Part III Mathematics
Fluid Dynamics of Climate (PHH part)
Michaelmas Term 2020

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[Lectures 1-10 JRT: §1 Fluid motion in a rotating reference frame, §2 Departures from geostrophy, §3 Rotating shallow water equations, §4 Small amplitude motions in rotating shallow water, §5 Geostrophic adjustment, §6 Quasi-geostrophic equations. §7 Large-scale ocean circulation. Lectures 11-15 PHH: §8 Density stratification and internal gravity waves, §9 Derivation of the 3-D quasigeostrophic equations, §10 Waves and instabilities in the 3-D quasi-geostrophic equations. Lectures 16-18 JRT §11 Fronts, §12 Internal waves and instability inside fronts: Lectures 19-24 PHH, §13 Wave mean-flow interaction, §14 Mean meridional circulations, §15 Equatorial waves.

These notes include a small number of Figures which show schematic diagrams. Other pictures etc. that I showed in lectures – e.g. pictures from published articles – I will put on Moodle – under 'Slides'. I've referred to these as 'Slides' in the notes below.]

8 Stratification and internal gravity waves

8.1 The Boussinesq approximation

We will consider stably stratified flow under the Boussinesq approximation. The latter is a simplification which excludes some of the complicating factors of variable density. A brief explanation is as follows.

We start from the equations in a rotating frame of reference for 3-D incompressible flow. Incompressibility implies that sound waves are excluded and that other effects of large pressure differences across the fluid domain are neglected. Thus the incompressible equations are not quantitatively accurate when considering motion in the atmosphere or indeed in the ocean across domains with height greater than a few km and modifications are needed. However they serve as useful model equations.

The governing equations are

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} \tag{8.1a}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{8.1b}$$

$$\frac{D\rho}{Dt} = 0 \tag{8.1c}$$

where velocity $\mathbf{u} = (u, v, w)$, $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$, ρ is density and p is pressure. $\mathbf{g}$ is the gravitational force per unit mass assumed to be constant and vertical. $\mathbf{f} = 2\boldsymbol{\Omega}$, with $\boldsymbol{\Omega}$ the vector angular velocity of the frame of reference.

The Boussinesq approximation follows by assuming that the density ρ may be split into two parts ρ_0 and ρ' with ρ_0 constant and $\rho'/\rho_0 \ll 1$. The pressure may then also be split into two parts, $p_0(z)$, such that $-dp_0/dz - \rho_0 g = 0$ (i.e. the pressure field $p_0(z)$ would be in hydrostatic balance with the constant density ρ_0) and the remainder p'.

Then the vertical component of the momentum equation is written as follows

$$\begin{aligned}\frac{Dw}{Dt} &= -\frac{1}{\rho_0+\rho'}\frac{\partial p_0}{\partial z} - \frac{1}{\rho_0+\rho'}\frac{\partial p'}{\partial z} - g\\ &= -\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} + \frac{\partial p_0}{\partial z}\left\{\frac{1}{\rho_0} - \frac{1}{\rho_0+\rho'}\right\} - \frac{\partial p'}{\partial z}\frac{1}{(\rho_0+\rho')} - g\\ &\simeq -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0}g,\end{aligned}$$

retaining only leading-order terms in ρ'/ρ_0 in each of the two terms in the final expression. In the horizontal components of the momentum equations $1/\rho$ is is replaced by $1/\rho_0$, i.e. again retaining only leading-order terms in ρ'/ρ_0.

The Boussinesq equations are therefore

$$D\mathbf{u}/Dt + \mathbf{f}\times\mathbf{u} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\mathbf{g} \tag{8.2a}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{8.2b}$$

$$\frac{D\rho'}{Dt} = 0 \tag{8.2c}$$

To summarise, under the Boussinesq approximation, the density ρ is replaced by the constant value ρ_0, except where it is multiplied by the gravitational acceleration g. The Boussinesq equations make clear the role of buoyancy, that a light fluid parcel experiences an upward force and a heavy fluid parcel experiences a downward force.

It is now further useful to write $\rho'(x, y, z, t) = \rho_s(z) + \tilde{\rho}(x, y, z, t)$, dividing it into a background or reference density $\rho_s(z)$ and a disturbance density $\tilde{\rho}$. The latter might, for example, be zero when the fluid is at rest.

An important quantity that measures the stability of the reference density state is the *buoyancy frequency* or *Brunt-Vaisala frequency* N defined, in the Boussinesq system, by

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_s}{dz}. \tag{8.3}$$

If $N^2 > 0$, i.e. if density increases downward then a displaced fluid parcel will tend to return to its original location. The reference density state is *statically stable*, vertical motion is inhibited and a vertically displaced fluid parcel will tend to return to its original location. N is the frequency of oscillation of vertically aligned slabs of fluid which are displaced vertically relative to their neighbours. (These properties will be demonstrated in §8.3 below.)

8.2 Stratification in the atmosphere and ocean

In the ocean the buoyancy frequency N is typically 10^{-2}s^{-1} in the upper ocean (the *thermocline* where the stratification is strong, and $5 \times 10^{-4}\text{s}^{-1}$ in the deep ocean, where the stratification is weak.

In the atmosphere calculating the buoyancy frequency needs to take account of compressibility, because the density ρ is not conserved by a fluid parcel in reversible, dissipationless motion. The quantity that is conserved is the potential temperature θ defined by $\theta = T(p/p_0)^{-2/7}$ where T is temperature and p is pressure. The corresponding expression for the buoyancy frequency is

$$N^2 = \frac{g}{\theta}\frac{d\theta}{dz}. \tag{8.4}$$

Note for example, that if T is constant in height then N^2 is positive, because p decreases with height. Indeed N^2 can be positive when T decreases with height, provide it does not do so too rapidly. In the Earth's atmosphere T decreases upwards for about 10km or so (the *troposphere*) and then begins to increase (the *stratosphere*). The corresponding buoyancy frequencies are 10^{-2}s^{-1} in the troposphere and $2 \times 10^{-2}\text{s}^{-1}$ in the stratosphere.

[See **Slides** showing vertical structure of density/temperature in the atmosphere and ocean.]

For the remainder of the course we will consider a fluid described by the Boussinesq equations (8.2a-c) and interpret the behaviour we find as a qualitative description of the atmosphere and the ocean. But a quantitative description of the atmosphere would need to take account of compressibility as outlined above. (Note that in fact an accurate description of the ocean, particular when considering when considering the whole depth range, also needs to take some account of compressibility, because the density depends on pressure and the pressure variation over several km is very large.)

8.3 Small-amplitude motion about a state of rest

To gain some insight into the effects of stable stratification we consider the Boussinesq equations (8.2a-c) linearised about a resting state with density structure represented by the buoyancy frequency N. For simplicity we neglect background rotation for the time being. It is convenient to introduce the variable $\sigma = -\rho' g/\rho_0$, sometimes called the *buoyancy*.

The equations are

$$\tilde{\mathbf{u}}_t = -\rho_0^{-1}\nabla\tilde{p} + \tilde{\sigma}\hat{\mathbf{z}}, \tag{8.5a}$$

$$\nabla\cdot\tilde{\mathbf{u}} = 0, \tag{8.5b}$$

$$\tilde{\sigma}_t + N^2\tilde{w} = 0. \tag{8.5c}$$

The $\tilde{\mathbf{u}}$ etc notation has been used to denote the fact that these are disturbance quantities away from the resting state. $\hat{\mathbf{z}}$ denotes the unit vector in the vertical direction.

These equations may be reduced to a single equation for $\tilde{w}$,

$$\nabla^2\tilde{w}_{tt} + N^2(\tilde{w}_{xx} + \tilde{w}_{yy}) = 0. \tag{8.6}$$

[Take ∇. of (8.5a), to give a relation between $\tilde{p}$ and $\tilde{\sigma}$, then combine the vertical component of (8.5a) with (8.5c) to eliminate $\tilde{p}$ and $\tilde{\sigma}$.]

Now consider the special case where N^2 is constant and seek plane-wave solutions with $\tilde{w} = \mathrm{Re}(\hat{w}e^{ikx+ily+imz-i\omega t})$. This leads to the dispersion relation:

$$\omega^2 = \frac{N^2(k^2+l^2)}{k^2+l^2+m^2}. \tag{8.7}$$

A first important implication of the dispersion relation is that if $N^2 > 0$ a small amplitude disturbance will lead to oscillatory motion rather than growing in time, i.e. the resting state is stable. If $N^2 < 0$ on the other hand , corresponding to density increasing upwards, then small amplitude disturbances will grow exponentially in time, i.e. the resting state is unstable. (The latter possibility is important, of course, e.g. in leading to atmospheric or oceanic convection, but we will now focus on the case with $N^2 > 0$.)

The dispersion relation implies that $0 \leq |\omega| \leq N$, with the lower limit achieved in the limit $k^2 + l^2 \ll m^2$. Define $\theta = \tan^{-1}(m/(k^2+l^2)^{1/2})$, so that θ is the angle that surfaces of constant phase, which are perpendicular to the wavenumber vector $\mathbf{k} = (k, l, m)$, make with the vertical, then $\omega = \pm N\cos\theta$. Note that $\nabla.\mathbf{u} = 0$ implies that the velocity vector is perpendicular to $\mathbf{k}$, i.e. the velocity vector lies in surfaces of constant phase. Therefore θ is also the angle that fluid parcel trajectories make with the vertical.

For θ close to 0, i.e. wave phase surfaces vertical, wavenumber vector horizontal, $\omega \simeq \pm N$. For θ close to $\pi/2$, i.e. wave phase surfaces horizontal, wavenumber vector vertical, $\omega \simeq 0$.

$|\omega| \leq N$ implies that only disturbances with sufficiently low frequency can propagate as waves. The response to a localised forcing at frequency greater than N will be localised near the forcing rather than taking the form of a propagating wave field. Note also that the angle of the phase surfaces depends only on the frequency, not on the scale of the waves, so forcing at a single frequency, but with a range of spatial scales, will lead to a wave field in which the phase surfaces are all oriented in the same direction.

The dispersion relation implies the group velocity

$$\mathbf{c}_g = (\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}) = \pm\frac{N}{(k^2+l^2)^{1/2}(k^2+l^2+m^2)^{3/2}}(km^2, lm^2, -m(k^2+l^2)). \tag{8.8}$$

Note that $\mathbf{c}_g.\mathbf{k} = 0$, i.e. the group velocity is parallel to phase surfaces and therefore perpendicular to the phase velocity and that $|\mathbf{c}_g| = N \sin\theta/|\mathbf{k}|$, i.e. the group velocity is an increasing function of wavelength and a decreasing function of frequency.

[See **Slides** showing schematic diagram of internal gravity wave response structure of density and velocity field for this solution.]

9 Derivation of the 3-D quasi-geostrophic equations

9.1 Some basic facts about rotation and stratification in 3-D

(i). In a model problem with stratification represented by constant buoyancy frequency N and with $\mathbf{f}$ vertical the dispersion relation for small amplitude waves is

$$\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}$$

where k is the horizontal wavenumber and m is the vertical wavenumber. (This is consistent with the dispersion relation in the case with $f = 0$ derived in §8.3. See Q6 on Example Sheet 2 for the case with $f \neq 0$.) It follows that the relative strength of stratification vs rotation is N/L vs f/D, where L is the horizontal length scale and D the vertical length scale.

(ii). In the atmosphere and the ocean N is typically much larger than f. f is $O(10^{-4}\mathrm{s}^{-1})$. N is $O(10^{-3}\mathrm{s}^{-1})$ (deep ocean) or $O(10^{-2}\mathrm{s}^{-1})$ (upper ocean and atmosphere).

(iii). It follows that rotation is important only if the vertical length scale D is much less than the horizontal length scale L, but this implies that vertical velocities are much less than horizontal velocities and that the hydrostatic approximation is valid (using the same scaling arguments as those applied in §3 to the shallow-water system). Alternatively to justify the hydrostatic approximation it could simply be observed that $D \ll L$ for a wide class of atmospheric and oceanic flows.

(iv). Given the above, the Coriolis force may be neglected in the vertical momentum equation and in the horizontal momentum equation only the part of the Coriolis force associated with the horizontal velocity need be included.

This can be seen as follows. Write $\mathbf{f} = \mathbf{f}_h + \mathbf{f}_v$ and $\mathbf{u} = \mathbf{u}_h + \mathbf{u}_v$, where $_h$ denotes horizontal and $_v$ denotes vertical.

Then (noting that $\mathbf{f}_v \times \mathbf{u}_v = \mathbf{0}$)

$$\mathbf{f} \times \mathbf{u} = (\mathbf{f}_h + \mathbf{f}_v) \times (\mathbf{u}_h + \mathbf{u}_v) = \mathbf{f}_h \times \mathbf{u}_h + \mathbf{f}_v \times \mathbf{u}_h + \mathbf{f}_h \times \mathbf{u}_v \simeq \mathbf{f}_h \times \mathbf{u}_h + \mathbf{f}_v \times \mathbf{u}_h$$

where the second step follows from $|\mathbf{u}_v| \ll |\mathbf{u}_h|$. (Note that this second step might have to be reexamined if $\mathbf{f}_v$ was very small relative to $\mathbf{f}_h$, which is true at very low latitudes.) The first part of the right-hand side is the vertical component of the Coriolis force and it may be shown that if the hydrostatic approximation applies then this is much smaller than the dominant terms in the vertical momentum equations and it may therefore be neglected. (This is called the 'traditional approximation' in §1.) Therefore the only part of the Coriolis force that needs be retained is the $\mathbf{f}_v \times \mathbf{u}_h$ contribution to the horizontal component. Retaining this part is equivalent to replacing the rotation vector by its vertical component only.

Note that $|\mathbf{f}_v| = 2\Omega \sin\phi$ where Ω is the rotation rate and ϕ is latitude.

(v). The sequence of equations that follow from last two points, plus the geometric simplication that the fluid layer is thin compared to the radius of the Earth, are called the *primitive equations*. These have, until very recently, been widely used as a basis for numerical modelling of atmosphere and ocean. (The most recent models are designed to be valid at very high horizontal resolution, where the hydrostatic approximation may no longer apply.)

A further very useful approximation is locally to define Cartesian co-ordinates (x, y, z) with x in the longitudinal direction, y in the latitudinal direction and z in the vertical direction, and to approximate $|\mathbf{f}_v| = 2\Omega \sin\phi$ by $f_0 + \beta y$, i.e. the local latitudinal variation of the Coriolis parameter is retained but approximated as linear in y. (Other geometric effects of spherical geometry are neglected.) This is called the β-plane approximation. (Recall §6.)

Under the sequence of approximations described above, the full 3-D equations lead to the Boussinesq primitive equations on a β-plane

$$\frac{Du}{Dt} - (f_0 + \beta y)v = -\frac{1}{\rho_0}p'_x \tag{9.1a}$$

$$\frac{Dv}{Dt} + (f_0 + \beta y)u = -\frac{1}{\rho_0}p'_y \tag{9.1b}$$

$$p'_z = -\rho' g \tag{9.1c}$$

$$\frac{D\rho'}{Dt} = 0 \tag{9.1d}$$

$$\nabla.\mathbf{u} = 0 \tag{9.1e}$$

where $\mathbf{u} = (u, v, w)$ and $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$.

This set of equations has five dependent variables and three prognostic equations plus two instantaneous constraints. There are strong similarities to the shallow-water equations on a β-plane, indeed these equations behave rather like a set of shallow-water systems communicating in the vertical through the hydrostatic equation. As for the shallow-water equations, for small Rossby number these equations have 'fast' modes (Poincaré waves for the shallow-water equations, hydrostatic inertio-gravity waves for the primitive equations) and 'slow' modes that are close to geostrophic balance (discussed in §6 for the shallow-water equations).

9.2 Thermal wind equation

When the Rossby number is small we expect the flow to be close to geostrophic balance, so that

$$-fv = -\frac{1}{\rho_0}p'_x,$$

$$fu = -\frac{1}{\rho_0}p'_y.$$

Then differentiating in the vertical and using the hydrostatic relation, it follows that

$$fv_z = -\frac{g}{\rho_0}\rho'_x \text{ and } fu_z = \frac{g}{\rho_0}\rho'_y$$

This is the *thermal wind equation.* (ρ' is density, but in many contexts can be seen as equivalent to temperature.)

This equation has been of practical interest since historically it has often been the density or temperature field that is observed and then the thermal wind equation may be used to deduce information about the velocity field. The vertical integration introduces an arbitrary function of x and y. This may be set in the atmosphere by low-level pressure observations or in the ocean by an ad hoc assumption of a 'level of no motion'. (The modern approach is to use 'data assimilation' – all available observations, of different variables and taken at different positions and times are used as input to a dynamical model and space-time fields of all model variables constructed.)

[See **Slides** showing variations of temperature and longitudinal velocity in the atmosphere, and variations of density in the ocean with strongly sloping density surfaces in the Antarctic Circumpolar Current.]

