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Similarity considerations for non-Boussinesq plumes in an unstratified environment

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We develop herein the similarity form of a non-Boussinesq gaseous plume. This is done by obtaining the equation for the conservation of enthalpy flux, and using it and the continuity equation to demonstrate that the flux of density deficiency is conserved, and not the flux of buoyancy as is the case for Boussinesq plumes. We then use this conservation relation to describe the form of the similarity solution in the non-Boussinesq case. The similarity solution is then used to derive the theoretical form of the entrainment velocity across the plume edge, which is seen to be in agreement with the 'modified entrainment assumption' suggested empirically from experiments by Ricou & Spalding (1961).

1. Introduction

A plume is defined as a flow that arises due to free convection from an isolated source of buoyancy. In a stationary ambient the flow is vertical in the mean, and can be laminar, or turbulent, or undergo transition to turbulence at some height. The vertical flow in the plume induces a secondary inward flow of ambient fluid towards the plume centreline.

In turbulent plumes, the ambient fluid is mixed across the plume edge and becomes incorporated into the body of the plume. This process is called turbulent entrainment and has the effect of increasing the plume volume flux and increasing (or decreasing) the plume density. The mechanism of turbulent entrainment, whereby ambient fluid is incorporated into the plume and attains vorticity and buoyancy, is poorly understood. Details of this process depend on transfers of mass and momentum at small scales which are impossible to compute. Instead the process is usually parameterized by relating the inflow velocity to the mean flow in the plume.

In the Boussinesq case, where the mean plume density is comparable to the ambient density, self-similarity of the plume in an unstratified ambient implies that the mean velocity of the entrained ambient fluid across the plume edge is proportional to the mean vertical velocity in the plume. However, there are many instances where plumes are non-Boussinesq, for example a small fire may reach temperatures of around 300°C, producing a plume above it with an initial density approximately half that of the ambient. It has been observed (Ricou & Spalding 1961) that in the non-Boussinesq case, where the initial density difference between the plume and the ambient is large, there is also a dependence of the entrainment velocity on the ratio of the plume density to the ambient density. Morton (1965) has interpreted these observations as an additional proportionality of the entrainment velocity to the square root of the density ratio of the plume and ambient fluids.

Plume theory has mainly been developed in the Boussinesq regime and non-Boussinesq plume properties have in the past been grafted onto existing theory when required in a rather *ad hoc* manner. Here we present a mathematical formulation of plumes without making the Boussinesq approximation, in order to provide a coherent description of non-Boussinesq plumes. This formulation also highlights the position of Boussinesq theory as a limiting, simplified case of the more general plume problem.

2. Boussinesq plumes

The initial work on isolated sources of buoyancy was performed by Batchelor (1954), and Morton, Taylor & Turner (1956, hereafter referred to as MTT), all of whom were concerned with describing isolated convection in a meteorological context. In such a context it is reasonable to make the Boussinesq approximation, in which variations in density are neglected except where they are responsible for the presence of the buoyancy force. (For a detailed description of the Boussinesq approximation, see Spiegel & Veronis 1960.)

In this section we proceed by outlining the main points of plume theory, following MTT and others.

In flows where density fluctuations have dynamical implications, they are incorporated into the momentum equations by means of the reduced gravity, given by $g' = g\rho'/\rho$, where ρ' is the density fluctuation from the basic state with density ρ_0, such that

$$\rho = \rho_0 + \rho'.$$

In the Boussinesq case, where $\rho \sim \rho_0$, the reduced gravity approximates to $g\rho'/\rho_0$.

For an axisymmetric plume we use a cylindrical coordinate system (r, θ, z), with the z-axis vertically along the axis of the plume. The time-averaged cross-plume profiles of vertical velocity $w(r,z)$ and density difference $\rho'(r,z)$ are observed to take the form of approximate Gaussian distributions with roughly equal widths (Turner 1979). The description of the plume is simplified by taking the cross-plume average of the relevant plume variables. In the Boussinesq plume, we may define the plume radius b, and averaged values of the reduced gravity $\bar{g}'$ and the vertical velocity $\bar{w}$ (as functions of height) by three relations. Following Turner (1979), these may be given by

$$\bar{w}b^2(z) = \int_0^\infty w\, r\, \mathrm{d}r\,,$$

$$\bar{w}^2b^2(z) = \int_0^\infty w^2\, r\, \mathrm{d}r\,,$$

$$\bar{g}'\bar{w}b^2(z) = \int_0^\infty g' w\, r\, \mathrm{d}r\,.$$

These averaged variables are known as 'top-hat' variables. In the rest of this section we shall consider only top-hat variables and so drop the overbar.

