(\dfrac{\xi_{m-1}}{\xi_1}\right)^{5/3}}\right\}\left(\frac{\xi_n}{\xi_1}\right)^{5/3}$$
$$\times\left(\frac{\xi_m}{\xi_1} - \frac{\xi_{m-1}}{\xi_1}\right)^{-1}\left[1 - \sum_{m=1}^{n} \Gamma_m \xi_m\right]^{-8/15}. \tag{47}$$

Substitution of (44) and (45) into (47) allows the stratification to be calculated.

Figures 9(*a*) and 9(*b*) show the stratification within the enclosure for $n = 10$ plumes for the cases where $\beta = 0.01$ and $\beta = 0.1$, respectively, calculated from the exact solution using (21) and (23)–(26). The dimensionless vent area $A^*/H^2 = 0.0167$ and the height H of the enclosure is 5 m. The buoyancy flux B of the strongest plume in this case is 20 kW, and when $\beta = 0.01$ (0.1) the total buoyancy flux into the enclosure is 20.9 (29) kW. We see that there is an unstratified ambient zone and then there is a strong stratification immediately above this zone. This stratification results from the large change in buoyancy across the lowest interface at $z = \zeta_1$. Above this the stratification decreases with height as a result of the smaller changes in buoyancy across the higher interfaces until a large step in density occurs at ξ_n where the strongest plume enters the upper layer which is assumed to be well-mixed.

It is useful to examine the asymptotic form of the stratification on the assumption that $\beta \ll 1$, so that $\psi_i \ll 1$, $i = 1, \ldots, n-1$. Then, using the limiting forms given in (40) and (41) we find ignoring $O(1)$ constants

$$N_m^2 \approx \frac{\beta^{1/3} n^{11/15}}{m^{1/3}} \frac{B^{2/3}}{H^{8/3}} \left(\frac{H^2}{A^*}\right)^{16/15}. \tag{48}$$

Thus, we see from (48) that the strength of the stratification as characterized by the vertical density gradient (proportional to N^2) is proportional to the buoyancy flux as $B^{2/3}$, and inversely proportional to the height of the space as $H^{-8/3}$. This dependence is a result of the dilution of the warm air in the individual plumes as indicated by (21). The strength of the stratification decreases as the dimensionless vent area increases, and the relationship is almost linear.

The feature of the flow that is sensitive to the number of sources is the depth of the ambient zone. From the results shown in figures 2 and 4 we expect this depth to be relatively insensitive to the specific values of ψ_i and a good estimate is given by the case when all the plumes have equal strengths. In this case the exact result is given by (3) of Cooper & Linden (1996) and when ξ is small this equation implies $\xi \sim n^{-2/5}$. Thus, distributing the buoyancy flux from a single source into 10 equal sources would reduce the height of the ambient zone by a factor of 2.5, and distributing it into 100 sources would reduce it by a factor of 6.3. In the limit $n \to \infty$, $\xi \to 0$ and there is no ambient zone within the enclosure.

The theoretical analysis described above has relevance to the design of natural ventilation systems in buildings and prediction of air and contaminant movement in large, naturally ventilated spaces. Many practical situations can be identified where multiple sources of buoyancy will be present in an enclosure, including the following examples. Large, multi-storey, glazed spaces in buildings, or atria, may have substantial solar gains that heat surfaces such as floors and walls, which, in turn, heat adjacent air that rises causing significant stratification within the space (Cooper 1993). Often there are several distinct hot objects and surfaces acting as separate sources of buoyancy in such spaces resulting in complex fluid flow and thermal behaviour. Industrial buildings are often naturally ventilated and contain multiple sources of hot gases and fumes which must be controlled to prevent exposure of occupants to unacceptable contaminant concentrations. In event of fire, considerations of smoke clearance from buildings by natural ventilation is important in the design of openable ventilation areas in atria and other large spaces. Current design guidelines are generally based on a single source of heat and smoke in the space. The results described above show that the height of a smoke-free zone in a naturally ventilated space will decrease in the event of two (or more) fires being present. However, the presence of other thermal plumes will confine the smoke from a strong fire plume to a thinner region near the ceiling.

The results shown in figure 3 show that the position of the lower interface (which is usually the most critical as regards control of contaminants such as heat, fumes or smoke) in an enclosure with multiple sources of buoyancy of the same sign is well predicted by the relation for n equal plumes for situations where $\psi_i > 0.2$. Thus, for many design purposes the assumption that multiple buoyancy sources have the same strength may be adequate in the determination of ventilation opening sizes *vis-à-vis* the thickness of the uncontaminated air layer above floor level.

6. Conclusions

In this paper we have presented an approximate model for the stratification and flow through an enclosure in which there are a number of sources of buoyancy. The enclosure is connected to an exterior ambient by openings at high and low levels so that a displacement ventilation flow is established with the buoyant fluid leaving at the upper level openings and ambient fluid entering at the lower level openings. The sources of buoyancy are considered to cause plumes which rise through the enclosure without interaction. The model assumes that, in the steady state, the interior stratification that results consists of a set of layers each of uniform density separated by sharp horizontal interfaces. The weakest plume terminates at the lowest interface and stronger plumes terminate at higher interfaces, with the strongest plume reaching the top of the enclosure. The model is approximate in the sense that the interior stratification is ignored when the dynamics of the plumes are calculated.

For the case of two plumes the approximate model is compared with the exact solution. It is found that the position of the lower interface is given accurately by the approximate model, but the height of the upper interface is under-predicted. This underestimate results from neglecting the change in the buoyancy flux in the stronger plume when it enters the buoyant layer above the lower interface. The model is then extended to n plumes and detailed calculations are given for the case $n = 3$. It is found that the behaviour in these more complex flows has the same general character as the two-plume case. In particular, the positions of the interfaces are found to be independent of the total buoyancy flux into the enclosure, and to depend only on the ratio of the fluxes from the plumes. Also it is observed that the position of the lower interface is relatively insensitive to the detailed fluxes and, to a good approximation, is given by the predicted value when all n plumes have equal strengths.

The form of the stratification within the space is found to be considerably more complex when there are multiple sources of buoyancy of different strengths. Immediately above the ambient layer there is a region of strong stratification and above that the form of the stratification depends on the buoyancy fluxes of the plumes. The main idea behind this approximate solution for the multiple plume case is that the stratification and flow are controlled by the entrainment into the plumes. The volume flux in the plumes is very strongly dependent on the height and only very weakly dependent on the buoyancy flux of the plume. Thus we are able to neglect changes in the plume buoyancy flux to obtain the approximate solution. We find, as in the simpler single- and two-plume cases, that the height of the ambient zone is independent of the total buoyancy flux into the enclosure.

From a design viewpoint, the height of the lowest interface is the critical parameter. These calculations show that this interface is well approximated by the n equal-plume result for $\psi_{n-1} > 0.2$. Below these values the stratification is dominated by the strong plume and the weaker plumes have only a minor effect. In particular, they do not entrain significant amounts of hot upper-layer fluid downwards. The comparison with Cooper & Linden (1996), in the case where the entrainment flux $Q^* = 0$, is a valid test of the approximate model in those cases. Entrainment is significant if ψ_{n-1} is large, but as stated above, in that case the lowest interface is determined by the n equal-plume value. We believe that this estimate will be adequate for many cases of practical interest. Therefore, in determining a design solution to a ventilation problem it is necessary to determine the number of significant sources of buoyancy in any particular case.

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