9.3 Potential vorticity

For the shallow-water equations we deduced in §6 a prognostic equation for the slow motion from the equation from potential vorticity conservation. Recall that the potential vorticity is conserved exactly, according to the full shallow-water equations, without any assumption that the motion is slow.

The equations for 3-dimensional density stratified flow under the Boussinesq approximation also imply, without approximation, material conservation of a potential vorticity – generally called Rossby-Ertel potential vorticity – in the absence of forcing and dissipation. The expression for the Rossby-Ertel potential vorticity (under the Boussinesq approximation) is

$$P = \frac{1}{\rho_0}(\mathbf{f} + \boldsymbol{\zeta}).\nabla\rho'.$$

The primitive hydrostatic equations also give conservation of potential vorticity, provided that $\mathbf{f} + \boldsymbol{\zeta}$ is suitably simplified consistent with hydrostatic scaling. For the Boussinesq β-plane primitive equations that we are using the relevant expression is

$$P = \frac{1}{\rho_0}\left\{(f_v + v_x - u_y)\rho'_z + u_z\rho'_y - v_z\rho'_x\right\}.$$

Exercise: Show from (9.1a), (9.1b), (9.1c), (9.1d) and (9.1e) that $DP/Dt = 0$ in the absence of forcing and dissipation.

Note that forcing and dissipation terms have not yet been included in any of the equations considered in this section. When these are non-zero then there is a non-zero term on the right-hand side of the equation for DP/Dt. This is potentially important because it gives rise to features in the P field that can affect (even drive) the evolution of the flow.

9.4 The quasi-geostrophic equations

Repeating the procedure already carried out for the shallow-water equations in §6, we now obtain a prognostic equation for the slow (i.e. close to geostrophic balance) motion from the Boussinesq primitive equations on a β-plane.

We write $\rho'(x, y, z, t) = \rho_s(z) + \tilde{\rho}(x, y, z, t)$, where $\rho_s(z)$ represents density variation in a hydrostatically balanced basic state where there is no motion. We therefore expect that $\tilde{\rho}$ is associated with motion of the fluid when it is disturbed from this resting basic state.

We then write pressure as the sum of two terms, $p'(x, y, z, t) = p_s(z) + \tilde{p}(x, y, z, t)$, where each term is in hydrostatic balance with the corresponding part of the density field, i.e.

$$\frac{dp_s}{dz} = -\rho_s g \text{ and } \frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g$$

The density equation (9.1d) therefore becomes

$$\frac{D\tilde{\rho}}{Dt} + w\frac{d\rho_s}{dz} = 0$$

The velocity field is divided into a part that is in geostrophic balance with the pressure field (assuming that the Coriolis parameter is the constant f_0) and a remainder, referred to as the ‘ ageostrophic’ velocity, i.e.

$$\mathbf{u} = \mathbf{u}_g + \mathbf{u}_a \quad \text{where} \quad f_0 \mathbf{k} \times \mathbf{u}_g = -\frac{1}{\rho_0} \nabla_h \tilde{p}$$

and ∇_h indicates the horizontal part of ∇.

Note that the vertical component of $\mathbf{u}_g$ is zero, and $\nabla.\mathbf{u}_g = 0$.

We shall also assume that the scale L_y in the y direction is sufficiently small that $\beta L_y/f_0 \ll 1$. Then if $Ro \ll 1$ it follows that $|\mathbf{u}_a| \ll |\mathbf{u}_g|$.

The primitive equations may now be written

$$\left\{\frac{\partial}{\partial t}+(\mathbf{u}_g+\mathbf{u}_a).\nabla\right\}(u_g+u_a)-f_0v_g-f_0v_a-\beta y(v_g+v_a)=-\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial x} \tag{9.2a}$$

$$\left\{\frac{\partial}{\partial t}+(\mathbf{u}_g+\mathbf{u}_a).\nabla\right\}(v_g+v_a)+f_0u_g+f_0u_a+\beta y(u_g+u_a)=-\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial y} \tag{9.2b}$$

$$-\frac{\partial\tilde{p}}{\partial z}-\tilde{\rho}g=0 \tag{9.2c}$$

$$\left\{\frac{\partial}{\partial t}+(\mathbf{u}_g+\mathbf{u}_a).\nabla\right\}\tilde{\rho}+w_a\frac{d\rho_s}{dz}=0 \tag{9.2d}$$

$$\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}+\frac{\partial w_a}{\partial z}=0 \tag{9.2e}$$

Given that $Ro \ll 1$ we may approximate $D\mathbf{u}/Dt$ by $D_g\mathbf{u}_g/Dt$, where $D_g/Dt = \partial/\partial t + \mathbf{u}_g.\nabla$, and $\beta y\mathbf{u}$ by $\beta y\mathbf{u}_g$. There are two aspects of these approximations that require further comment.

The first is that $f_0(u_a, v_a)$ and $\beta y(u_g, v_g)$ are considered to be of similar size. This suggests that the requirement $\beta L_y/f_0 \ll 1$ noted above is better expressed as $\beta L_y/f_0 \sim Ro$.

The second is that $w_a d\rho_s/dz$ is retained, but $w_a d\tilde{\rho}/dz$ is not. This requires $|d\tilde{\rho}/dz| \ll |d\rho_s/dz|$. If the horizontal scale is L and the vertical scale is D, the thermal wind equation implies that $g\tilde{\rho}/L\rho_0 \sim f_0U/D$, where U is a typical horizontal velocity. Then it follows that

$$\frac{\tilde{\rho}_z}{(\rho_s)_z}\sim\frac{f_0UL\rho_0}{gD^2(\rho_s)_z}=(\frac{U}{f_0L})(\frac{Lf_0}{ND})^2=Ro(\frac{Lf_0}{ND})^2.$$

The dimensionless quantity $(Lf_0/DN)^2$ is sometimes called the *Burger Number* and denoted by Bu. The approximation that has been made is therefore justified if $Ro\,Bu \ll 1$ or equivalently $Bu \ll Ro^{-1}$, i.e. if Bu is not too large. If $Bu \sim 1$ then this is implied by the fact that $Ro \ll 1$.

It remains to eliminate $\mathbf{u}_a$, which is done by calculating $\partial(9.2b)/\partial x$ - $\partial(9.2a)/\partial y$ and then using the nondivergence of the geostrophic velocity, to form a vorticity equation

$$\frac{D_g}{Dt}\left\{\frac{\partial v_g}{\partial x}-\frac{\partial u_g}{\partial y}\right\}+\beta v_g+f_0\left\{\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}\right\}=0.$$

(9.2d) and (9.2e) are now used to eliminate u_a, v_a and w_a to leave

$$\frac{D_g}{Dt}\left\{\frac{\partial v_g}{\partial x}-\frac{\partial u_g}{\partial y}\right\}+\beta v_g+f_0\frac{\partial}{\partial z}\left\{\frac{D_g\tilde{\rho}}{Dt}/\frac{d\rho_s}{dz}\right\}=0$$

Defining $\psi = \tilde{p}/\rho_0 f_0$, so that $u = -\psi_y$, $v = \psi_x$ and, using (9.2c), $\tilde{\rho} = -\rho_0 f_0\psi_z/g$, this reduces to

$$\frac{D_g}{Dt}\left\{\psi_{xx}+\psi_{yy}+\left\{\frac{f_0^2\psi_z}{N^2}\right\}_z\right\}+\beta\psi_x=0 \tag{9.6a}$$

or

$$\frac{D_g q}{Dt} = 0 \quad \text{where} \quad q = \left\{ \psi_{xx} + \psi_{yy} + \left\{ \frac{f_0^2 \psi_z}{N^2} \right\}_z + \beta y \right\}. \tag{9.6b}$$

Note $N^2 = -g\rho_0^{-1} d\rho_s/dz$ and that, written in terms of ψ,

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}.$$

(9.6b) is the *quasigeostrophic potential vorticity equation.* The quantity q is the *quasi-geostrophic potential vorticity.* Under the quasi-geostrophic approximation it is conserved following the (horizontal) geostrophic flow.

It is possible show that (9.6a) or (9.6b) are approximations to the statement of material conservation of Rossby-Ertel potential vorticity following the flow along ρ' surfaces (in a Boussinesq fluid) or θ surfaces (in a compressible fluid).

Just as for shallow-water quasi-geostrophic flow, if q is known then ψ can be calculated, in this case through the relation

$$\psi = \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right) \right\}^{-1} (q - \beta y).$$

The operator being applied to $q - \beta y$ is sometimes called a *potential vorticity inversion operator.* Note that the application of this operator requires requires boundary conditions on ψ or its derivatives.

(i). At rigid side boundaries, i.e. boundaries on each horizontal level, we typically require than the normal component of $\mathbf{u}$ is zero, equivalent to ψ being constant along the boundary.

(ii). At rigid top or bottom boundaries we again require the kinematic boundary condition to be satisfied, i.e. $Dz/Dt = w = Dh/Dt$ on the boundary $z = z_b + h$, where z_b is constant and h represents a topographic perturbation. The variable w_a has been eliminated from the quasi-geostrophic potential vorticity equation, but may be expressed in terms of other variables via the density equation, implying that the boundary condition may be approximated as

$$w \sim w_a = -\frac{D_g\tilde{\rho}}{Dt} / \frac{d\rho_s}{dz} = \frac{Dh}{Dt} \simeq \frac{D_g h}{Dt}.$$

This suggests that $h \sim \tilde{\rho}/(d\rho_s/dz) \sim D(U/f_0L)(L^2 f_0^2/N^2D^2) = Ro\, Bu\, D$, i.e. $h \ll D$. Therefore to be consistent with the other dynamical assumptions h must be small compared with the vertical length scale D. This allows 'linearisation' of the boundary condition, i.e. that it may be applied at $z = z_b$ rather than at $z = z_b + h$. Finally, substituting for $\tilde{\rho}$ in terms of ψ, it follows that

$$\frac{D_g}{Dt}\psi_z = -\frac{N^2}{f_0}\frac{D_g}{Dt}h. \tag{9.7}$$

at $z = z_b$.

The boundary condition therefore takes the form of a prognostic equation for ψ_z. Note that the physical interpretation of this condition is as a statement of material rate of change of density or temperature (and material conservation in the case where there is no topography).

The density or temperature at horizontal boundaries therefore have similar status in the quasi-geostrophic equations to the quasi-geostrophic potential vorticity in the interior of the flow. Indeed there are some formulations in which the boundary density or temperature is explicitly incorporated into the quasi-geostrophic potential vorticity, as delta functions localised at the boundaries. This is analogous to the way in which in electrostatics surface charge can either be separated from interior charge, with the surface charge implying a boundary condition for the normal component of the interior electric field, or incorporated within the interior charge distribution as a thin δ-function sheet localised at the boundary, in which case the boundary condition on the 'interior' electric field is that the normal component is zero at the boundary.

Note the physical interpretation of different contributions to the quasi-geostrophic potential vorticity q:

$$q = \underbrace{\psi_{xx} + \psi_{yy}}_{\text{relative vorticity}} + \underbrace{\left(\frac{f_0^2}{N^2}\psi_z\right)_z}_{\text{stretching term}} + \underbrace{\beta y}_{\text{planetary vorticity}} \tag{9.8}$$

The stretching term measures vertical gradients in density perturbations, hence the amount by which nearby density surfaces move apart or together.

The ratio of the typical size of the relative vorticity term to that of the stretching term is $N^2D^2/f_0^2L^2$. If $ND/f_0L \gg 1$ then relative vorticity dominates. If $ND/f_0L \ll 1$ then stretching dominates. When $ND/f_0L \sim 1$ then relative vorticity and stretching are comparable. The ratio of horizontal to vertical scale, $L/D \sim N/f_0$ implied by this condition is sometime called Prandtl's ratio of scales.

Recalling similar considerations in §6 for the shallow-water equations, the scale ND/f might be described as the Rossby deformation scale (or 'Rossby radius of deformation') associated with the vertical scale D. (In the shallow-water system there is a single deformation scale determined by the layer thickness. In the continuously stratified system the deformation scale depends on the vertical scale D, which might be determined by initial conditions, or forcing, rather than being uniquely determined for the system as a whole.)

The three-dimensional quasi-geostrophic equations have a strong structural similarity to the equations for two-dimensional vortex dynamics in the sense that there is a non-local dependence of the streamfunction ψ on q. In two-dimensional vortex dynamics the non-locality is purely in the horizontal (e.g. the velocity field associated with a point vortex extends in the horizontal away from the position of the vortex). In the three-dimensional quasi-geostrophic equations the non-locality is also in the vertical and the q field in a localised region on a given level not only affects the ψ field outside that region on the same level, but also affects the ψ field at other levels.

If N is constant in height then the operator appearing in (9.8) is isotropic in scaled co-ordinates x, y, Nz/f_0. However the evolution equations (9.6a) or (9.6b) are not isotropic since the flow only has components in the horizontal. We might therefore expect solutions of the quasi-geostrophic equations to have some tendency towards isotropy in the scaled co-ordinates just defined, but the isotropy may not be exact. This has been an ongoing topic of discussion and research amongst geophysical fluid dynamicists.

[See **Slides** giving examples of two-dimensional turbulence and three-dimensional quasi-geostrophic turbulence.]

9.5 Illustrative calculation of a simple three-dimensional quasi-geostrophic flow

We consider a quasi-geostrophic 'point vortex', i.e. we assume $q = UL^2\delta(x, y, Nz/f_0)$, where $\delta(\mathbf{x})$ is the Dirac delta function and solve

$$\psi_{xx} + \psi_{yy} + \left\{\frac{f_0^2}{N^2}\psi_z\right\}_z = UL^2\delta(x, y, z)$$

to give the corresponding $\psi(x, y, z)$. We assume that N is constant and that the βy term appearing in (9.8) may be neglected. U and L are respectively, a constant velocity and a constant length, included simply to give dimensional consistency. Note that the 'dimensions' of the δ-function in 3-D are (length)$^{-3}$.

We first rescale z by defining $\overline{z} = Nz/f_0$. Then in Cartesians $(x, y, \overline{z})$ the operator on the left-hand side of (9.5) is the three-dimensional Laplacian and we deduce that a solution satisfying the boundary condition $\psi \to 0$ as $|x|$, $|y|$ and $|\overline{z}|$ tend to infinity is

$$\psi(x, y, z) = -\frac{UL^2}{4\pi}\frac{1}{(x^2 + y^2 + \overline{z}^2)^{1/2}} = -\frac{1}{4\pi}\frac{1}{(x^2 + y^2 + N^2z^2/f_0^2)^{1/2}}$$

It follows that the horizontal velocity components (u, v) are given by

$$(u, v) = \frac{UL^2}{4\pi}\frac{(-y, x)}{(x^2 + y^2 + N^2z^2/f_0^2)^{3/2}}$$

and the density perturbation is given by

$$\tilde{\rho} = -\frac{f\rho_0}{g}\psi_z = -\frac{f\rho_0 UL^2}{4\pi g}\frac{z}{(x^2 + y^2 + N^2z^2/f_0^2)^{3/2}}$$

Away from the point vortex the sum of the relative vorticity and the stretching term in q is zero, but, consistent with the fact that the circulation (both velocity and density anomalies) extend away from the point vortex, they are individually non-zero.

[See **Slides** showing structure of density and velocity field for this solution plus examples of atmospheric and oceanic vortices.]

10 Waves and instabilities in the three-dimensional quasi-geostrophic equations

10.1 Introduction

We take the quasi-geostrophic equations (9.6a,9.6b) as a starting point and consider small-amplitude disturbances superimposed on a background flow that is itself a self-consistent solution of the quasi-geostrophic equations. To be specific the background (geostrophic) flow is assumed to be only in the x-direction and to depend only on y and z, i.e. $(u_g, v_g) = (U(y, z), 0)$. There is a corresponding background state quasi-geostrophic stream function $\Psi(y, z)$ such that $-\Psi_y = U$ and quasi-geostrophic potential vorticity $Q(y, z) = \Psi_{yy} + (f_0\Psi_z/N^2)_z + \beta y$.

In §7, when developing the Boussinesq approximation, the $'$ (prime) notation was used to denote difference from constant density and the $\tilde{}$ (tilde) notation was then used to denote horizontally varying part. Having implemented the Boussinesq approximation it is now convenient to redefine the $'$ notation and use it for a different purpose. In this section it will be used to denote disturbance quantities.