2.1. *Similarity theory*

The specific buoyancy flux in a plume is given by the product of the volume flux and reduced gravity of the plume. If one assumes that the buoyancy flux is conserved with height (an assumption which will later be shown to be correct for an unstratified ambient) and that the plume arises from a source of small dimension, then one can construct a similarity solution for w, ρ, and g' from the buoyancy flux B, and the height z. For a geometrically self-similar flow, the width of the plume is proportional to the distance from the source and dimensional analysis implies the following relationships for the (top hat) plume properties (see MTT):

$$w \propto B^{1/3} z^{-1/3}, \quad b \propto z, \quad g' \propto B^{2/3} z^{-5/3}. \tag{1}$$

2.2. *Basic equations*

The plumes we wish to describe are turbulent but, within certain limitations, can be adequately described by the mean flow variables alone. The momentum equations and the continuity equation for steady, axisymmetric flow with the hydrostatic approximation are

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial r}, \tag{2}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} - g\frac{\rho'}{\rho}, \tag{3}$$

$$\frac{\partial}{\partial r}(ru\rho) + \frac{\partial}{\partial z}(rw\rho) = 0, \tag{4}$$

where (ρ', p') are perturbation density and pressure fields from the hydrostatically balanced basic state (ρ_0, p_0) which satisfies

$$\frac{1}{\rho_0}\nabla p_0 = \boldsymbol{g}.$$

In the Boussinesq approximation we may assume that the fluid is incompressible, and so the continuity equation (4) may be written more simply as

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0. \tag{5}$$

2.3. *Plume equations*

In terms of top-hat variables, the equations of mean motion can be obtained from the basic equations by integrating across the plume. In conjunction with the Boussinesq approximation, this yields (MTT)

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(b^2 w) &= -ru|_{r=\infty}, && \text{volume flux}, \\ \frac{\mathrm{d}}{\mathrm{d}z}(b^2 w^2) &= b^2 g', && \text{momentum flux}, \\ \frac{\mathrm{d}}{\mathrm{d}z}(b^2 w g') &= -b^2 w \frac{\mathrm{d}\rho_0}{\mathrm{d}z}, && \text{buoyancy flux}, \end{aligned}\right\} \tag{6}$$

where ρ_0 is the density of the ambient fluid.

It can be seen that for an unstratified ambient we have $\mathrm{d}\rho_0/\mathrm{d}z = 0$, so in this case the buoyancy flux $B = b^2 w g'$ is conserved, as was assumed previously.

2.4. *Entrainment assumption*

The volume flux equation (in (6)) is obtained by integrating the incompressible continuity equation (5) radially, and indicates that the increase in plume volume flux is supplied by a radial influx from the far field. This influx from infinity clearly implies a flow across the plume boundary b. The velocity of fluid across the plume boundary, which is termed the entrainment velocity u_e, is then given by

$$bu_e = -ru|_\infty \, .$$

Thus the volume flux equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}(b^2 w) = bu_e \, .$$

For a self-similar Boussinesq plume, (1) imply that the rate of increase of the volume flux $\mathrm{d}/\mathrm{d}z(b^2 w) \sim z^{2/3}$, and we have that this increase in volume flux is supplied by entrainment with velocity u_e around the perimeter of the plume, radius $b\,(\propto z)$. Thus in the Boussinesq case the vertical velocity, w, and the horizontal inflow velocity across the plume edge, u_e, have the same height dependence ($\propto z^{-1/3}$). Thus, the equations of motion are consistent with similarity theory if $w \propto u_e$. This relationship is the basis of the *entrainment assumption* introduced by MTT, which states that the velocity of the entrained fluid across the plume edge is proportional to the plume velocity,

$$u_e = \alpha w \, ,$$

where α is a constant known as the entrainment constant, which has been estimated to be approximately 0.1 (Turner 1979). The similarity solution then satisfies the plume equations (6).

3. Non-Boussinesq plumes

3.1. *Mixing in the non-Boussinesq case*

For a mixture of two fluids of different densities, the commonest and easiest assumption to make is that of 'linear mixing', i.e. the density of the mixture is a volume-weighted average of the densities of the components. This is the assumption that will be used throughout the rest of this paper. However, we will briefly mention here the other possible behaviour of such a mixture.