We now write down the quasi-geostropic equations retaining only linear terms in disturbance quantities (and noting that the terms involving only background-state quantities balance, because the background state is a self-consistent solution of the equations). The linearised quasi-geostrophic potential vorticity equation is

$$(\frac{\partial}{\partial t} + U(y, z)\frac{\partial}{\partial x})\{\psi'_{xx} + \psi'_{yy} + (\frac{f_0^2}{N^2}\psi'_z)_z\} + \{\beta - U_{yy} - (\frac{f_0^2}{N^2}U_z)_z\}\psi'_x = 0 \tag{10.1}$$

and the boundary condition at any e.g. top or bottom boundary, illustrated here for $z = 0$,

$$(\frac{\partial}{\partial t} + U(y, 0)\frac{\partial}{\partial x})\psi'_z - U_z(y, 0)\psi'_x = -\frac{N^2}{f_0}(\frac{\partial}{\partial t} + U(y, 0)\frac{\partial}{\partial x})h'. \tag{10.2}$$

Note the potentially important role played by the quantity $\beta - U_{yy} - (f_0^2 U_z/N^2)_z$ appearing in (10.1). This is the y-gradient of quasi-geostropic potential vorticity in the background state. Correspondingly the quantity $U_z(y, 0)$ appearing in (10.2), which is proportional to the y-gradient of density at the boundary $z = 0$, may also be important. (Recall that (10.2) has been derived from the density equation and also that the density distribution at the boundary has similar status in the equations to the interior distribution of quasi-geostrophic potential vorticity.)

10.2 Rossby waves – vertical modes

Consider the 3-D quasi-geostrophic equations in an oceanic configuration with a free surface at $z = 0$ in the resting undisturbed state, with a flat bottom at $z = -H$ and with buoyancy frequency $N(z)$.

Assume that in the disturbed state the height of the free surface is displaced to $z = \eta'(x, y, t)$. If we assume that disturbances are small then we may estimate $p(x, y, 0, t) = p_{atm} + \rho_0 g\eta'$ and hence $\rho_0 gw = D_g\tilde{p}(x, y, 0, t)/Dt$ at $z = 0$. Now using the expression for the pressure and for the vertical velocity under the quasi-geostrophic approximation, it follows that the boundary condition at $z = 0$ is

$$\frac{D_g\psi'_z}{Dt} + \frac{N^2}{g}\frac{D_g\psi'}{Dt} = 0.$$

At $z = -H$, the boundary condition is

$$\frac{D_g\psi'_z}{Dt} = 0.$$

The quasi-geostrophic potential vorticity equation (10.1) reduces to

$$\left(\psi_{xx} + \psi_{yy} + (\frac{f_0^2}{N^2}\psi_z)_z\right)_t + \beta\psi_x = 0.$$

We now seek solutions of the form $\psi(x, y, z, t) = \phi(x, y, t)P(z)$, where

$$\frac{d}{dz}(\frac{1}{N^2}\frac{dP}{dz}) = -\frac{1}{gh}P \tag{10.3}$$

with h a suitable constant and with boundary conditions $P' + (N^2/g)P = 0$ at $z = 0$ and $P' = 0$ at $z = -H$. This is an eigenvalue equation for h, sometimes called *the vertical structure equation*, and we may expect a countable sequence of possible values $h_1 > h_2 > \cdots > 0$, with the maximum value h_1 corresponding to the simplest possible structure for $P(z)$.

Note furthermore that the height g/N^2 is typically large compared to the depth H (or the vertical length scale associated with variations in stratification) and therefore the boundary condition at $z = 0$ may be approximated by $P' = 0$. This is the so-called *rigid lid approximation*. (It is equivalent to imposing zero vertical velocity.) Solving with this boundary condition gives P non-zero at $z = 0$ and the solution may therefore be used to give an good first estimate of the pressure variation at $z = 0$ and hence the variation in free-surface height.

If $N = N_0$ (constant) then the largest value h_1 is $N_0^2H^2/g\pi^2$, i.e. $(gh_1)^{1/2} = N_0H/\pi$. $P_1(z)$ for this case has a single zero in the interior of the layer. $P_1'(z)$, corresponding to the vertical displacement, has a single maximum in the interior of the layer. This corresponds to the *first baroclinic mode*. For realistic oceanic stratification, the first baroclinic mode is typically found to have $(gh_1)^{1/2} \simeq 3\text{ms}^{-1}$ and the second baroclinic mode $(gh_2)^{1/2} \simeq 1\text{ms}^{-1}$.

Given h_i and $P_i(z)$ the corresponding equation for $\phi_i(x, y, t)$, describing the horizontal structure of the ith mode will be

$$\left(\phi_{ixx} + \phi_{iyy} - \frac{f_0^2}{gh_i}\phi_i\right)_t + \beta\phi_{ix} = 0,$$

i.e. the quasi-geostrophic equation for a single layer of fluid of depth h_i as derived in §6.2. We have therefore reduced the three-dimensional problem to an equivalent single-layer problem, or a set of such problems, one for each mode, with the layer depths being determined as the eigenvalues of the vertical structure equation.

Note that for each vertical mode there is a corresponding Rossby radius of deformation, given by

$$L_{iR} = \frac{(gh_i)^{1/2}}{f_0}.$$

For the first baroclinic mode $L_{1R} \simeq 30\text{km}$. For the second $L_{2R} \simeq 10\text{km}$. (These are both estimates for midlatitudes.) (Note that §6 uses R_D to denote Rossby radius of deformation.)

For scales much larger than L_R then the single-layer dispersion relation given in §6.2 implies that the phase and group velocities are westward and given by βL_R^2. (Note that the waves are non-dispersive in this limit.) This implies that at midlatitudes the phase/group speed for the first baroclinic mode Rossby wave is about 1.5 $\times 10^{-2}$ ms^{-1}. (This implies about 10 years to cross the Atlantic Ocean.)

At latitude λ the first baroclinic mode Rossby wave speed is $gh_1 \cos\lambda/2\Omega a \sin^2\lambda$. The speed of propagation therefore increases towards the equator. (This formula clearly breaks down as the equator is approached and the correct value applying close to the equator may be deduced from the dispersion relation for equatorial Rossby waves – see later.)

This vertical-mode decomposition is most relevant to oceanic Rossby waves. Oceanic Rossby waves have now been clearly observed from satellite observations of sea-surface height . Oceanic Rossby waves are an important mechanism for propagation of information in the ocean (on time scales of months to years).

[See **Slides** for satellite observations interpreted as Rossby-wave propagation.]

10.3 Topographically forced Rossby waves in a flow with vertical shear

Now consider a basic flow $U(z)$ in $z > 0$. To simplify the problem we assume that N^2 is constant and, additionally, that $U'(0)$, i.e. the vertical shear vanishes at the lower boundary. As previously, note that the basic flow is itself a solution of the equations of motion.

We consider the effect of a steady topographic perturbation $h'(x, y)$ at the lower boundary and seek solutions that are steady. In the interior we have the steady form of the linearised quasi-geostrophic PV equation (10.1), giving

$$U(z)\frac{\partial}{\partial x}\{\psi'_{xx} + \psi'_{yy} + (\frac{f_0^2}{N^2}\psi'_z)_z\} + \{\beta - \frac{f_0^2}{N^2}U''(z)\}\psi'_x = 0.$$

Note that the y-gradient of quasi-geostrophic potential vorticity has been simplified by the assumption that U is independent of y and that N is constant, but still includes a term depending on the second derivative $U''(z)$. The boundary condition (10.2) at $z = 0$, simplified by the assumption of vanishing vertical shear, takes the form

$$-\frac{f_0}{N^2}U(0)\frac{\partial^2\psi'}{\partial x\partial z} = U(0)\frac{\partial h'}{\partial x} \text{ at } z = 0.$$

For convenience we assume that the topographic perturbation h is sinusoidal in x and y

$$h' = \mathrm{Re}\left(\hat{h}e^{ikx+ily}\right).$$

The form of steady disturbances forced by the topography may be written in the form $\psi' = Re\left(\hat{\psi}(z)e^{ikx+ily}\right)$ where

$$\frac{f_0^2}{N^2}\frac{d^2\hat{\psi}}{dz^2} - \left(k^2 + l^2\right)\hat{\psi} + \frac{(\beta - f_0^2U''(z)/N^2)}{U(z)}\hat{\psi} = 0,$$

i.e. $\hat{\psi}'' + m(z)^2\hat{\psi}$ or $\hat{\psi}'' - \mu(z)^2\hat{\psi} = 0$ where

$$m(z)^2 \text{ or } -\mu(z)^2 = \left\{ \frac{\beta - f_0^2 U''(z)/N^2}{U(z)} - \left(k^2 + l^2\right) \right\} \frac{N^2}{f_0^2}$$

Solutions are wave-like in the vertical if $(m(z))^2 > 0$, i.e. if

$$0 < U(z) < \frac{\beta - f_0^2 U''(z)/N^2}{k^2 + l^2} = U_c,$$

otherwise they have exponential behaviour in the vertical.

These statements can be made more precise if we assume that $U(z)$ is slowly varying in the vertical so that WKBJ-type theory is valid. In the 'wave-like' case $m(z)$ is then a local value of the vertical wavenumber. The sign of $m(z)$ is determined by a *radiation condition* that the group velocity is upward and detailed calculation shows that $m(z)$ must take the same sign as k. Correspondingly when there is exponential behaviour the sign of $\mu(z)$ is chosen so that waves decay in the vertical.

Note that:

(i). Disturbances are trapped in the vertical if $U < 0$ or $U > U_c$;

(ii). Disturbances can propagate up into westerlies (eastward flow, $U > 0$) providing that these are not too strong;

(iii). U_c is a decreasing function of k. If $U(z)$ increases upwards, then the longest waves (with smallest k) will propagate through the greatest range of heights.

These results were first noted by Charney and Drazin in 1961. They are highly relevant to the circulation in the stratosphere, which may be disturbed from a symmetric state, where the flow is around latitude circles, by large-scale Rossby waves that are forced in the troposphere (by flow over topography and other processes) and, under suitable conditions, propagate up into the stratosphere.

Particular implications of the points above are as follows.

(i). appears to explain the disturbed circulation in the winter stratosphere compared to the summer stratosphere;

(ii). may explain why the waves are so weak in the midwinter southern hemisphere stratosphere, where the winds are very strong;

(iii). explains the increased scale of the waves with height in the winter troposphere and stratosphere.

Note that the problem above is often formulated in a channel with side walls at $y = 0$, $y = L$ implying the boundary conditions $\partial\psi'/\partial x = 0$ at $y = 0$ and $y = L$. This easily incorporated into the solution presented above by assuming that the y- structure of the disturbances is proportional to $\sin\frac{\pi y}{L}$ and hence that $l^2 = \pi^2/L^2$.

[See **Slides** showing differences between scales of variation of tropospheric circulation and stratospheric circulation and differences between the winter and summer stratospheric circulation.]

10.4 Baroclinic instability

We now consider a flow for which disturbances that are small-amplitude initially may grow substantially in amplitude, corresponding to instability of the flow. One important instability mechanism in the atmosphere and the ocean is associated with sloping density surfaces, i.e. horizontal density gradients, which exist in the basic state when there is vertical shear. A traditional description of this instability is that the sloping density surfaces in the basic flow imply a store of potential energy, which may be released and converted into disturbance energy as the disturbances grow. This is half correct, but note for example that there may be flows with sloping density surfaces that are not unstable.

A relevant very simple flow configuration here is that considered by Eady. We consider a basic state on a f-plane (β is neglected), with constant buoyancy frequency N and with flow in the x-direction $U = \Lambda z$, so that $\psi_0 = -\Lambda zy$, with Λ a positive constant, with constant buoyancy frequency N. The flow is taken to be bounded above and below by horizontal rigid boundaries at $z = 0$ and $z = H$.

Given the above, it follows that the the y-gradient of quasi-geostrophic potential vorticity appearing in (10.1),

$$\beta - U_{yy} - (\frac{f_0^2}{N^2}U_z)_z = -(\frac{f_0^2}{N^2}\Lambda)_z = 0.$$

(The interior quasi-geostrophic potential vorticity is constant in the basic state.) Therefore the linearised quasi-geostrophic potential vorticity equation (10.1) reduces to the form

$$q'_t + \Lambda z q'_x = 0.$$

This has the solution $q'(x, y, z, t) = q'(x - \Lambda zt, y, z, 0)$, i.e. the initial q' field at any level is simply advected by the horizontal flow at that level. The q' field is therefore of no consequence in the analysis of possible instability and it is convenient simply to assume that $q'(x, y, z, 0) = 0$ implying q' is zero at all times.

The important ingredients in this problem are the boundary conditions at $z = 0$ and $z = H$, following from (10.2) as

$$\psi'_{zt} - \Lambda\psi'_x = 0 \qquad \text{at } z = 0 \qquad (10.4)$$

$$\psi'_{zt} + \Lambda H\psi'_{zx} - \Lambda\psi'_x = 0 \qquad \text{at } z = H. \qquad (10.5)$$

These are combined with the interior equation

$$q' = \psi'_{xx} + \psi'_{yy} + \frac{f_0^2}{N_0^2}\psi'_{zz} = 0 \qquad (10.6)$$

We seek solutions of the form

$$\psi' = \text{Re}\left(\hat{\psi}(z)e^{ikx+ily-ikct}\right).$$

Note that the time dependence is incorporated via the e^{-ikct} term, with c being a potentially complex phase speed in the x-direction. c will be determined by solution of an eigenvalue problem. There will be instability if $k\text{Im}(c) > 0$.

Then $q' = 0$ implies that $\hat{\psi}_{zz} - \mu^2\hat{\psi} = 0$ where $\mu = N_0(l^2 + k^2)^{1/2}/f_0$ and hence that

$$\hat{\psi} = Ae^{\mu z} + Be^{-\mu z}$$

where A and B are constants.

Substituting into the boundary conditions it follows that

$$-c\,(\mu A - \mu B) - \Lambda(A + B) = 0$$

$$(\Lambda H - c)(\mu Ae^{\mu H} - \mu Be^{-\mu H}) - \Lambda(Ae^{\mu H} + Be^{-\mu H}) = 0.$$

Combining these equations gives the solution of the eigenvalue problem

$$c = \tfrac{1}{2}\Lambda H \pm \tfrac{1}{2}H\Lambda\left(1 - \frac{4\coth\mu H}{\mu H} + \frac{4}{\mu^2 H^2}\right)^{1/2} = \tfrac{1}{2}\Lambda H \pm \tfrac{1}{2}\Lambda H(F(\mu H))^{1/2}. \qquad (10.7)$$

where the function $F(.)$ is defined by the second equality. There will be instability if $F(\mu H) < 0$ (since one of the two possible values for c will give $k\mathrm{Im}(c) > 0$). It is simple to show that $F(.)$ is an increasing function, with $F(0) = -\frac{1}{3}$ and $F(\mu H) \to 1$ as $\mu H \to \infty$. Hence there is instability if $\mu H < \hat{\mu}_c = 2.399$ where $F(\hat{\mu}_c) = 0$.

The growth rate for any value of k and l is given by

$$k\mathrm{Im}(c) = \frac{f_0\Lambda}{N}\,\frac{k}{(k^2 + l^2)^{1/2}}\{\tfrac{1}{2}\mu H(-F(\mu H))^{1/2}\}.$$

It is clear from this formula that for given μ the growth rate is maximised when l is zero. Therefore the maximum growth rate will occur when $\frac{1}{2}\mu H(-F(\mu H))^{1/2}$ is maximised, which determines μ and, for that value of μ, when $l = 0$. (Recall that μ is proportional to $(k^2 + l^2)^{1/2}$.) The maximum value of $\frac{1}{2}\mu H(-F(\mu H))^{1/2}$ is 0.31 and occurs at $\mu H = 1.61$. Hence in the Eady problem the maximum growth rate is $0.31 f_0\Lambda/N$, i.e. it is proportional to the vertical shear Λ multiplied by the ratio f_0/N. The maximum growth occurs with x-wavenumber k equal to $1.61 f_0/NH$, i.e. the corresponding length scale is the height of the domain H multiplied by the ratio N/f_0, and the y-wavenumber $l = 0$.

The stability properties are affected by the geometry of the domain. If there were side boundaries at $y = 0$ and $y = L$ these could be satisfied by superposing solutions of the form given above with $l = \pm n\pi/L$ where $n = 1, 2, \ldots$, i.e. the minimum possible value of $|l$ is π/L. Hence the minimum possible value of μH is $\pi NH/f_0 L$ and there will be no instability if $L < 2.399\pi NH/f_0$. [The fact that reduced channel width can suppress instability suggests that the mechanism is not quite as simple as release of potential energy.]

Returning to the dispersion relation, we note that in the limit $\mu H \to \infty$, $c/\Lambda H \to 1/\mu H$ or $c/\Lambda H \to 1 - 1/\mu H$. In the first case it follows that $B \gg A$, and therefore that the wave is bottom trapped. In the second it follows that $A \sim B$ and hence, taking account that $\mu H \gg 1$, that the wave is top trapped. We thus have a wave trapped at the bottom boundary, propagating in the positive x-direction against the flow, and another trapped at the top, propagating in the negative x-direction against the flow.

It is useful to consider the case where there is only a lower boundary. Then, putting $\psi' = \mathrm{Re}(e^{ikx - ikct + ily}\hat{\psi})$, it again follows from the potential vorticity equation that $\hat{\psi}_{zz} -$

$\mu^2\hat{\psi} = 0$, where again μ has been defined by $\mu = N_0(k^2 + l^2)^{1/2}/f_0$. Requiring $\hat{\psi}$ to be bounded as $z \to \infty$ gives that $\hat{\psi} = Ae^{-\mu z}$. Substituting in the lower boundary condition, $kc\hat{\psi}_z + \Lambda\hat{\psi}k = 0$, it follows that the dispersion relation is

$$c = \frac{\Lambda}{\mu}.$$

So we have an eastward travelling wave trapped at the lower boundary when there is positive vertical shear (or equivalently, a poleward increase in density). The propagation mechanism for such a wave may be understood in terms of the circulation induced by a surface density change. It is appropriate to regard such a wave as a Rossby wave, but propagating on a surface density gradient, rather than on an interior potential vorticity gradient.

Returning to the case with two boundaries it can be argued that the instability mechanism results from the phase locking of two such waves (one trapped on the top boundary, the other on the bottom boundary) in a configuration such that the velocity field associated with one tends to increase the amplitude of the other, and vice versa. The fact that instability occurs only for sufficiently small μH is because at larger values of μH the interaction between the boundary waves is too weak. This mechanism is discussed in detail in many papers and textbooks, e.g. in §9.7.2 of Vallis (2017).

Finally we examine the density transport by the growing wave. The relevant quantity is the density flux in the y-direction, $\overline{\rho' v'}$ where represents an x-average. This can be written in terms of the quasi-geostrophic streamfunction,

$$\overline{\rho' v'} = -\frac{\rho_0 f_0}{g}\overline{\psi'_x\psi'_z} = -\tfrac{1}{2}\frac{\rho_0 f_0}{g}\mathrm{Re}\left(ik\hat{\psi}\hat{\psi}^*_z\right) = -\tfrac{1}{2}\frac{\rho_0 N^2}{g f_0}\mathrm{Re}(ik\mu(A^*B - AB^*))$$

$$= \frac{k\mu}{f_0}\frac{d\rho_s}{dz}\mathrm{Im}(AB^*) = \frac{k\mu}{f_0}\frac{d\rho_s}{dz}\mathrm{Im}\left(AA^*\frac{c^*\mu + \Lambda}{c^*\mu - \Lambda}\right) = \frac{2k\mu^2}{f_0}\frac{d\rho_s}{dz}\frac{|A|^2\Lambda\mathrm{Im}(c)}{|c^*\mu - \Lambda|^2} < 0$$

So the density flux is negative, i.e. light fluid is transported in the positive y-direction and heavy fluid in the negative y-direction, tending to weaken the y-gradient of density and hence release some of the potential energy in the background state.

Thus in a growing disturbance, on a basic state where there is a positive density gradient from equator to pole, there is a poleward flux of light fluid and an equatorward flux of heavy fluid. It follows that there is release of potential energy from the basic state and this is one criterion that is used to identify the instability as *baroclinic instability*. However, the release of energy from the basic state again does not seem to be a complete explanation for the instability. After all, we know that the presence of rotation tends to inhibit the conversion from potential energy to kinetic energy, so just because the potential energy is there does not mean that instability must result.

The Eady problem was first studied as a paradigm for the waves that provide day-to-day and week-to-week variations in the atmospheric flow associated with weather systems. We may insert realistic atmospheric values into the parameters of the Eady problem, setting H to be 10km, which is the approximate height of the tropopause (the notional boundary between the troposphere and the stratosphere). (The static stability is much larger in the stratosphere than in the troposphere, so the tropopause might in some ways act as a rigid lid for the tropospheric circulation.) It follows that $NH/f_0 \simeq 1000$km, $f_0\Lambda/N \simeq 5 \times 10^{-5}\text{s}^{-1}$ (taking $\Lambda \simeq 5\text{ms}^{-1}\text{km}^{-1}$), giving maximum growth rates of about 1 per day and the wavelength of the fastest growing mode about 4000 km. This seems in reasonable accordance with observed weather disturbances. (See JRT lectures for further comments on the atmospheric and oceanic cases.)

[See **Slides** showing numerical simulations of baroclinic instability including the nonlinear regime where the amplitude of disturbances is too large to be described by linearised theory and other nonlinear mechanisms are important.]

13 Wave mean-flow interaction

13.1 Definitions

We consider means/averages taken over x. We use the notation $\overline{()}$ to be the x-average, e.g.

$$\overline{\chi} = \frac{1}{L}\int_0^L \chi\, dx$$

where the extent in x of the flow domain is $0 \leq x < L$. We typically assume periodicity in x, e.g. motivated by atmospheric or circumpolar ocean geometry, with x corresponding to longitude. Then we define $\chi' = \chi - \overline{\chi}$, with χ' often being called the 'wave' or 'eddy' part. If the mean state is the background state then this matches use of $()'$ in §8. (But note that the mean state may evolve in time, whereas in §8 the background state has generally been a specified time-independent flow.)

Note that $\overline{(\chi_x)} = 0$ (following from periodicity in x). This may be used to derive results such as $\overline{\theta_x \psi} = \overline{(\theta\psi)_x} - \overline{\theta\psi_x} = -\overline{\theta\psi_x}$.

Note also that $\overline{()}$ is an Eulerian average, i.e. it is taken at fixed values of y and z.

For convenience, in this and the following Section the $\tilde{}$ (tilde) notation for ρ and p will be dropped, i.e. ρ and p will be interpreted as, respectively, the variation of density and pressure away from the state with density $\rho_s(z)$.

13.2 Key facts about wave propagation and wave activity

(1) *Dispersion relations*: For plane waves, i.e. waves with sinusoidal structure in the spatial coordinates x, y and z, or some subset of those coordinates, the dispersion relation gives the frequency ω as a function of the spatial wavenumber $\mathbf{k}$. The phase velocity $\mathbf{c}_p$, with components ω/k_i and the group velocity $\mathbf{c}_g$, with components $\partial\omega/\partial k_i$ follow from the dispersion relation. Use of these various quantities requires a scale separation between the waves and the background state, with the length scale of variation of the background state being much larger than the wavelength.

See **Slides** for a summary of the dispersion relations for Rossby waves in a single-layer fluid and in a continuously stratified fluid, with background flow at rest. The single-layer result was given in §6.2 and the continuously stratified result follows from a very similar calculation to that in §8.3.

(2) *Wave activity conservation relation*: This is an equation of the form

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla.\mathbf{F} = \mathcal{D}$$

where $\mathcal{A}$ is a wave-activity density, $\mathbf{F}$ is a flux and $\mathcal{D}$ is a term representing dissipation of wave activity, associated with physical processes that are dissipative or otherwise non-conservative. (If the term on the right-hand side is non-zero for non-dissipative/conservative flows, as is the case for wave energy, for example, then the equation would *not* be a wave activity conservation relation.)

The following is a derivation of a wave activity relation from the quasi-geostrophic equations, which describes the propagation and dissipation of Rossby waves.

Start with the quasi-geostrophic potential vorticity equation (10.1) linearised about a basic state flow in the x-direction. We use $\overline{u}$ for the x-component of the velocity and $\overline{q}_y$ for the corresponding gradient of quasi-geostrophic potential vorticity in the y direction. Note that this implies that the instantaneous x-mean can be considered as a basic state, whereas in writing down (10.1) it was assumed that the basic state flow was a self-consistent steady solution of the equations. This apparent inconsistency can be resolved by assuming that any time evolution of quantities such as $\overline{u}$ and $\overline{q}$ is slow.

Re-write (10.1) in the form

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\overline{q}_y = \mathcal{D}'$$

where $\mathcal{D}'$ represents the effect of dissipation on q'. Now multiply by $q'/\overline{q}_y$, assume that $\overline{q}_y$ is varying slowly in time to give and take the x-mean to give

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\frac{\overline{q'^2}}{\overline{q}_y}\right\} + \overline{v'q'} = \frac{\overline{q'\mathcal{D}'}}{\overline{q}_y}. \tag{13.1}$$

Now exploit what is sometimes called the generalised Taylor identity which follows from multiplying q' by v':

$$\begin{aligned} v'q' &= \psi'_x\left(\psi'_{xx} + \psi'_{yy} + (\psi'_z\frac{f_0^2}{N^2})_z\right) \\ &= \left(\tfrac{1}{2}\psi'^2_x\right)_x + \left(\psi'_x\psi'_y\right)_y - \left(\tfrac{1}{2}(\psi'_y)^2\right)_x + \left(\psi'_x\psi'_z\frac{f_0^2}{N^2}\right)_z - \left(\tfrac{1}{2}(\psi'_z)^2\frac{f_0^2}{N^2}\right)_x. \end{aligned}$$

Taking the x-mean of the above gives an expression for $\overline{v'q'}$ which can be substituted into (13.1) to give

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\frac{\overline{q'^2}}{\overline{q}_y}\right\} + \frac{\partial}{\partial y}\left\{-\overline{u'v'}\right\} + \frac{\partial}{\partial z}\left\{-\frac{gf_0}{\rho_0 N^2}\overline{v'\rho'}\right\} = \frac{\overline{q'\mathcal{D}'}}{\overline{q}_y}. \tag{13.2}$$

Note that the terms within the y and z derivatives have been re-expressed in terms of u', v' and ρ'.

The above equation has the structure

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial F^{(y)}}{\partial y} + \frac{\partial F^{(z)}}{\partial z} = \mathcal{D}_{\mathcal{A}}$$

which expresses the fact that $\overline{\mathcal{A}}$ is the density of a quantity that can be transported by a flux with components $\overline{F}^{(y)}$ and $\overline{F}^{(z)}$ in the y- and z-directions and destroyed or created in situ at a rate $\overline{\mathcal{D}_{\mathcal{A}}}$ per unit volume. The quantities appearing in this wave activity relation are often called the *Eliassen-Palm wave activity* and the *Eliassen-Palm flux*.

A wave activity conservation relation need not require a scale-separation assumption, but it is helpful if the conservation relation is consistent with the cases where there is scale-separation, in the sense that it then satisfies the *group-velocity property* $\langle\mathbf{F}\rangle = \langle\mathcal{A}\rangle\mathbf{c}_g$, where $\langle . \rangle$ denotes a phase average. (This is a non-trivial requirement, since the flux $\mathbf{F}$ appearing in the conservation relation is unique only up to addition of non-divergent vectors $\mathbf{G}$. The group velocity property can be helpful in resolving this non-uniqueness.) The Eliassen-Palm wave activity and flux satisfy the group-velocity property.

(3) *Eddy fluxes and wave propagation*: The flux $\mathbf{F}$ associates correlations between different wave or eddy quantities with directions of wave propagation. Important results are that for the Eliassen-Palm flux the y-component $\overline{F^{(y)}} = -\overline{u'v'}$ and the z-component $\overline{F^{(z)}} = -gf_0\overline{v'\rho'}/N^2\rho_0$ implying that the eddy flux in the y-direction of the x-component of momentum, $\overline{u'v'}$, satisfies

$$\overline{u'v'} \begin{cases} < 0 & \text{for northward (group) propagation} \\ > 0 & \text{for southward (group) propagation} \end{cases}$$

and that the eddy flux in the y direction of density, $\overline{v'\rho'}$, satisfies

$$\overline{v'\rho'} \begin{cases} < 0 & \text{for upward (group) propagation} \\ > 0 & \text{for downward (group) propagation.} \end{cases}$$

[Note that the results given above relating components of the Eliassen-Palm flux to eddy fluxes and hence eddy fluxes to the direction of group propagation hold for Rossby waves under the usual small-Ro assumption and also assuming that the y-gradient of potential vorticity in the basic state is positive as is always the case in the real atmosphere or ocean if this gradient is dominated by β. For other waves, e.g. Poincaré waves, internal gravity waves or equatorial waves (see later) the results will be different.]

13.3 Mean-flow evolution equations and eddy forcing terms

Using the division into 'mean' and 'eddy' parts, we may apply the averaging operator to the Boussinesq β-plane primitive equations to give

$$\overline{u}_t + (\overline{uv})_y + (\overline{uw})_z - \overline{v}(f_0 + \beta y) = 0 \tag{13.3}$$

$$\overline{v}_t + \left(\overline{v^2}\right)_y + (\overline{wv})_z + (f_0 + \beta y)\overline{u} = -\frac{\overline{p}_y}{\rho_0} \tag{13.4}$$

$$-\overline{p}_z - \overline{\rho} g = 0 \tag{13.5}$$

$$\overline{v}_y + \overline{w}_z = 0 \tag{13.6}$$

$$\overline{\rho}_t + (\overline{\rho v})_y + (\overline{\rho w})_z = 0 \tag{13.7}$$

[Note that to get from (7.4a)-(7.4e) to the above it is necessary to use the non-divergence of the velocity field (7.4e). The simplest approach is to re-write Du/Dt as $u_t + (u^2)_x + (uv)_y + (uw)_z$ (and do the same thing for Dv/Dt and $D\rho/Dt$)].

The definition of the averaging operator, together with (13.6), implies that

$$(\overline{uv})_y + (\overline{uw})_z = (\overline{u}\ \overline{v})_y + (\overline{u}\ \overline{v})_z + \left(\overline{u'v'}\right)_y + \left(\overline{u'w'}\right)_z = \overline{v}\ \overline{u}_y + \overline{w}\ \overline{u}_z + (\overline{u'v'})_y + (\overline{u'w'})_z$$

(Note once again that the primes are now used solely to denote disturbances from the x-average, not the pressure and density perturbations associated with the Boussinesq approximations. The primes are dropped from the latter quantities and ρ_0 is still the constant background density.)

Now applying small Rossby number scaling as in §7.5, dividing horizontal velocities into geostrophic and ageostrophic parts and noting that $\overline{v}_g$ is zero, it follows that

$$\overline{u}_t - f_0\overline{v}_a = -(\overline{u'v'})_y \tag{13.8}$$

$$f_0\overline{u} = -\frac{\overline{p}_y}{\rho_0} \tag{13.9}$$

$$-\overline{p}_z - \overline{\rho}g = 0 \tag{13.10}$$

$$\overline{v}_{ay} + \overline{w}_{az} = 0 \tag{13.11}$$

$$\overline{\rho}_t + \overline{w}_a\frac{d\rho_s}{dz} = -(\overline{\rho'v'})_y. \tag{13.12}$$

The above are a coupled set of equations for the five Eulerian-mean quantities $\overline{u}_t$, $\overline{\rho}_t$, $\overline{p}$, $\overline{v}_a$ and $\overline{w}_a$. (Note no subscript on u or v implies that this means the geostrophic part of the flow.)

There are two 'eddy forcing' terms, $-(\overline{u'v'})_y$ and $-(\overline{\rho'v'})_y$, determined respectively by the eddy momentum flux and the eddy density (or heat) flux. The response in each of $\overline{u}_t$ and $\overline{\rho}_t$ depends on some combination of these two terms. In some cases both forcing terms may be non-zero but the response in each of $\overline{u}_t$ and $\overline{\rho}_t$ may be zero.

13.4 The transformed Eulerian mean equations

We now make a transformation defined as follows:

$$\overline{w}_a^* = \overline{w}_a + \frac{\left(\overline{\rho'v'}\right)_y}{d\rho_s/dz} \tag{13.13}$$

and then define $\overline{v}_a^*$ such that

$$\overline{w}_{az}^* + \overline{v}_{ay}^* = 0, \tag{13.14}$$

Hence

$$\overline{v}_a^* = \overline{v}_a - \frac{\partial}{\partial z}\left[\frac{\overline{\rho'v'}}{d\rho_s/dz}\right]. \tag{13.15}$$

The x-momentum and density equations then become

$$\overline{u}_t - f_0\overline{v}_a^* = -\left(\overline{u'v'}\right)_y + \left(\frac{f_0\overline{\rho'v'}}{d\rho_s/dz}\right)_z = \nabla\cdot\overline{\mathbf{F}} \tag{13.16}$$

$$\overline{\rho}_t + \overline{w}_a^*\frac{d\rho_s}{dz} = 0 \tag{13.17}$$

These, together with (13.9), (13.10) and (13.14) are the so-called transformed Eulerian mean equations.

The transformation has combined the two separate eddy forcing terms into a single eddy forcing term $\nabla.\overline{\mathbf{F}}$ in the x-momentum equation and has removed the eddy forcing terms in the density equation.

The vector $\overline{\mathbf{F}}$ has components $(0, -\overline{u'v'}, f_0\overline{\rho'v'}/(d\rho_s/dz))$ and is identical to the flux vector that appears in the Eliassen-Palm wave activity relation. Here $\nabla.\overline{\mathbf{F}}$ is interpreted as the force acting on the mean flow, due to the eddies.