Firstly, the initial plume fluid may be immiscible with the ambient (e.g. in the case of two-phase fluid systems). In such a case the initial plume fluid will remain at a constant density throughout its vertical trajectory (in the absence of compression or decompression effects). Depending on the properties of the initial plume fluid and the ambient fluid, such emissions may break up and take the form of a stream of bubbles or drops. Whilst no small-scale mixing or density change takes place, drag effects of the ambient on these drops will mean that ambient fluid is still accelerated in the plume direction. Therefore, at a sufficient vertical displacement one may find that a high volume fraction of moving fluid within the plume boundary is ambient fluid. Thus the plume may still be thought of as entraining, although this entrainment depends on different parameters from those of ordinary plume theory (see, for example Leitch & Baines 1989).

Alternatively, the plume fluid and ambient fluid may have some other, nonlinear mixing behaviour. This may occur if the densities are functions of two or more variables, for example temperature and salinity. Such a situation is described in Turner & Campbell (1987). This behaviour could possibly be incorporated into the plume analysis via a more complex equation of state for the mixture than the one considered here.

3.2. *Entrainment in the non-Boussinesq case*

Batchelor (1954) mentions that for $\rho/\rho_0 \sim 1$ we have vigorous entrainment of the ambient, but for $\rho/\rho_0 \sim 0$, entrainment falls to zero, and he asserts that "as the density ratio varies there will be a smooth transition between these extremes".

Consistent with this behaviour, the experiments of Ricou & Spalding (1961) suggest that for an arbitrary density ratio, the entrainment assumption should be modified to become

$$u_e = \alpha \left(\frac{\rho}{\rho_0}\right)^{1/2} w \quad \longrightarrow \quad \begin{cases} 0, & \rho/\rho_0 \to 0 \\ \alpha w, & \rho/\rho_0 \to 1. \end{cases} \tag{7}$$

Morton (1965) attempted to justify this dependence on the density ratio. He assumed that the rate of entrainment into a strongly buoyant plume is some function of ρ/ρ_0, but also depends on the Reynolds stresses which have local magnitude $\propto \rho w^2$. Hence, on dimensional grounds, it seems reasonable to assume a local entrainment velocity of $\alpha\left(\rho/\rho_0\right)^{1/2} w$.

It is not possible to extrapolate the modified entrainment assumption into the limit of $\left(\rho/\rho_0\right) \to \infty$, owing to the fundamental changes in the processes of plume entrainment in this limit, as mentioned in the previous section.

3.3. *Mass flux*

In the non-Boussinesq case, we cannot use the simplified continuity equation (5) to describe the volume flux of the plume. Instead we look at the mass flux of the plume using the continuity equation (4). Integrating (4) radially gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty rw\rho \,\mathrm{d}r = -ru\rho|_{r=\infty} = -ru|_{r=\infty}\rho_0 \,.$$

Assuming that w is negligible for $r > b$, integrating (4) for $b < r < \infty$ gives

$$\int_b^\infty \frac{\partial}{\partial r}(ru\rho)\,\mathrm{d}r = 0\,,$$

$$\Rightarrow bu_e = -ru|_\infty\,,$$

where u_e denotes the inflow velocity at the plume edge. Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty rw\rho\,\mathrm{d}r = bu_e\rho_0\,. \tag{8}$$

3.4. *Radial and vertical momentum*

We may gain some insight into the importance of the pressure perturbation field p' by examining the radial and vertical momentum equations in the interior and the exterior of the plume. We make use of the fact that plumes are tall and thin, so that the ratio of the radial lengthscale l of the plume to its vertical lengthscale $\mathscr{L}$ can be regarded as small. The vertical velocity w in the interior of the plume and the radial velocity u in the exterior of the plume both scale as given by the continuity equation (4). We additionally assume that both the radial velocity in the interior and the vertical velocity in the exterior are negligible compared with the radial velocity in the exterior.

Thus, in the radial momentum equation (2) for the region outside the plume, we may neglect the term containing the vertical velocity w to give

$$u\frac{\partial u}{\partial r} \sim -\frac{1}{\rho_0}\frac{\partial p'}{\partial r}, \tag{9}$$

which can be integrated to give the radial pressure gradient caused by the entrainment inflow.