Note that the transformation introduced above does not imply a different response in $\overline{u}_t$ or $\overline{\rho}_t$. But the result of the transformation is that the eddy density flux $\overline{\rho'v'}$ appears to play quite a different role. In the standard Eulerian-mean formalism $\overline{\rho'v'}$ appears as a forcing term in the density equation. In the transformed-Eulerian-mean formalism $\overline{\rho'v'}$ appears as part of the force, in fact $f_0\overline{\rho'v'}/(d\rho_s/dz)$ appears to act as a vertical momentum flux. Therefore in the transformed Eulerian-mean formalism, just as horizontally propagating Rossby waves transfer momentum in the horizontal, vertically propagating Rossby waves can be considered to transfer momentum in the vertical.

The Eulerian-mean circulation and the transformed Eulerian-mean circulation satisfy different boundary conditions – e.g. if $(\overline{\rho'v'})_y$ is non-zero at a lower boundary. See §14 for more details.

13.5 Non-acceleration conditions

We have the divergence of the mean Eliassen-Palm flux $\nabla.\overline{\mathbf{F}}$ appearing as the complete eddy forcing on the mean flow. We also have the Eliassen-Palm wave activity conservation relation

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \mathbf{F} = \mathcal{D} \tag{13.18}$$

(where again $\mathcal{D}$ representing non-conservative effects) and hence that

$$\nabla.\overline{\mathbf{F}} = \overline{\mathcal{D}} - \frac{\partial \overline{\mathcal{A}}}{\partial t}.$$

If follows that if $\partial\mathcal{A}/\partial t = 0$, i.e. the waves are steady, and $\mathcal{D} = 0$, i.e. there are no dissipative or other non-conservative effects acting, then $\nabla.\overline{\mathbf{F}} =$ and hence $\overline{u}_t = 0$, i.e. there is no acceleration of the mean flow. Such a result is often called a *non-acceleration theorem.*

The importance of non-acceleration theorems is that they focus attention on what is needed for there to be a mean flow acceleration. [Many early calculations assumed steady non-dissipative waves and then, after significant work, deduced that $\overline{u}_t = 0$.]

Note that what is referred to as wave dissipation here may correspond to a range of physical effects. It might be that explicitly dissipative processes such as viscosity or some other frictional effect, thermal (or density) diffusion or other thermal damping act on their own. It might be that the waves *break*, i.e. they become strongly nonlinear, generating turbulence and thereby enhancing explicitly dissipative processes that would have otherwise have been weak.

Under quasi-geostrophic scaling it may be shown that

$$\nabla \cdot \overline{\mathbf{F}} = \overline{v'q'}, \tag{13.19}$$

the quantity on the right-hand side being the northward flux of quasi-geostrophic potential vorticity. Indeed a different (but consistent approach) to quantifying the effect of the eddies on the mean flow would be via the quasi-geostrophic PV equation, which would take the form

$$\overline{q}_t + \frac{\partial}{\partial y}(\overline{v'q'}) = 0. \tag{13.20}$$

The mean acceleration $\overline{u}_t$ and the rate of change of mean density $\overline{\rho}_t$ could then be deduced by applying the appropriate inversion operator. However, one advantage of using the transformed Eulerian-mean equations is that the response in the mean circulation $(\overline{v}_a^*, \overline{w}_a^*)$, which is often itself of interest, is visible.

13.6 Wave dissipation

To illustrate some of the consequences of the results presented previously, consider two-dimensional flow on a β-plane (i.e. governed by the quasi-geostrophic potential vorticity equation for a single-layer, with the deformation radius $L_D \to \infty$), with waves propagating in the y-direction. In this case the term *absolute vorticity* is often used for the sum of relative vorticity plus βy which is conserved by fluid elements in the absence of dissipation.

First consider a case where there is no dissipation (so that $\mathcal{D} = 0$). If a wave packet propagates through some region, then as it arrives $\partial\mathcal{A}/\partial t > 0$ and hence $\nabla.\overline{\mathbf{F}} < 0$, as it leaves $\partial\mathcal{A}/\partial t < 0$ and hence $\nabla.\overline{\mathbf{F}} > 0$. The force on the mean flow in this region is first negative, but then positive, so that the time-integrated force is zero. The net effect of the waves on the mean flow is zero.

For there to be a net effect in some region then the waves must arrive but not leave, e.g. if the waves dissipate in that region. In that case as the wave arrives $\partial\mathcal{A}/\partial t > 0$ and hence $\nabla.\overline{\mathbf{F}} < 0$, as before. But then the wave dissipates, with $\partial\mathcal{A}/\partial t = \mathcal{D} < 0$ and hence $\nabla.\overline{\mathbf{F}} = 0$. Therefore a net negative force is possible.

Example: Rossby wave critical layer

A useful example of wave dissipation and the resulting effect on the mean flow is provided by the case of the forced Rossby wave on a shear flow $U(y)$. The equation describing the waves is (10.1) and if it is assumed that $\psi' = \mathrm{Re}(\hat{\psi}(y)e^{ik(x-ct)})$, i.e. the waves have x-wavenumber k and phase speed c then the equation for $\hat{psi}(y)$ is

$$(U - c)(\hat{\psi}_{yy} - k^2\hat{\psi}) + (\beta - U_{yy}) = 0. \tag{13.21}$$

The locations where $U = c$ are called *critical lines*. The sign of $\hat{\psi}_{yy}/\hat{\psi}$ changes from one side of the critical line to the other, so that one side there is propagation and on the other evanescence. Also the equation is singular at $U = c$ implying there is no possible steady linear non-dissipative balance. Therefore other processes must be included to describe the dynamics which lead to wave dissipation. The small but finite region about the critical line in which (13.21) is insufficient and other processes become important is called the *critical layer*. In a configuration, depicted in Figure 1, where the waves are generated some distance away from the critical line and propagate towards it then the dissipation of the waves in the critical layer will imply a systematic force on the flow in that region.

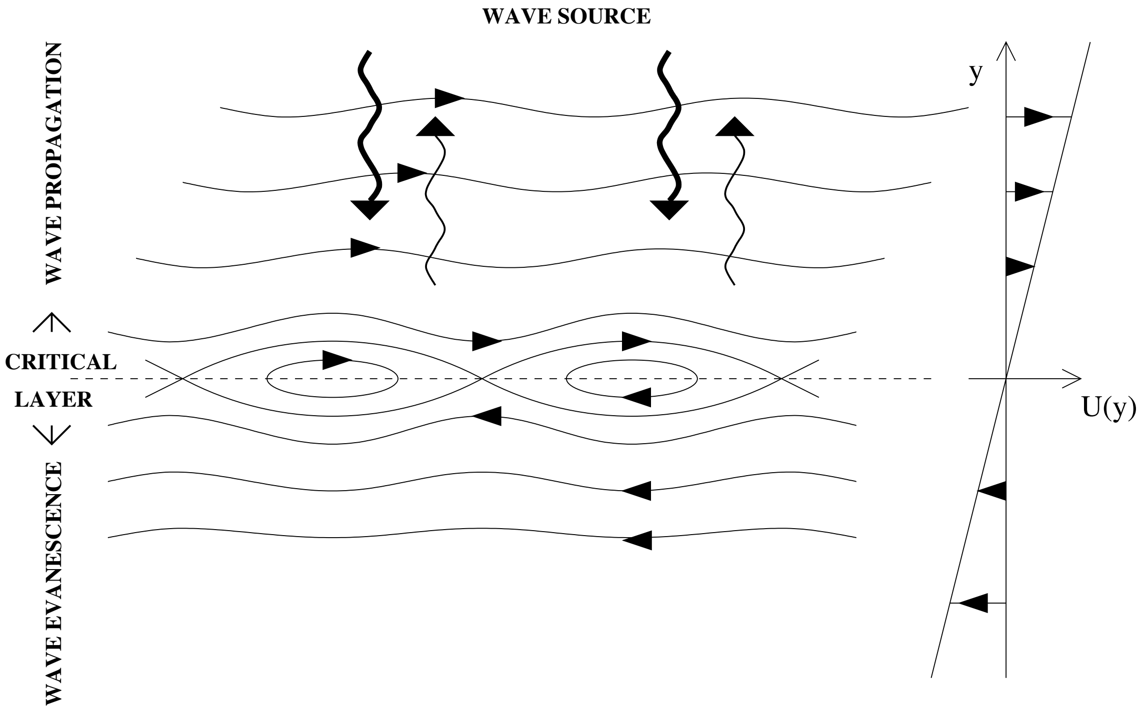

Figure 1: Schematic diagram of Rossby-wave propagation on a shear flow $U(y)$ with a critical line. The flow is positive (i.e. eastward) in $y > 0$ (upper portion of the diagram) and negative (i.e. westward) in $y < 0$ (lower portion of the diagram). The waves are forced, with zero phase speed in x-direction, in $y > 0$ and propagate towards $y = 0$. In $y < 0$ the waves are evanescent (i.e. non-propagating and decaying as y becomes more negative). The critical line is at $y = 0$, where $U(y) = 0$. In the neighbourhood of $y = 0$ the streamlines are closed and form a Kelvin's cat's eye pattern. The width of the closed streamline region, which increases as the wave amplitude increases, defines the width of the nonlinear critical layer. If dissipation were strong enough then dissipative effects would dominate over a relatively broad region near $y = 0$ and the closed streamlines would essentially be irrelevant to the dynamics. (The critical layer would then be linear and dissipative, rather than nonlinear.) There may be some reflected wave in $y > 0$, but the amount of reflection can be determined only by considering the detailed dynamics of the critical layer.

Example: rearrangement of absolute vorticity

When nonlinearity is an important part of the processes taking place in the critical layer then the effect may be understood in terms of rearrangement of the pre-existing absolute vorticity profile. (This is case that might be described as 'wave breaking'.) In the model problem considered above and depicted in Figure 1, the rearrangement occurs within a *nonlinear critical layer* centred on $y = 0$ through advection around the 'cat's eye' streamlines.

A very simple model might be that before the waves arrive the absolute vorticity is simply equal to βy everywhere and after the waves have arrived the absolute vorticity is 'mixed' within some region $|y| < \delta$ to be equal to zero (i.e. the average value in $|y| < \delta$ prior to the arrival of the waves). Thus the change in relative vorticity $\Delta\zeta = -\beta y$ in $|y| < \delta$ and, assuming that the change in the x-component of velocity Δu is zero outside of $|y| < \delta$ it follows that $\Delta u = \frac{1}{2}\beta(y^2 - \delta^2)$ in $|y| < \delta$. The change in the x-component of momentum within the region $|y| < \delta$ is therefore

$$\int_{-\delta}^{\delta} \Delta u \, dy = -\tfrac{2}{3}\beta\delta^3.$$

Note that the rearrangement of absolute vorticity within a localised region does not locally conserve momentum – an external force is required – but we expect that the force is supplied by the transport of momentum by the waves.

[See **Slide** showing example of advective rearrangement of absolute vorticity and corresponding change in mean flow. More details of the Rossby wave critical layer problem are given in the 'PHH critical layers article' available on the course Moodle page.]

13.7 Summary

Two important general principles about wave propagation and wave mean-flow interaction are:

(i). There can be 'long-range' transfer of momentum. Propagating waves transfer momentum from the region where they are generated to the region where they dissipate or break.

(ii). Dissipating or breaking waves change the potential vorticity distribution in the region where the dissipation or breaking occurs. This change is not usually consistent with local conservation of momentum in this region, but it is consistent with the long-range transfer of momentum by the waves into or out of the region.

In the above we have established that $\overline{u}_t = -(\overline{u'v'})_y$ (and we have a corresponding expressions for $\overline{u}_t$ and $\overline{\rho}_t$ in the 3-D case), but the most difficult part of the wave mean-flow interaction problem is to predict the dependence of $\overline{u'v'}$ on $\overline{u}$. Ideally we would have $-(\overline{u'v'})_y = \mathcal{F}[\overline{u}]$ (where the right-hand side might well be a non-local function of $\overline{u}$ – i.e. $\mathcal{F}$ at a particular value of y might depend on $\overline{u}$ at many values of y) and could then solve a self-contained equation for the evolution of $\overline{u}$. $\mathcal{F}$ would have to incorporate the effects of flow-dependent wave propagation and flow-dependent transport of potential vorticity. Certainly in many cases down-gradient diffusion of momentum would be a very poor model for $\mathcal{F}$ – for example if Rossby waves were being generated where the flow was strong and positive and dissipated where the flow was weak then there would be a wave activity flux from the strong-flow region to the weak-flow region and hence a momentum flux from the weak-flow region to the strong-flow region – i.e. momentum transport *up* the gradient of momentum.

The above arguments are particularly important in understanding the effect of baroclinic instability on the mean flow. Recall from the discussion of the Eady problem in §10 that $\overline{\rho' v'} < 0$ for the growing wave. It is straightforward to show that $\overline{u'v'} = 0$. We now recognise this as implying Eliassen-Palm flux that is purely upward. In more complicated basic states with eastward jet-like structure in the y-direction $\overline{u'v'} \neq 0$ and the pattern of $\overline{u'v'}$ implies wave-activity flux out of the jet and hence flux of (eastward) horizontal momentum into the jet. This implies that momentum fluxes are up-gradient. This can be explained on the basis that the growing instability corresponds to a source of wave activity within the jet and hence wave propagation out of the jet. [See **Slides** showing example of eddy fluxes in numerical simulation of a 'baroclinic lifecycle'. Another slide shows spontaneous formation of jets in 'β-plane turbulence' as a result of two-way interaction between waves (and hence eddy fluxes) and mean flow.]

14 Mean meridional circulations

14.1 Introduction

We examine in more detail the mean response of the fluid to wave forcing, comparing the Eulerian-mean viewpoint, expressed by the coupled equations (13.8-13.12), and the transformed Eulerian-mean viewpoint expressed by the coupled equations (13.16, 13.17, 13.14, 13.9, 13.10).

Each of the velocity fields $(\overline{v}_a, \overline{w}_a)$ and $(\overline{v}^*_a, \overline{w}^*_a)$ represents a circulation in the 'meridional' (y, z) plane. Since each is non-divergent we may define stream functions $\overline{\chi}_a$ and $\overline{\chi}^*_a$ such that

$$(\overline{v}_a, \overline{w}_a) = (\overline{\chi}_{az}, -\overline{\chi}_{ay}) \qquad \text{and} \qquad (\overline{v}^*_a, \overline{w}^*_a) = (\overline{\chi}^*_{az}, -\overline{\chi}^*_{ay}).$$

It may be shown from the Eulerian-mean and transformed Eulerian-mean equations that

$$f_0^2 \overline{\chi}_{azz} + N^2 \overline{\chi}_{ayy} = f_0 (\overline{u'v'})_{yz} - \frac{g}{\rho_0} (\overline{\rho' v'})_{yy} = -f_0 \overline{F^{(y)}}_{yz} + \frac{N^2}{f_0} \overline{F^{(z)}}_{yy} \tag{14.1}$$

and

$$f_0^2 \overline{\chi}^*_{azz} + N^2 \overline{\chi}^*_{ayy} = f_0 (\overline{u'v'})_{yz} - f_0^2 \left(\frac{\overline{\rho' v'}}{d\rho_s/dz} \right)_{zz} = -f_0 (\nabla . \overline{\mathbf{F}})_z, \tag{14.2}$$

where $\overline{\mathbf{F}} = (0, \overline{F^{(y)}}, \overline{F^{(z)}})$ is the x-averaged EP flux.

These equations express the forcing of the mean meridional circulation by the eddy fluxes of momentum and density (or, equivalently, by the EP flux divergence).

14.2 Boundary conditions

The equations (14.1) and (14.2) require boundary conditions on $\overline{\chi}_a$ or $\overline{\chi}_a^*$. The side boundary condition is usually straightforward, but the bottom boundary condition needs to be derived with care, particularly when the bottom boundary is not flat. Consider for example the case where there is topographic forcing at the lower boundary, so that the full nonlinear boundary condition is

$$w = \frac{Dh}{Dt} \quad \text{at} \quad z = h,$$

where $h(x, y, t)$ is the topographic height.

If the topography is small amplitude a Taylor series expansion may be used to express the full boundary condition in terms of quantities at $z = 0$ and it follows that

$$w + hw_z = h_t + uh_x + vh_y + u_z hh_x + v_z hh_y + O(h^3),$$

where all quantities on the right-hand side are evaluated at $z = 0$.

The continuity equation may be used to rewrite the last term on the left-hand side as

$$\begin{aligned} hw_z &= -hu_x - hv_y \\ &= -(hu)_x - (hv)_y + h_x u + h_y v \end{aligned}$$

and substituting into the previous equation it follows that

$$w = h_t + u_z hh_x + v_z hh_y + (hu)_x + (hv)_y + O(h^3).$$

Now assuming that the velocity in the absence of topography is purely in the x-direction, so that $v = O(h)$, and that $\overline{h} = 0$ for all t, it follows on taking x-averages that , at leading order, replacing $\overline{w}$ by $\overline{w}_a$ as consistent with quasi-geostrophic scaling,

$$\overline{w}_a(y, 0, t) = (\overline{h'v'})_y.$$

Thus the Eulerian mean velocity is not necessarily zero at $z = 0$. Note that the corresponding boundary condition on the transformed Eulerian mean circulation is that

$$\overline{w}_a^*(y, 0, t) = (\overline{h'v'})_y + \frac{(\overline{\rho'v'})_y}{d\rho_s/dz}.$$

If the waves are steady, and there is no density dissipation at the lower boundary, then the lower boundary condition may be written in the form

$$\overline{u} h'_x \frac{d\rho_s}{dz} = -\overline{u}\rho'_x - \psi'_x \overline{\rho}_y.$$

Multiplying by ψ' and averaging, it follows after some manipulation that $f_0 \overline{h'\psi'_x} = -\overline{F^{(z)}}$ and hence that the lower boundary conditions on $\overline{w}_a$ and $\overline{w}_a^*$ may be written as

$$\overline{w}_a = -f_0^{-1}(\overline{F^{(z)}})_y \quad \text{and} \quad \overline{w}_a^* = 0 \quad \text{on} \quad z = 0.$$

Note that under the above conditions it is $\overline{w}_a^*$ and not $\overline{w}_a$ that is zero at the lower boundary.

14.3 Model problem

Consider a specific example in the small-Ro regime, a flow confined to a β-plane channel with rigid walls at $y = 0$ and $y = L$, with waves forced by topographic perturbations of the lower boundary, of the form $h = \mathrm{Re}(h_0 e^{ikx} \sin \pi y/L)$. The basic state flow is assumed to be in the x-direction, and the velocity a function of height $u_0(z)$.

The details of the vertical variation of the waves depend on $u_0(z)$ and on the buoyancy frequency N. If the latter is only a function of z, as usual in quasi-geostrophic theory, then ψ' may be written in the form $\psi' = \mathrm{Re}(\hat{\psi}(z)e^{ikx} \sin \pi y/L)$ and it immediately follows that $\overline{\psi'_x \psi'_y} = 0$, so that the EP flux is purely vertical and

$$\overline{F^{(z)}} = \frac{f_0^2}{N^2}\overline{\psi'_x\psi'_z} = \frac{f_0^2}{N^2}\mathrm{Im}(k\hat{\psi}(z)^*\hat{\psi}'(z))\sin^2(\frac{\pi y}{L}) = F_0\Theta(z)\sin^2(\frac{\pi y}{L})$$

where the constant F_0 and the function $\Theta(z)$ are defined by the last equality. Note that to be consistent with the basic properties of Rossby waves, for upward propagation the product $F_0\Theta(z)$ should be positive.

$\Theta(z)$ is determined by solving the equation for $\hat{\psi}(z)$ given the z variation of the background state and of any dissipative processes. To avoid this complication of solving this equation we shall assume a simple form for $\Theta(z)$, that

$$\Theta(z) = \begin{cases} 1 & z < H_d \\ 0 & z > H_d. \end{cases}$$

This is a simple representation of a situation where the waves are generated a long way below $z = H_d$, propagate upwards and then dissipate in a very thin layer localised around $z = H_d$.

Now the problem has reduced to solving (14.1) or (14.2) with a given forcing term on the right-hand side and with specified boundary conditions. Note that the assumed form for $\overline{F^{(z)}}$ gives a forcing term in (14.1) that remains non-zero as $z \to -\infty$, but a forcing term in (14.2) which tends to zero as $z \to \pm\infty$. Therefore is most straightforward to solve (14.2) and seek a solution $\overline{\chi}^*$ that tends to zero as $z \pm \infty$. Once $\overline{\chi}^*$ is known, $\overline{\chi}$ can be straightforwardly deduced.

The boundary conditions in y are that $\overline{\chi}_{az} = \overline{\chi}^*_{az} = 0$ on $y = 0$ and $y = L$ and it is therefore natural to expand the forcing and the solution in sine Fourier series, writing

$$\overline{F^{(z)}} = F_0\,\Theta(z)\sum_{n=1} c_n \sin(\frac{n\pi y}{L})$$

$$\overline{\chi}^*_a = \sum_{n=1}^{\infty} \overline{\chi}^*_n(z)\sin(\frac{n\pi y}{L})$$

where the c_ns are simply the coefficients in the Fourier series for $\sin^2(\frac{\pi y}{L})$.

(14.2) then implies that each $\overline{\chi}^*_n(z)$ satisfies the ordinary differential equation

$$f_0^2\overline{\chi}^{*\prime\prime}_n(z) - \frac{N^2\pi^2n^2}{L^2}\overline{\chi}^*_n(z) = f_0F_0c_n\delta'(z - H_d),$$

where $\delta(.)$ is the Dirac delta function. The solution satisfying the boundary conditions $\overline{\chi}_n^* \to 0$ as $z \to \pm\infty$ is

$$\overline{\chi}_n^*(z) = \frac{1}{2}\frac{F_0 c_n}{f_0}\exp(-\frac{N\pi n}{Lf_0}|z - H_d|)\,\mathrm{sgn}(z - H_d).$$

All flow variables can now be deduced from the above. Using the notation $(.)_n$ to denote the nth coefficient in the Fourier series, it follows that

$$(\overline{v}_a^*)_n = -\frac{1}{2}\frac{F_0 c_n}{f_0}\frac{N\pi n}{Lf_0}\exp(-\frac{N\pi n}{Lf_0}|z - H_d|) + \frac{F_0 c_n}{f_0}\delta(z - H_d),$$

and, noting that $(\nabla.\overline{\mathbf{F}}) = -F_0 c_n \delta(z - H_d)$, that

$$(\overline{u}_t)_n = -\frac{1}{2}F_0 c_n \frac{N\pi n}{Lf_0}\exp(-\frac{N\pi n}{Lf_0}|z - H_d|).$$

Similar expressions may be derived for the Fourier coefficients for $\overline{w}_a^*$ and for $\overline{\rho}_t$. In particular note that $\overline{\chi}_a = \overline{\chi}_a^* + f_0^{-1}\overline{F^{(z)}}$.

Schematic pictures of the response in various quantities to the eddy forcing specified above are shown in the Figure below. (A good qualitative approximation to the response is obtained by considering only the $n = 1$ Fourier coefficients.)

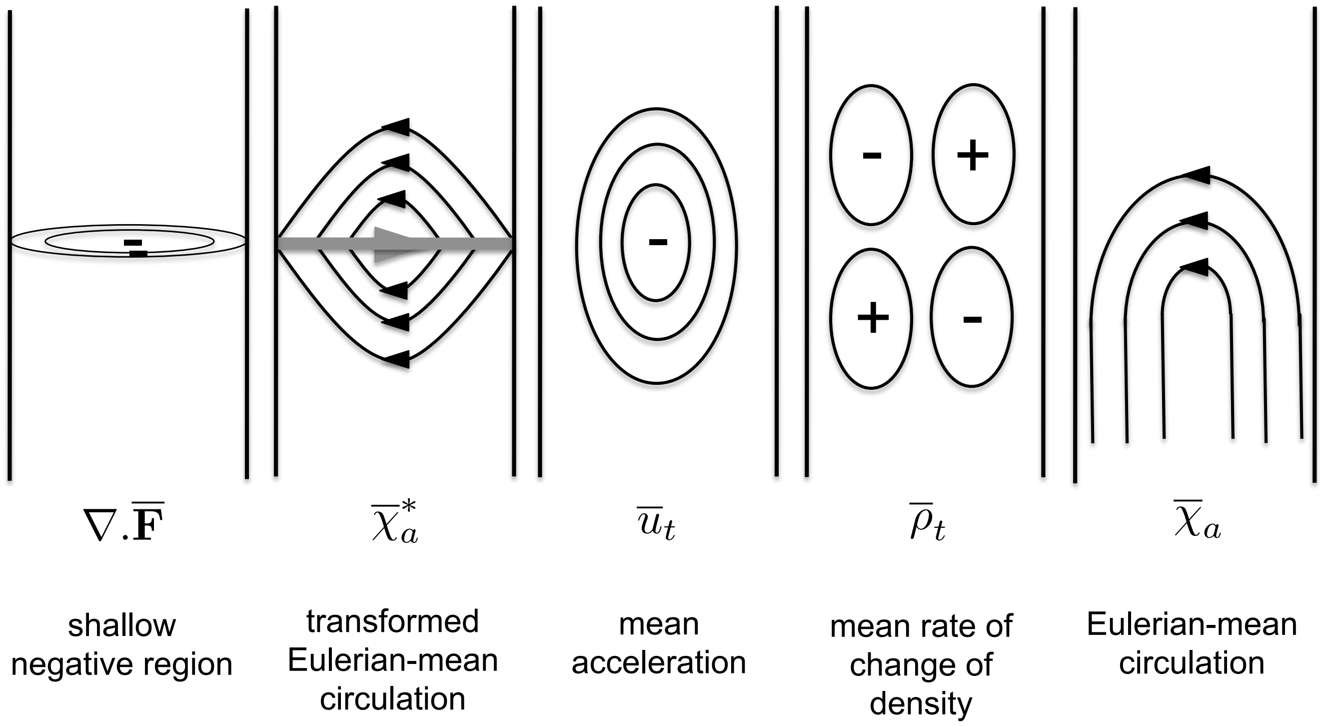

Figure 2: Schematic diagram showing response of the transformed Eulerian-mean circulation, the mean acceleration and mean rate of change of density and the Eulerian-mean circulation, to an eddy forcing represented by a shallow layer of negative $\nabla.\overline{\mathbf{F}}$.

Note the contrast between the responses in the transformed Eulerian-mean circulation and the Eulerian-mean circulation and the implications for the balance in the x-momentum and density equations.

Transformed Eulerian-mean view: In the *wave propagation* region below $z = H_d$ the vertical velocity $\overline{w}_a^*$ is zero and there is vertical transport of momentum via $\overline{F}^{(z)}$. In the *wave dissipation* region centred on $z = H_d$ there is a localised wave force $\nabla.\overline{\mathbf{F}}$ which is redistributed in the vertical by the meridional circulation $(\overline{v}_a^*, \overline{w}_a^*)$.

Eulerian-mean view: In the *wave propagation* region below $z = H_d$ the vertical velocity $\overline{w}^*$ is non-zero. The effect of $\overline{w}^*$ on the mean density field is cancelled by the effect of the eddy flux $\overline{\rho' v'}$. There is vertical transport of planetary angular momentum (because the fluid moving upwards on one side of the channel has a different value of planetary angular momentum to that moving downwards on the other side of the channel). In the *wave dissipation* region there is a latitudinal velocity $\overline{v}_a^*$ which provides a Coriolis force and hence leads to acceleration.

The transformed Eulerian-mean view is arguably simpler because it removes the cancellation between the effect of vertical advection by the mean flow and the effect of eddy density fluxes and also because it combines the eddy fluxes into a single forcing term. Additionally it may be shown that the transformed Eulerian-mean flow is more relevant to the transport of tracers (e.g. chemicals). Vertical motion in the Eulerian-mean circulation does not imply corresponding vertical motion of tracers, e.g. the upward motion at high latitudes in the 'Ferrel Cell' does not imply that tracers are transported upwards.

The transformed Eulerian-mean formalism can be interpreted as an approximation to taking averages not at fixed z, but over very thin layers between neighbouring density surfaces. The fact that the thickness and the z-position of the layer are both variable affects the calculated average. Momentum can be exchanged between neighbouring layers by pressure forces acting on their boundaries. This is examined in more detail in an Example Sheet question.

14.4 Dependence of response on vertical scale of $\nabla.\overline{\mathbf{F}}$

In the model problem above $\nabla.\overline{\mathbf{F}}$ is non-zero only in a layer with very small vertical scale. Suppose instead that this 'forcing layer' has vertical scale D. Then in order-of-magnitude terms (14.2) implies

$$\max\{\frac{f_0^2\overline{\chi}_a^*}{D^2}, \frac{N^2\overline{\chi}_a^*}{L^2}\} \sim \frac{f_0 F_0}{D}.$$

The *shallow* forcing regime is when $ND/f_0L \ll 1$. Then $f_0^2\overline{\chi}_a^*/D^2 \sim f_0F_0/D$, hence $\overline{\chi}_a^* \sim F_0D/f_0$ and $\overline{v}_a^* \sim F_0/f_0$. The dominant balance in the momentum equation within the 'forcing layer' is therefore that most of $\nabla.\overline{\mathbf{F}}$ is balanced by the Coriolis force. The mean meridional circulation redistributes in the vertical the effect of $\nabla.\overline{\mathbf{F}}$ and the resulting acceleration occurs over a region that is much deeper than the forcing layer. (This is the situation shown the schematic Figure.)

The *deep* forcing regime is when $ND/f_0L \gg 1$. Then $N^2\overline{\chi}_a^*/L^2 \sim f_0F_0/D$, hence $\overline{\chi}_a^* \sim (f_0L/ND)^2F_0D/f_0$ and $\overline{v}_a^* \sim (f_0L/ND)^2F_0/f_0$. The Coriolis force therefore plays only a minor role in the momentum equation within the 'forcing layer' and at each level $\nabla.\overline{\mathbf{F}}$ is balanced by the x-component of the mean acceleration.

These different regimes can be illustrated by replacing $\Theta(z)$ in the example described in the previous section with a simple function that varies from 1 to 0 over some finite vertical scale.

The mean meridional circulation may be regarded as arising in order to maintain, under the effect of eddy forcing, the constraints of geostrophic and hydrostatic balance . Thus if a force is applied to a rotating system, the response cannot appear purely as an acceleration, but there must be an accompanying change in the density field. Broadly speaking, if a force is deep, in coordinates scaled by Prandtl's ratio, then most of the response will appear as acceleration, but if it is shallow, then most of the response will appear as a meridional circulation and hence a density change. Similarly, if an applied heating field is shallow, then most of the response appears as a change in temperature or density, but if it is deep, then most will appear as a meridional circulation, and hence as a change in velocity.

[See **Slides** showing Eulerian-mean and transformed Eulerian-mean circulations in the Southern Ocean and in the atmosphere. In each case the Eulerian-mean flow shows regions where there is vertical motion, but where the effect of this on the density/temperature is largely cancelled by the eddy fluxes.]

15 Equatorial waves

We have previously considered the wave motion that is possible when rotation and stratification co-exist and the shallow-water model provided a good context in which to do that. What emerged from the analysis of the shallow-water model was that when f is constant, f_0 say, (and the background state is at rest) the system supports Poincaré waves and, in the presence of boundaries, Kelvin waves. When f is allowed to be non-constant, e.g. via the β-plane assumption $f = f_0 + \beta y$ then, if the Rossby number Ro $\ll 1$ the system supports Rossby waves. More generally Rossby waves are allowed by a variation in background state potential vorticity. This might be due to the β-effect or it might be due, in the shallow-water system, to variation in the height of the lower boundary, which gives rise to topographic Rossby waves.

In very simple terms, a typical frequency of Poincaré waves is f_0 and a typical frequency of Rossby waves is βL_D where L_D is the deformation radius $(gH)^{1/2}/f_0$. In the case where the deformation radius L_D is much less than the radius of the Earth a (which is the regime assumed by quasi-geostrophic theory) then Rossby wave frequencies are much smaller than Poincaré wave frequencies. Corresponding the Rossby wave phase speed βL_D^2 is much less than the Kelvin wave phase speed $f_0 L_D$. On this basis Poincaré and Kelvin waves are sometimes described as 'fast waves' and Rossby waves are sometimes described as 'slow waves'.

This distinction between 'fast waves' and 'slow waves' becomes much less clear at low latitudes. Note, for example, that $\beta L_D \sim f_0$ when the distance from the equator is comparable to L_D. Therefore dynamics at low latitudes requires a different analysis from much of what has been presented in earlier parts of the course.

15.1 Horizontal structure and horizontal propagation

The shallow-water equations are a convenient model on which to base a study of low-latitude dynamics. The latitudinal variation of the Coriolis parameter turns out to be important in the dynamics and the $\beta-$plane approximation is a convenient way to include this. The equatorial $\beta-$plane approximation is that the Coriolis parameter $f = \beta y$ (i.e. $f_0 = 0$).

Then the shallow-water equations linearised about a state of rest are:

$$u_t - \beta y v = -g\eta_x \tag{15.1}$$

$$v_t + \beta y u = -g\eta_y \tag{15.2}$$

$$\eta_t + H\{u_x + v_y\} = 0 \tag{15.3}$$

where u and v are respectively horizontal velocity components in x and y directions, η is free surface displacement, H is the layer depth in the resting state and g is the gravitational acceleration.