Inside the plume, the fact that u is negligible compared with both u in the exterior and w in the interior means that the radial pressure gradient across the plume is much smaller than that in the exterior, and so we have that the pressure perturbation is approximately constant in the cross-plume direction, and equal to the pressure perturbation outside the plume at any height.

The fact that the pressure perturbation is constant across the plume implies that the vertical pressure gradient must be comparable inside and outside the plume. Therefore, given that the radial pressure gradient outside the plume scales as in (9), and that the vertical pressure gradient varies little between the plume interior and exterior, we can deduce that the pressure term in the vertical momentum equation (3) for the plume interior will be smaller than the other terms by at least a factor of $(l/\mathscr{L})^2$, and hence can be neglected.

3.5. *Momentum flux*

Having shown that we can neglect the pressure perturbation term, and using continuity (4), the z momentum equation (3) becomes

$$\frac{\partial}{\partial r}(r\rho uw) + \frac{\partial}{\partial z}(r\rho w^2) = -gr\rho' .$$

Integrating this equation radially gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty rw^2\rho\,\mathrm{d}r = -g\int_0^\infty r\rho'\,\mathrm{d}r . \tag{10}$$

3.6. *Buoyancy, enthalpy and volume*

The third of the MTT equations describes the rate of change of the buoyancy flux with height, and shows that for a Boussinesq plume in an unstratified ambient (the case we are considering) the buoyancy flux is conserved. In this section we will show that *without making the Boussinesq approximation*, we may first obtain an equation for the rate of change of the volume flux. We then use the mass flux equation to describe the rate of change of the density deficiency $\bar{\rho}'$, and discover the conserved quantity for non-Boussinesq plumes.

For the case of a gas plume, the equivalent of the buoyancy flux equation is best described in thermodynamical terms. Delichatsios (1981) identifies the buoyancy flux with the enthalpy flux and assumes that it is conserved with height. Here, we will also look at the enthalpy flux in the plume, but will consider the factors that contribute to its rate of change.

As we have indicated earlier, we may approximate the pressure across the plume to be uniform and equal to the ambient pressure just outside the plume boundary. Additionally, if we restrict our considerations to plumes with vertical lengthscales much smaller than the ambient scale height ($\sim$ 10 kilometres for the atmosphere), we can also neglect the variation of the ambient pressure with height. The plume then becomes an isobaric system, so we would expect enthalpy flux only to vary with height because of entrainment and heating effects.

We begin by examining a generalized system, and then make our discussion specific to the case of an ideal gas. The internal energy equation for a system capable of expansion work only is given by

$$dU = dQ - P\,dV\,,$$

or in terms of enthalpy $H = U + PV$,

$$dH = dQ + V\,dP\,,$$

$$\Rightarrow \; \dot{H} = \dot{Q} + V\dot{P}\,, \tag{11}$$

where $\dot{Q}$ is the total heating power input to the system, and dot superscripts denote time derivatives.

When the system in question is a plume, the total time rate of change of enthalpy in the system is equal to the net enthalpy flux across the boundary of the plume,

$$\dot{H} = \int_{S_P} \boldsymbol{I} \cdot d\boldsymbol{S}\,, \tag{12}$$

where $\boldsymbol{I}$ is the enthalpy flux per unit area, and S_P is a surface bounding the plume, consisting of a (circular) horizontal surface covering the plume source, a similar surface at some large height above the source where the plume velocity is tending to zero, and all the surface covering the mean plume boundary between these two planes. In the steady state the pressure has no time dependence, that is

$$\dot{P} = 0\,,$$

and so the flux of enthalpy (11) is due to the rate of heat release within the plume.

The enthalpy flux (12) can be written as

$$\int_{S_P} \boldsymbol{I} \cdot d\boldsymbol{S} = \int_{S_P} \rho h \boldsymbol{v} \cdot d\boldsymbol{S}\,, \tag{13}$$

where $\boldsymbol{v}$ is the velocity, and the specific enthalpy is given by

$$h = u + \frac{P}{\rho}\,.$$

The specific internal energy u is defined by $u = c_v T$, and thus, (11) becomes

$$\int_{S_P} (c_v T \rho + P)\boldsymbol{v} \cdot d\boldsymbol{S} = \dot{Q}\,.$$

We now focus on the gaseous case. For an ideal gas, $P = \rho R T$, so

$$c_v T \rho + P = \rho T (c_v + R) = \rho T c_p\,.$$

When the specific heat capacity c_p is constant,

$$\frac{c_p}{R} \int_{S_P} P \boldsymbol{v} \cdot d\boldsymbol{S} = \dot{Q}\,, \tag{14}$$