The equations contain two dimensional parameters, β and $c = (gH)^{1/2}$ and these can be used to form time and length scales $T_{eq} = (c\beta)^{-1/2}$ and $L_{eq} = (c/\beta)^{1/2}$. L_{eq} is usually called the equatorial deformation radius, i.e. it is the analogue of the extratropical $L_D = c/f_0$.

Taking $\partial/\partial x$ (15.2) - $\partial/\partial y$ (15.1) - $\beta y \times$ (15.3) gives that

$$\frac{\partial}{\partial t}(v_x - u_y - \frac{\beta y \eta}{H}) + \beta v = \frac{\partial}{\partial t}(\zeta - \frac{\beta y \eta}{H}) + \beta v = 0, \tag{15.4}$$

where $\zeta = v_x - u_y$ is the relative vorticity, which expresses conservation of potential vorticity (in the linearised approximation).

Now consider $\partial/\partial t$ (15.2) - $\beta y \times$ (15.1) giving

$$v_{tt} + \beta^2 y^2 v = -g\eta_{yt} + \beta y g \eta_x.$$

Now substitute for η_t from (15.3) to give

$$v_{tt} + \beta^2 y^2 v = gH(u_{xy} + v_{yy}) + \beta y g \eta_x = c^2(u_y - v_x + \frac{\beta y \eta}{H})_x + c^2(v_{xx} + v_{yy}).$$

Note that the first term on the right hand side includes the potential vorticity associated with the disturbance, therefore differentiate with respect to time and substitute from (15.4) to give

$$v_{ttt} + \beta^2 y^2 v_t - c^2(v_{xx} + v_{yy})_t - \beta c^2 v_x = 0. \tag{15.5}$$

This equation is similar in structure to equations obtained for the extratropics. For example, if βy is replaced by f_0 and then the remaining β is replaced by zero then the resulting equation is simply that for Poincaré waves plus a zero-frequency wave (the latter corresponding to the fact that the potential vorticity is uniform).

The procedure now is to seek plane wave solutions of the form $v = \mathrm{Re}(\hat{v}(y)e^{i(kx-\omega t)})$ with k and ω respectively the constant x-wavenumber and frequency, where the function $\hat{v}(y)$ is bounded as $|y| \to \pm\infty$. Substituting for v and then dividing by ω gives

$$(\frac{\omega^2}{c^2} - \frac{\beta^2 y^2}{c^2} - k^2 - \frac{\beta k}{\omega})\hat{v} + \hat{v}_{yy} = 0. \tag{15.6}$$

This second-order ordinary differential equation, plus the boundary conditions at $|y| \to \pm\infty$, define an eigenvalue problem which is the same, for example, as that for the energy levels of the harmonic oscillator potential in quantum mechanics. The eigenvalue condition is that

$$\omega^2 - c^2 k^2 - \frac{\beta k c^2}{\omega} = (2n+1)\beta c \qquad \text{for } n = 0, 1, 2, \ldots \tag{15.7}$$

and the corresponding eigenfunctions are

$$\hat{v}_n(y) = H_n(y(\beta/c)^{1/2}) \exp(-y^2\beta/2c) \tag{15.8}$$

where the $H_n(.)$ are the Hermite polynomials, with $H_0(s) = 1$, $H_1(s) = 2s$, $H_2(s) = 4s^2 - 2$, etc.

In order to consider further the dispersion relation defined by (15.7) it is helpful to non-dimensionalise, writing $\omega = (\beta c)^{1/2}\hat{\omega}$ and $k = (\beta/c)^{1/2}\hat{k}$. Then

$$\beta c\hat{\omega}^2 - \beta c\hat{k}^2 - \beta c^2(\frac{\beta}{c})^{1/2}\hat{k}\frac{1}{(\beta c)^{1/2}}\frac{1}{\hat{\omega}} = (2n+1)\beta c$$

and hence

$$\hat{\omega}^2 - \hat{k}^2 - \frac{\hat{k}}{\hat{\omega}} = (2n+1). \tag{15.9}$$

This is a quadratic equation for $\hat{k}$, given $\hat{\omega}$, with roots

$$\hat{k} = -\frac{1}{2\hat{\omega}} \pm \sqrt{\hat{\omega}^2 + \frac{1}{4\hat{\omega}^2} - (2n+1)}.$$

At this point it is useful that for any pair $(\hat{k}, \hat{\omega})$ there is a corresponding pair $(-\hat{k}, -\hat{\omega})$. These two pairs do not represent different wave modes. They are required in order to generate a real solution of the equations. Therefore it is necessary to consider only half of the $(\hat{k}, \hat{\omega})$ plane and the convention is to consider only $\hat{\omega} > 0$, but to allow $\hat{k}$ of either sign.

Returning to the solution of the quadratic equations, for $n = 0$ this gives $\hat{k} = \hat{\omega} - 1/\hat{\omega}$ or $\hat{k} = -\hat{\omega}$. The second root turns out to be non-physical. (Consider the corresponding equations for u, v and η and deduce that there is exponential growth away from the equator. Therefore for $n = 0$ the only possibility is

$$\hat{k} = \hat{\omega} - \frac{1}{\hat{\omega}}.$$

Note that it follows that $\hat{\omega} > 1$ corresponds to $\hat{k} > 0$ and $0 < \hat{\omega} < 1$ corresponds to $\hat{k} < 0$.

For $n = 1, 2, \ldots$ there are real roots for $\hat{k}$ only if $\hat{\omega}^4 - (2n+1)\hat{\omega}^2 + 1/4\hat{\omega}^2 > 0$, implying that either $\hat{\omega}^2 < n + \frac{1}{2} - \sqrt{n(n+1)}$ or that $\hat{\omega}^2 > \sqrt{n(n+1)} + n + \frac{1}{2}$. This implies a 'frequency gap' which increases in size as n increases.

The above covers all solutions of the equations with $v \neq 0$. It turns out that there are interesting solutions with $v = 0$ and of course these have been missed by the approach above, which eliminates u and η to leave a single equation (15.5) for v. To consider this possibility it is simplest to return to the original equations (15.1), (15.2) and (15.3) and investigate the consequences of setting $v = 0$. The equations are

$$u_t = -g\eta_x \tag{15.10}$$

$$\beta y u = -g\eta_y \tag{15.11}$$

$$\eta_t + Hu_x = 0. \tag{15.12}$$

(15.10) and (15.12) together require that $\eta_{tt} - c^2\eta_{xx} = 0$, suggesting solutions of the form $[u, \eta] = \mathrm{Re}([\hat{u}(y), \hat{\eta}(y)]e^{i(kx-\omega t)})$ where $\omega^2 = c^2k^2$ and $\hat{u} = (gk/\omega)\hat{\eta}$. Then substituting into (15.11), which expresses geostrophic balance in the y direction, implies $(\beta y k/\omega)\hat{\eta} = -\hat{\eta}_y$. This implies exponential decrease in $|y|$ if $k/\omega > 0$ and exponential increase in $|y|$ if $k/\omega < 0$. The former is allowed and the latter is not, therefore $\omega^2 = c^2k^2$ reduces to $\omega = ck$. The functions $\hat{u}$ and $\hat{\eta}$ both have a Gaussian form in y, with e.g.

$$\hat{\eta}(y) \propto \exp(-\beta y^2/2c).$$

This wave is the *equatorial Kelvin wave* analogous to the boundary Kelvin wave on an f-plane. The propagation is only in the positive x-direction, i.e. only eastward. There is no latitudinal velocity and the balance in the latitudinal momentum equation is geostrophic, between the Coriolis force associated with u and the pressure gradient associated with η_y.

Note that the relation $\omega = kc$, or equivalently $\hat{\omega} = \hat{k}$ corresponds to a root of (15.9) when $n = -1$. Therefore this wave is sometimes described as the $n = -1$ mode.

Summary

This completes the dispersion relation for equatorial wave modes.

For $n = 1, 2, \ldots$, there are for each $\hat{\omega} > \sqrt{n(n+1)} + n + \frac{1}{2}$ two values of $\hat{k}$. These are high frequency waves, essentially inertio-gravity/Poincaré waves. There are also for each $\hat{\omega} < n + \frac{1}{2} - \sqrt{n(n+1)}$ two values of $\hat{k}$. There are low frequency equatorial Rossby waves.

For $n = 0$ the dispersion relation is $\hat{\omega} - 1/\hat{\omega} = \hat{k}$, with $\hat{\omega} \sim \hat{k}$ as $\hat{k} \to \infty$ ('gravity-like') and $\hat{\omega} \sim 1/\hat{k}$ as $\hat{k} \to -\infty$ ('Rossby-like'). The $n = 0$ mode is therefore often called a (mixed) Rossby-gravity wave. (It is also sometimes called a Yanai wave.)

For $n = -1$ the dispersion relation is $\hat{\omega} = \hat{k}$. This is an equatorial Kelvin wave.

[See **Slides** showing (ω, k) dispersion diagram for equatorial waves.]

15.2 Horizontal propagation in the real atmosphere or ocean

The above detailed analysis has been for a single-layer fluid. As was previously noted in §8 for Rossby waves, the theory may be applied to continuously stratified fluid by assuming a vertical structure of the disturbances such that

$$u(x,y,z,t) = \tilde{u}(x,y,t)P(z), v(x,y,z,t) = \tilde{v}(x,y,t)P(z), p(x,y,z,t) = (\rho_0/g)\tilde{\eta}(x,y,t)P(z)$$

where

$$\frac{d}{dz}\left(\frac{1}{N^2(z)}\frac{dP}{dz}\right) = -\frac{1}{gh}P = -\frac{1}{c^2}P. \qquad (15.13)$$

The constant $gh = c^2$ is an eigenvalue, possible values of which are determined by the equation together with the boundary conditions in the vertical. The horizontal structure and propagation characteristics of the disturbances are then the same as those for a shallow-water system with layer depth h. (The constant h is therefore often called the 'equivalent depth'.)

In some configurations, e.g. 'ocean-like' or 'troposphere-like' there is a discrete set of eigenvalues, with each corresponding to a different speed c and to a different vertical structure, e.g. for the ocean the 1st baroclinic modes, the 2nd baroclinic mode, etc. The same ideas would be relevant to the atmosphere if the tropopause (the boundary between troposphere and stratosphere) were viewed as a rigid lid on the basis that the buoyancy frequency is much larger in the latter than the former. (A similar argument is used to justify the Eady model presented in §8.4 as relevant to the extratropical atmosphere.) This might be a good approach for describing tropospheric weather systems, but of course if vertical propagation of waves from troposphere to stratosphere were of interest then a different approach would required.

For the equatorial ocean the 1st baroclinic mode has $c \sim 2$ ms^{-1}, equivalent to $h \sim 0.4$ m.

The corresponding equatorial deformation scale is

$$L_{eq} = (\frac{c}{\beta})^{1/2} = (\frac{2}{2.3 \times 10^{-11}})^{1/2}\text{m} \sim 3 \times 10^5\text{m} \sim 3\,\text{degrees of latitude}$$

The time scale $T_{eq} = L_{eq}/c \sim 1.5$ days.

For the tropical troposphere it appears that $c \sim 25$ ms^{-1} is a reasonable fit to the observations implying

$$L_{eq} = (\frac{c}{\beta})^{1/2} = (\frac{25}{2.3 \times 10^{-11}})^{1/2}\text{m} \sim 10^6\text{m} \sim \text{ 10 degrees of latitude}$$

The time scale $T_{eq} = L_{eq}/c \sim 0.4$ days.

Note that an estimate of c based on dry dynamics and the depth of the tropical troposphere would be about 50 ms^{-1}, i.e. substantially greater than the value suggested by tropical observations. This is potentially explained by the important role of moisture in tropical dynamics, e.g. the fact that upward motion tends to lead to condensation and hence to internal heating, which can be argued to reduce the effective static stability from the 'dry' value. There is ongoing research into the subject of *convectively coupled equatorial waves*.

[See **Slides** showing examples of oceanic equatorial waves and space-time spectra of tropical cloudiness, showing good correspondence with dispersion diagram if gravity-wave speed is suitably chosen. A further slide shows complicated, but structured, time-longitude variation of tropical cloudiness.]

15.3 The forced problem

We now consider the response of the equatorial atmosphere or ocean to a specified forcing. The problem to be considered was motivated by the observed longitudinal structure of the tropical atmosphere and considers the response of the atmosphere to a specified heating. But the methods could be applied more generally.

Following the previous section, we start with the shallow-water equations on a equatorial β-plane as a model, expecting that the layer thickness H can be chosen appropriately for the tropical atmosphere. Heating is represented as a specified source term $g^{-1}Q(x, y, t)$ on the right-hand side of the mass continuity equation. (The g^{-1} factor is simply included for algebraic convenience.) For simplicity the *long-wave approximation* is made, assuming that the length-scale of x variation is much larger than the equatorial deformation radius L_{eq}. This implies that the v_t term may be neglected in the y-momentum equation, i.e. that there is geostrophic balance in the y direction. The equations are therefore:

$$u_t - \beta y v = -g\eta_x \tag{15.14}$$

$$\beta y u = -g\eta_y \tag{15.15}$$

$$\eta_t + H\{u_x + v_y\} = g^{-1}Q(x, y, t) \tag{15.16}$$

The unforced form of these equations allow the waves described by the full equatorial dispersion relation, confined to small values of $\hat{k}$ and $\hat{\omega}$, i.e. Rossby waves and Kelvin waves (but not the mixed Rossby-gravity waves or the inertio-gravity waves).

Now introduce the new variables q and r with $q = gc^{-1}\eta + u$ and $r = gc^{-1}\eta - u$. In terms of the new variables the equations (15.14), (15.15) and (15.16) transform to:

$$q_t + cq_x + cv_y - \beta yv = \frac{1}{c}Q \tag{15.17}$$

$$r_t - cr_x + cv_y + \beta yv = \frac{1}{c}Q \tag{15.18}$$

$$cq_y + \beta yq = \beta yr - cr_y \tag{15.19}$$

Now write the three variables q, r and v and the forcing Q as sums of the latitudinal eigenfunctions (15.8), i.e.

$$[v(x,y,t), q(x,y,t), r(x,y,t), Q(x,y,t)] = \sum_{n=0}^{\infty}[v_n(x,t), q_n(x,t), r_n(x,t), Q_n(x,t)]\tilde{D}_n(Y) \tag{15.20}$$

where the $\tilde{D}_n(Y) = H_n(Y)\exp(-\frac{1}{2}Y^2)$, with $Y = y(\beta/c)^{1/2}$, are the eigenfunctions identified in §13.1, with the properties $\tilde{D}_n' + Y\tilde{D}_n = 2n\tilde{D}_{n-1}$ and $\tilde{D}_n' - Y\tilde{D}_n = -\tilde{D}_{n+1}$.

Then substituting the series (15.20) into the equations (15.17), (15.18) and (15.19) gives

$$\frac{\partial q_0}{\partial t} + c\frac{\partial q_0}{\partial x} = \frac{1}{c}Q_0 \tag{15.21}$$

$$q_1 = 0 \tag{15.22}$$

and, for $n = 1, 2, \ldots$

$$\frac{\partial q_{n+1}}{\partial t} + c\frac{\partial q_{n+1}}{\partial x} - (c\beta)^{1/2}v_n = \frac{1}{c}Q_{n+1}, \tag{15.23}$$

$$\frac{\partial r_{n-1}}{\partial t} - c\frac{\partial r_{n-1}}{\partial x} + 2n(c\beta)^{1/2}v_n = \frac{1}{c}Q_{n-1}, \tag{15.24}$$

$$2(n+1)q_{n+1} = r_{n-1}. \tag{15.25}$$

The variables v_n and r_{n-1} may be eliminated from the last three equations to give:

$$(2n+1)\frac{\partial q_{n+1}}{\partial t} - c\frac{\partial q_{n+1}}{\partial x} = \frac{1}{c}(nQ_{n+1} + \tfrac{1}{2}Q_{n-1}). \tag{15.26}$$

From the above structure it can be seen that the variable q_0 is associated with the Kelvin wave ((15.21) allows a wave travelling in the positive x-direction with speed c) and the variables q_{n+1}, v_n and r_{n-1}, for $n = 1, 2, \ldots$ are associated with the n-th Rossby wave ((15.26) allows a wave travelling in the negative x-direction with speed $c/(2n+1)$).

The response to a forcing $Q(x,y)$ that is non-zero in a localised region and switched on at $t = 0$ wiull therefore consist (if $Q_0 \neq 0$) of a Kelvin wave propagating eastward away from the forcing region and a set of Rossby waves propagating westward away from the forcing region. As currently defined the solution will not tend to a steady state. A simple way to allow a steady state solution is to add a identical linear damping to each of the equations (15.14) and (15.15), i.e. to replace the $\partial/\partial t$ by $\partial/\partial t + \alpha$. The q_n variables then satisfy the equations:

$$\alpha q_0 + c\frac{dq_0}{dx} = \frac{1}{c}Q_0 \tag{15.27}$$

$$(2n+1)\alpha q_{n+1} - c\frac{dq_{n+1}}{dx} = \frac{1}{c}(nQ_{n+1} + \tfrac{1}{2}Q_{n-1}), \qquad (n = 1, 2, \dots). \tag{15.28}$$

All other information about the solution can be obtained from the q_n, the r_n from (15.25) and the v_n from the steady form of (15.23).