and by the divergence theorem we have

$$\frac{c_p}{R} \int_{\tau_P} P \nabla \cdot \boldsymbol{v}\, d\tau = \dot{Q} - \frac{c_p}{R} \int_{\tau_P} \boldsymbol{v} \cdot \nabla P\, d\tau\,, \tag{15}$$

where τ_P is the volume enclosed by the surface S_P. If the pressure is constant across the plume, which is the approximation described above for thin plumes, $P = P(z)$ and

$$\frac{c_p}{R}\int_{\tau_P} P\nabla\cdot\boldsymbol{v}\,\mathrm{d}\tau = \dot{Q} - \frac{c_p}{R}\int_{\tau_P} w\frac{\partial P}{\partial z}\,\mathrm{d}\tau\,.$$

In cylindrical polar coordinates (with axisymmetry), taking $\mathrm{d}/\mathrm{d}z$, we obtain

$$P\int_0^b \nabla\cdot\boldsymbol{v}\,r\,\mathrm{d}r = \frac{R}{c_p}\frac{\dot{Q}'}{2\pi} - \frac{\mathrm{d}P}{\mathrm{d}z}\int_0^b w\,r\,\mathrm{d}r\,,$$

where $\dot{Q}' = (\mathrm{d}/\mathrm{d}z)\dot{Q}$ is the heating power input per unit height. That is,

$$P\int_0^b \left(\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw)\right)\,\mathrm{d}r = \frac{R}{c_p}\frac{\dot{Q}'}{2\pi} - \frac{\mathrm{d}P}{\mathrm{d}z}\int_0^b w\,r\,\mathrm{d}r\,.$$

Taking w to be negligible outside the plume boundary, in a similar manner to the derivation of the mass flux equation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty w\,r\,\mathrm{d}r = \frac{R\dot{Q}'}{2\pi P c_p} + bu_e - \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}z}\int_0^\infty w\,r\,\mathrm{d}r\,. \tag{16}$$

Equation (16) describes the rate of change of volume flux in a plume due to the effects of external heating (such as the heat evolved in a combusting plume), entrainment, and expansion due to the decrease in ambient pressure with height, respectively.

In a hydrostatic ambient, we have that $(1/P)(\mathrm{d}P/\mathrm{d}z) \approx l_{sh}^{-1}$ where $l_{sh} = P_0/g\rho_0$ is the scale height. As we have previously stated, we shall only consider plumes at lengthscales much smaller than the scale height, so that $\mathrm{d}/\mathrm{d}z \gg l_{sh}^{-1}$, and so the last term may be ignored. If we further assume that there is no heat input to the plume, that is we let

$$\dot{Q}' = 0\,,$$

then we see that the considerations of enthalpy lead us to a straightforward conservation of volume flux,

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty rw\,\mathrm{d}r = bu_e\,. \tag{17}$$

3.7. *Top-hat variables*

We wish to define the mean density, density difference, velocity and plume radius, $\bar{\rho}, \bar{\rho}', \bar{w}$ and b, respectively, in the non-Boussinesq case. We could do this using similar definitions to those of Turner as described in §2, but replacing g' by ρ', that is

$$\bar{w}b^2 = \int_0^\infty w\,r\,\mathrm{d}r\,, \tag{18}$$

$$\bar{\rho}'\bar{w}b^2 = \int_0^\infty \rho' w\,r\,\mathrm{d}r\,, \tag{19}$$

$$\bar{w}^2 b^2 = \int_0^\infty w^2\,r\,\mathrm{d}r\,, \tag{20}$$

and to make this system complete we must also define

$$\bar{\rho}' = \bar{\rho} - \rho_0\,. \tag{21}$$

Since the plume equations are for volume flux, mass flux and momentum flux, it is desirable that these quantities are well-defined in terms of the top-hat variables.

Volume flux has already been defined and (18), (19) and (21) give the mass flux to be what we would expect, namely

$$\bar{\rho}\bar{w}b^2 = \int_0^\infty \rho w\, r\, \mathrm{d}r\,.$$

However, we must also obtain an expression for momentum flux $\int_0^\infty \rho w^2\, r\, \mathrm{d}r$. We recall that as mentioned previously, the time-averaged profiles of velocity and density difference are Gaussian profiles of roughly equal width,

$$\frac{w(r,z)}{\hat{w}(z)} \approx \frac{\rho'(r,z)}{\hat{\rho}'(z)} \approx \mathrm{e}^{-\beta r^2} \tag{22}$$

(with the requirement that $\hat{\rho}' > -\rho_0$). If we substitute these into the above top-hat definitions we find that

$$\int_0^\infty \rho w^2\, r\, \mathrm{d}r = \bar{\rho}\bar{w}^2 b^2 + \tfrac{1}{3}\bar{\rho}'\bar{w}^2 b^2\,.$$

This is rather untidy for our present purposes, so instead of using the above modification of Turner's definitions, we shall define the top-hat variables directly in terms of the volume flux, mass flux and momentum flux. That is

$$\bar{w}b^2 = \int_0^\infty w\, r\, \mathrm{d}r\,, \tag{23}$$

$$\bar{\rho}\bar{w}b^2 = \int_0^\infty \rho w\, r\, \mathrm{d}r\,, \tag{24}$$

$$\bar{\rho}\bar{w}^2 b^2 = \int_0^\infty \rho w^2\, r\, \mathrm{d}r\,, \tag{25}$$

$$\bar{\rho}' = \bar{\rho} - \rho_0\,. \tag{26}$$

With the above choice of top-hat variables we have that

$$\int_0^\infty r\rho'\, \mathrm{d}r = \bar{\rho}' b^2 \left(\frac{\rho_0 + \frac{2}{3}\hat{\rho}'}{\rho_0 + \frac{1}{2}\hat{\rho}'} \right) \approx \bar{\rho}' b^2\,, \tag{27}$$

for a typical range of variation of $\hat{\rho}'$.

Thus we may rewrite the mass, momentum and volume flux equations, (8), (10) and (17), in terms of these top-hat variables as

$$\frac{\mathrm{d}}{\mathrm{d}z}(\bar{\rho}\bar{w}b^2) = bu_e\rho_0\,, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(\bar{\rho}\bar{w}^2 b^2) = -gb^2\bar{\rho}'\,, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(\bar{w}b^2) = bu_e\,. \tag{30}$$

3.8. *Density deficiency*

We now examine the density deficiency, $\bar{\rho}'$, by considering the mass flux equation (28) and the volume flux equation (30). It is clear that in the case of an unstratified ambient, $\rho_0 = \text{const.}$, these equations can be combined and integrated to give

$$b^2\bar{w}\bar{\rho}' = \text{const.} = D, \tag{31}$$

say. That is, we have conservation of the flux of density deficiency D. We can give this the dimensions of buoyancy flux upon taking the product with g/ρ_0, but note that in the non-Boussinesq case this is *not* the same as the buoyancy flux given by

$$B = gb^2\bar{w}\frac{\bar{\rho}'}{\bar{\rho}},$$

which is *not*, in general, conserved. It clearly does tend, however, to the conservation of buoyancy flux in the Boussinesq limit.

3.9. *Similarity*

In what follows we will make use of the fact that

$$\frac{\bar{\rho}'}{\bar{\rho}} = \frac{g'}{g}, \tag{32}$$

by definition.

Having shown that the flux of density deficiency is conserved, we may now proceed to consider a similarity solution for the non-Boussinesq case. We can then use the plume equations to derive its exact form.

In general, it appears that the solution will be given in terms of the density deficiency flux D, the height z, the acceleration due to gravity g, and the plume and ambient densities $\bar{\rho}$ and ρ_0. We may combine D, g and ρ_0 into a quantity F with the units of buoyancy flux, namely

$$F = Dg/\rho_0 .$$

Dimensional analysis gives solutions for $\bar{w}$ and b of the form

$$\bar{w} = LF^{1/3}z^{-1/3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{\lambda},$$

$$b = Mz\left(\frac{\bar{\rho}}{\rho_0}\right)^{\mu},$$

where L, M, λ and μ are unknown constants. (Note that by combining the parameters g and D into a single similarity variable, we preclude the free-fall solution $\bar{w} \propto (gz)^{1/2}$.)