The addition of damping means that the wave response to a localised forcing $Q(x, y)$ decays with distance away from the forcing. The solution to (15.27) represents a Kelvin wave which appears only to the east of the forcing region and decays in amplitude with distance away from it, with decay scale c/α. Note that this part of the response is excited only by Q_0, i.e. by the part of $Q(x, y)$ that projects onto $\tilde{D}_0$. The solution to (15.28) represents a set of Rossby waves which appear only to the west of the forcing region and, again, decay with distance away from it. The larger the value of n the smaller the propagation speed and therefore the more rapid the decay. The lowest value of n, $n = 1$, gives a propagation speed of $\frac{1}{3}c$, therefore the fastest Rossby wave decays to the west on a scale $\frac{1}{3}c/\alpha$, i.e. one third of the decay scale of the Kelvin wave to the west. Note that the nth Rossby wave (corresponding to the variables q_{n+1}, r_{n-1} and v_n) is excited by Q_{n-1} and Q_{n+1}. A forcing for which Q_0 is the only component therefore excites both a Kelvin wave and the $n = 1$ Rossby wave.

The model presented above, the Matsuno-Gill model, is now considered a classical model for the '1st baroclinic mode' of the tropical troposphere, i.e. a circulation driven by a heating maximising in mid-troposphere in which there is ascent or descent in mid-troposphere, and the upper-level horizontal flow is in the opposite direction to the low-level horizontal flow. If the horizontal velocity in the shallow-water system is interpreted as representing the low-level horizontal flow, then the mid-tropospheric vertical velocity corresponds to the convergence of the low-level horizontal flow, i.e. in the steady-state system described above, to $-u_x - v_y = \alpha\eta + Q$.

The solution for a localised positive forcing with Q_0 component only shows eastward low-level flow to the west of the forcing region (associated with the Rossby-wave component) and a westward low-level flow to east (associated with the Kelvin-wave component). The Rossby-wave low level flow has cyclonic flow on either side of the equator (anti-clockwise to the north and clockwise to the south). Correspondingly at upper levels there are anticyclones on either side of the equator.

For the tropical troposphere the heating is primarily latent heating associated with precipitation. There is relatively large precipitation over the Indonesian and West Pacific regions and the above model therefore provides a simple representation of the latitude-longitude structure of the flow in this region.

[See **Slides** showing simple solutions of the Matsuno-Gill model and observed structure of tropical tropospheric circulation.]

Whilst the Matsuno-Gill model is important and often-cited, some aspects of it remain mysterious. For example, the assumption of constant damping rate α, acting on horizontal velocities and on η, which may be regarded as representing temperature, is difficulty to justify physically. Damping of temperature is a simple representation of radiative effects, damping of horizontal velocities is a simple representation of frictional effects, which are plausible near the surface, but less plausible at upper levels.

15.4 Vertical propagation

To consider vertical propagation, assume that the vertical structure is oscillatory, and for convenience that the buoyancy frequency N is independent of z. Then the vertical structure is defined by a constant vertical wavenumber m. The vertical structure equation (15.13) implies that the horizontal structure implied by the single-layer models will apply provided that $gh = c^2 = N^2/m^2$. The dispersion relation (15.7) becomes

$$\omega^2 - \frac{N^2k^2}{m^2} - \frac{\beta k N^2}{\omega m^2} = (2n+1)\frac{\beta N}{|m|} \qquad \text{for } n = -1, 0, 1, 2, \ldots \tag{15.29}$$

where the previous analysis of the Kelvin wave has been exploited to allow the $n = -1$ case, and, as was noted in the previous analysis, only one of the roots of this equation (regarded as a quadratic equation for k) will be allowed in each of the $n = -1$ and $n = 0$ cases. Note that the $|m|$ is required on the right-hand because it was assumed in deriving (15.7) that c was positive.

It is helpful in analysing (15.29) to define the non-dimensional x- and z- wavenumbers $\hat{k}$ and $\hat{m}$ by $\hat{k} = k\omega/\beta$ and $\hat{m} = m\omega^2/\beta N$. (Note that this non-dimensionalisation is different to that used in §13.1.) (15.29) then becomes

$$|\hat{m}|^2 - (2n+1)|\hat{m}| - \hat{k}^2 - \hat{k} = 0, \qquad n = 1, 2, \ldots \tag{15.30a}$$

$$|\hat{m}| - \hat{k} - 1 = 0, \qquad n = 0 \tag{15.30b}$$

$$|\hat{m}| - \hat{k} = 0 \qquad n = -1. \tag{15.30c}$$

These equations define curves in $\hat{k}$-$\hat{m}$ space, on which ω may be determined in terms of k and m. The curves are symmetric about $\hat{m} = 0$, i.e. if $\hat{m}$ is a solution for given $\hat{k}$ then $-\hat{m}$ is also a solution. (15.30a) has the solutions

$$|\hat{m}| = M_{\pm}(\hat{k}) = (n + \tfrac{1}{2}) \pm [(\hat{k} + \tfrac{1}{2})^2 + n(n+1)]^{1/2}. \tag{15.31}$$

According to (15.30ab,c) the M_- solution is considered only for $n \geq 1$. The requirement that $|\hat{m}| > 0$ implies that for $n \geq 1$ the minus sign is relevant only when $-1 < \hat{k} < 0$.

The group velocity of the different waves can be deduced from the relation between $\hat{m}$ and $\hat{k}$. For simplicity assume that $m > 0$ (and therefore that $\hat{m} > 0$), then (15.31) implies that

$$\frac{m\omega^2}{\beta N} = M_{\pm}(\frac{k\omega}{\beta}) \tag{15.32}$$

Now take partial derivatives with respect to k and with respect to m.

$$2\frac{|m|\omega}{\beta N}\frac{\partial\omega}{\partial k} = M'_{\pm}(\frac{k\omega}{\beta})[\frac{\omega}{\beta} + \frac{k}{\beta}\frac{\partial\omega}{\partial k}] \tag{15.33}$$

$$\frac{\omega^2}{\beta N} + \frac{2m\omega}{\beta N}\frac{\partial\omega}{\partial m} = M'_{\pm}(\frac{k\omega}{\beta})\frac{k}{\beta}\frac{\partial\omega}{\partial m}. \tag{15.34}$$

Now rearrange (and substitute for m) to give the components of the group velocity as

$$\frac{\partial\omega}{\partial k} = \frac{\omega^2}{\beta}\frac{M'_{\pm}(\hat{k})}{2M_{\pm}(\hat{k}) - \hat{k}M'_{\pm}(\hat{k})} \tag{15.35}$$

$$\frac{\partial\omega}{\partial m} = -\frac{\omega^3}{\beta N}\frac{1}{2M(\hat{k}) - \hat{k}M'_{\pm}(\hat{k})}. \tag{15.36}$$

Now consider the denominator $2\hat{m} - \hat{k}M'_{\pm}(\hat{k})$ in the above expressions (and recall that $\hat{m} > 0$ is being assumed). Is the denominator ever less than or equal to zero?

First consider $M_+(\hat{k})$. From (15.31) $M_+(\hat{k}) \geq |\hat{k}+\frac{1}{2}|+\frac{1}{2}$. Then $|\hat{k}+\frac{1}{2}|+\frac{1}{2} = |\hat{k}+\frac{1}{2}|+|-\frac{1}{2}| \geq |\hat{k}|$ by the triangle inequality. Also $|M'_+(\hat{k})| = |\hat{k}+\frac{1}{2}|/[(\hat{k}+\frac{1}{2})^2 + n(n+1)]^{1/2} \leq 1$. Hence $M_+(\hat{k}) - \hat{k}M'_+(\hat{k}) \geq M_+(\hat{k}) - |\hat{k}| \geq 0$ and therefore $2M_+(\hat{k}) - \hat{k}M'_+(\hat{k}) \geq M_+(\hat{k})$.

Now consider $M_-(\hat{k})$, noting that this is relevant only for $-1 < \hat{k} < 0$. Consider

$$\frac{d}{d\hat{k}}(M_-(\hat{k}) - \hat{k}M'_-(\hat{k})) = \hat{k}M''_-(\hat{k}) = -\frac{n(n+1)}{[(\hat{k}+\frac{1}{2})^2 + n(n+1)]^{3/2}} < 0.$$

Hence, for $\hat{k} < 0$ $M_-(\hat{k}) - \hat{k}M'_-(\hat{k})$ is a decreasing function of $\hat{k}$ and $M_-(\hat{k}) - \hat{k}M'_-(\hat{k}) > M_-(0) = 0$, implying again that $2M_+(\hat{k}) - \hat{k}M'_+(\hat{k}) \geq M_+(\hat{k})$.

Therefore in $\hat{m} >$, $\partial\omega/\partial k$ has the same sign as $M'_{\pm}(k)$ and $\partial\omega/\partial m$ has the opposite sign to $\hat{m}$, i.e. it is negative. It is straightforward to show that for $\hat{m} < 0$ the sign of $\partial\omega/\partial k$ stays the same and the sign of $\partial\omega/\partial m$ reverses.

In summary, the function $\hat{m} = M_{\pm}(\hat{k})$ specifies the relation between ω, k and m and hence the dispersion relation. Each branch of the function $M_{\pm}(.)$, shown in the figure taken from Vallis (2017) as curves in the $\hat{k}$-$\hat{m}$ plane, corresponds to a different type of equatorial wave. The shape of the curves $M(\hat{k})$ implies the direction of the group velocity, which is indicated by the direction of the arrows in the figure. An important aspect of the behaviour is that the vertical wavenumber m has the opposite sign to the vertical component of the group velocity, i.e. the group velocity is upward for $m < 0$ and downward for $m > 0$.

[See **Slides** showing $(\hat{k}, \hat{m})$ dispersion diagram for equatorial waves (Fig. 17.9 from Vallis 2017). Note that in Vallis (2017) the discussion of the $(\hat{k}, \hat{m})$ diagram asserts that the curves in the diagram are curves of constant frequency and from that makes deductions about the group velocity. I suggest the following clarification. (As above, consider $m > 0$ and deduce results for $m < 0$ by symmetry.)

Consider the variation of ω with k and m. Each branch of the dispersion relation (for each n and for given k and m there are either one, two or three values of ω) corresponds to a set of contours in the (k, m) plane. The contours in the (k, m) plane can be generated by choosing a particular curve in the $(\hat{k}, \hat{m})$ plane and then redrawing it many times in the (k, m) plane, choosing a different value of ω, (and hence different values for k and m) in each case. The group velocity is perpendicular to these contours and in the direction of increasing ω. Given that each contour is a re-scaled version of one of the curves in the $(\hat{k}, \hat{m})$ plane, and taking account of the rescaling required to go from the (k, m) to the $(\hat{k}, \hat{m})$ plane, the rescaled group velocity in the $(\hat{k}, \hat{m})$ plane is indeed perpendicular to the curves. However the sense of the arrows is determined by the direction in which ω increases. It turns out that, whichever choice of curve in the $(\hat{k}, \hat{m})$ plane is used to generate the contours, the value of ω always decreases as m increases. This corresponds to the condition $2\hat{m} - \hat{k}M'_{\pm}(\hat{k}) > 0$ derived above. However note that this condition depends on the details of the curves in the $(\hat{k}, \hat{m})$ plane, i.e. on the properties of the function $M_{\pm}(.)$. The condition does not hold for all functions.]

Example: Seasonal variation in the equatorial ocean

There is seasonal variation of winds at the surface, providing forcing at the annual frequency. Consider the possible $n = 2$ Rossby wave response. It is convenient to return to the dimensional form of the dispersion relation (15.29). Consider the relative sizes of the c^2k^2 and $\beta kc^2/\omega$ terms on the left-hand side. The ratio of the first to the second is $k\omega/\beta$. For the annual frequency $\omega/\beta \sim$ (6400 / 700) km $\sim$ 10 km and if the spatial scale is much larger than this first term can be neglected. The ω^2 term can be neglected on the basis that the frequency is very small. (Actually neglecting this term requires an assumption that $m \ll Nk/\omega$, i.e. that the vertical scale is not too small.) The dispersion relation, for $n = 2$,then reduces to

$$\omega = -\frac{kN}{3|m|} > 0$$

implying that $\partial\omega/\partial k = -\frac{1}{3}N/|m| = \omega/k$ and $\partial\omega/\partial m = \frac{1}{3}Nk\,\mathrm{sgn}(m)/|m|^2 = -\omega/m = \omega^2\,\mathrm{sgn}(m)/Nk$.

Consistent with the general considerations above, the vertical group velocity has the opposite sign to m, therefore for the oceanic response to surface forcing, take $m > 0$ and for consistency with the dispersion relation, take $k < 0$. $\partial\omega/\partial k$ is negative, implying group propagation to the west. Each of the physical fields is proportional to the factor $\exp(i(kx + mz - \omega t)$. Therefore the phase speed is negative in the x direction and positive in the z direction. In an (x, z) plot phase lines therefore propagate westward and upward. Note also that for given frequency, the vertical component of the group speed increases as N reduces, i.e. it will increase with depth. These features are all visible in the observed structure of the seasonal cycle in the equatorial Pacific.

[See bf Slides showing latitudinal and vertical propagation of annual variation communicated by equatorial Rossby waves.]

Example: Atmospheric Kelvin waves

The dimensional form of the dispersion relation for Kelvin waves is $\omega = Nk/|m|$, with $k > 0$. The vertical group velocity is $-Nk\mathrm{sgn}(m)/m^2 = -\omega/m$, so, as predicted by the general theory above, $m < 0$ for upward group propagation. Noting that each physical field is proportional to $\exp(i(kx + mz - \omega t))$ it follows that waves with upward group velocity have downward phase propagation. This is clearly visible in time-height records of, e.g. temperature perturbations in the lower stratosphere associated with Kelvin waves that are forced by convection in the troposphere.

[See **Slides** showing time-height structure of Kelvin waves in the lower stratosphere. In this picture the vertical wavelength is about 4km and the period about 4 days, implying a vertical group velocity of about 1 km/day.]

Just as horizontally propagating Rossby waves can transport momentum in the horizontal, vertically propagating equatorial Kelvin waves can transport momentum in the vertical through the momentum flux $\overline{u'w'}$. (This term was neglected in the quasi-geostrophic approximation applied in the extratropics, but it is of leading-order importance for equatorial Kelvin waves.)

Recall that there is a solution of the three-dimensional equations with $u(x, y, z, t) = \tilde{u}(x, y, t)P(z)$ and $p(x, y, z, t) = \rho_0 g\tilde{\eta}(x, y, t)P(z)$ where $P(z)$ is a solution of the vertical structure equation (15.13) and $\tilde{u}$ and $\tilde{\eta}$ are solutions of the single-layer shallow water equations. Therefore $\rho(x, y, z, t) = -\rho_0 P'(z)\tilde{\eta}$ (from hydrostatic balance) and $w(x, y, z, t) = -(g/N^2)P'(z)\tilde{\eta}_t$ (from the density equation). Now take $P(z) = e^{imz}$ representing a wave with vertical wavenumber m, implying $c = N/|m|$. Combine this with the (x, y, t) structure of the single-layer Kelvin wave solution to give

$$u(x, y, z, t) = \mathrm{Re}(\hat{u}e^{i(kx+imz-kct)}\tilde{D}_0(Y))$$

$$w(x, y, z, t) = (g/N^2)\mathrm{Re}(m\omega\hat{\eta}e^{i(kx+imz-kct)}\tilde{D}_0(Y)) = \mathrm{Re}(-(k/m)\hat{u}e^{i(kx-kct)})D_0(Y)).$$

This implies that

$$\overline{u'w'} = -\tfrac{1}{2}\frac{k}{m}|\hat{u}|^2 \exp(-\beta y^2/c).$$

Equatorial Kelvin waves therefore transport eastward momentum in the direction of their group propagation, i.e. upward propagating waves transport eastward momentum upwards. This is in contrast to Rossby waves, which transport westward momentum in the direction of their group propagation. Correspondingly, where Kelvin waves dissipate there is an eastward force exerted on the mean flow.

This is part of the explanation of the quasi-biennial oscillation – a reversal of the winds in the tropical stratosphere once every 12 months or so. Some waves, e.g. Kelvin waves, drive the flow to the east, others drive the flow to the west. The fact that whether or not these different waves can propagate in the vertical depends on the winds leads to a two-way feedback between waves and mean flow and hence to an oscillation (the period of which is determined by the amplitudes of the waves).