Conservation of the flux of density deficiency (31) then implies that we will have an expression for $\bar{\rho}'$ of the form

$$\bar{\rho}' = \frac{\rho_0}{g}M^{-2}L^{-1}F^{2/3}z^{-5/3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{-\lambda-2\mu}, \tag{33}$$

so that using (32) we obtain the similarity solution for g' in the form

$$g' = NF^{2/3}z^{-5/3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{\nu}, \tag{34}$$

where

$$N = M^{-2}L^{-1},$$

and

$$\nu = -2\mu - \lambda - 1 .$$

We may now rewrite the momentum equation (29) in terms of the similarity solutions for $\bar{w}, b$ and g' to obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z^{4/3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{2\lambda+2\mu+1}\right] = \frac{-N}{L^2} z^{1/3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{-\lambda}, \tag{35}$$

which has solution

$$\frac{4}{3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{3\lambda+2\mu+1} + \frac{N}{L^2} = \mathrm{e}^{K_1} z^{-\frac{4}{3}\left[(3\lambda+2\mu+1)/(2\lambda+2\mu+1)\right]}, \tag{36}$$

where K_1 is a constant of integration.

It is possible to get a relationship between λ and μ from (36) by requiring that its Boussinesq limit be consistent with the Boussinesq similarity solution for g'. In the Boussinesq case, we have that

$$g' = NF^{2/3} z^{-5/3}, \tag{37}$$

and applying the similarity solution to the momentum equation in this case also gives that

$$-\frac{N}{L^2} = \frac{4}{3}. \tag{38}$$

In the Boussinesq limit we may let

$$\frac{\bar{\rho}}{\rho_0} = 1 + \epsilon,$$

where $|\epsilon|$ is small. Then to $O(\epsilon)$,

$$\frac{\bar{\rho}'}{\bar{\rho}} = \epsilon \Rightarrow \frac{g'}{g} = \epsilon.$$

Using all these relations, we may rewrite (36) as

$$\tfrac{4}{3}(1+\epsilon)^{3\lambda+2\mu+1} - \tfrac{4}{3} = \mathrm{e}^{K_1} z^{-\frac{4}{3}\left[(3\lambda+2\mu+1)/(2\lambda+2\mu+1)\right]},$$

and for $|\epsilon|$ small we have that

$$(1+\epsilon)^{3\lambda+2\mu+1} \approx 1 + (3\lambda+2\mu+1)\epsilon,$$

so that (36) becomes

$$\epsilon = \frac{\mathrm{e}^{K_1}}{3\lambda+2\mu+1} z^{-\frac{4}{3}\left[(3\lambda+2\mu+1)/(2\lambda+2\mu+1)\right]},$$

that is

$$g' = g\frac{\mathrm{e}^{K_1}}{3\lambda+2\mu+1} z^{-\frac{4}{3}\left[(3\lambda+2\mu+1)/(2\lambda+2\mu+1)\right]}. \tag{39}$$

Comparing this with the similarity solution in the Boussinesq regime (37), we see that in order to have the same z-dependence we require that

$$-\frac{4}{3}\left[\frac{3\lambda+2\mu+1}{2\lambda+2\mu+1}\right] = -\frac{5}{3},$$

which simplifies to

$$2\lambda - 2\mu = 1. \tag{40}$$

Matching the coefficients also gives us

$$e^{K_1} = \frac{5\lambda N F^{2/3}}{g}. \tag{41}$$

Thus we can now write (36) more simply using (38), (40) and (41) as

$$\frac{4}{3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{5\lambda} - \frac{4}{3} = \frac{5\lambda N F^{2/3}}{g} z^{-5/3}. \tag{42}$$

However, the similarity solution for g' in the non-Boussinesq regime (34) is another relationship between $\bar{\rho}$ and z, which must be compatible with (42). We have from (33) and (40) that

$$\frac{\bar{\rho}'}{\rho_0} = \frac{N F^{2/3}}{g} z^{-5/3} \left(\frac{\bar{\rho}}{\rho_0}\right)^{1-3\lambda},$$

or in terms of $\bar{\rho}$,

$$\left(\frac{\bar{\rho}}{\rho_0} - 1\right) = \frac{N F^{2/3}}{g} z^{-5/3} \left(\frac{\bar{\rho}}{\rho_0}\right)^{1-3\lambda},$$

or,

$$5\lambda \left(\frac{\bar{\rho}}{\rho_0}\right)^{3\lambda-1} \left(\frac{\bar{\rho}}{\rho_0} - 1\right) = \frac{5\lambda N F^{2/3}}{g} z^{-5/3}. \tag{43}$$

Now (42) and (43) imply an identity for $(\bar{\rho}/\rho_0)$, that is

$$\frac{4}{3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{5\lambda} + \frac{N}{L^2} = 5\lambda \left(\frac{\bar{\rho}}{\rho_0}\right)^{3\lambda} - 5\lambda \left(\frac{\bar{\rho}}{\rho_0}\right)^{3\lambda-1}, \text{ for all } \left(\frac{\bar{\rho}}{\rho_0}\right), \tag{44}$$

which can only hold in the trivial case

$$\lambda = 0, \tag{45}$$

and so we have, from (40),

$$\mu = -\tfrac{1}{2}, \tag{46}$$

and

$$\nu = 0. \tag{47}$$

Thus the similarity solution in the non-Boussinesq case must be given by

$$\bar{w} = L F^{1/3} z^{-1/3},$$
$$b = M z \left(\frac{\bar{\rho}}{\rho_0}\right)^{-1/2},$$
$$g' = N F^{2/3} z^{-5/3}.$$

Consequently, we have shown that non-Boussinesq effects only enter the similarity solution for the radius. However, these values of λ, μ and ν now imply that (35) contains no more information than that given in (38).

3.10. *Entrainment velocity*

Finally, we use this similarity solution to obtain an expression for the entrainment velocity u_e in the non-Boussinesq regime.

Substituting the solutions for $\bar{w}$ and b into the mass flux equation (28) we obtain

$$LMF^{1/3}\frac{\mathrm{d}}{\mathrm{d}z}(z^{5/3}) = z\left(\frac{\bar{\rho}}{\rho_0}\right)^{-1/2} u_e\,, \tag{48}$$

and so differentiating yields

$$u_e = \tfrac{5}{3}LMF^{1/3}z^{-1/3}\left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2}\,, \tag{49}$$

or in terms of the plume velocity, from the similarity solution,

$$u_e = \tfrac{5}{3}M\bar{w}\left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2}\,, \tag{50}$$

which is in agreement with the entrainment behaviour observed experimentally by Ricou & Spalding (1961) as given in (7).

4. Conclusions

We have presented a theoretical description of gaseous plumes in the non-Boussinesq regime.

Firstly, we have obtained expressions for the conservation of mass flux and momentum flux from the basic equations in the usual manner.

We have then derived an equation for the evolution of the volume flux by examining the thermodynamics of the system. Consideration of the aspect ratio of a typical plume leads us to neglect the variation of radial pressure in its interior. This suggests that under certain conditions the enthalpy in a plume should be conserved. Specifically, for an ideal gas, this enables us to describe how the rate of change of volume flux (with height) depends on the effects of heating, entrainment and expansion. For unheated plumes at vertical lengthscales much smaller than the scale height, we obtain a simple conservation of volume flux, which we use along with the conservation of mass flux to show that the flux of density deficiency is conserved in such a case.

Finally, we have obtained the plume similarity solutions and an expression for the entrainment velocity in the non-Boussinesq regime. In order to do this, we form a quantity with the units of buoyancy flux from the density deficiency flux, which tends to the buoyancy flux in the Boussinesq limit. We may then proceed by dimensional analysis to derive similarity solutions for the plume variables (mean vertical velocity, reduced gravity and radius) which are similar to those in the Boussinesq case. However, in this case we must also include in each of the solutions a factor of the ratio of the mean plume and ambient densities raised to an arbitrary power. By considering the constraints placed upon these solutions by the plume momentum flux equation, agreement with the Boussinesq limit, and internal self-consistency, we are able to fix the arbitrary index of the density ratio in each solution. It appears that the only similarity solution that changes in the non-Boussinesq case is that of the plume radius. This acquires a proportionality to the density ratio to the $-\frac{1}{2}$ power. In physical terms, this means that a positively buoyant non-Boussinesq plume will be wider near the source than a similar Boussinesq plume but will be otherwise unaffected, and similarly a negatively buoyant plume will be thinner in the non-Boussinesq case.

We then substitute from these solutions into the mass flux equation to obtain a form for the entrainment velocity, expressed as a function of mean plume velocity and density ratio. We find that it is proportional to the plume velocity, as was already well-known, and we also find that is is proportional to the square root of the density ratio, a dependence which had been previously postulated from experimental observations by Ricou & Spalding (1961) and which, until now, had no theoretical basis.

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