Mechanics Lecture Notes

1 Lecture 1: Statics — equilibrium of a particle

1.1 Introduction

This lecture deals with forces acting on a particle which does not move, i.e. is in *equilibrium.* The important concept is the resolution of forces to obtain the equations determining equilibrium. It is essential when solving such problems to start with a **good diagram** showing all the forces.

The example introduces the idea of friction which, although simple at first sight, turns out to be quite subtle. The idea of limiting friction is introduced: this occurs when a body is just on the point of slipping.

1.2 Key concepts

- Reduction of a number of forces to one *resultant* force by vector addition.
- Condition for equilibrium: the resultant force is zero.
- Resolution of forces in orthogonal directions to determine an unkown force.
- Frictional force; *limiting friction* and relation via the *coefficient of friction* to the normal reaction.

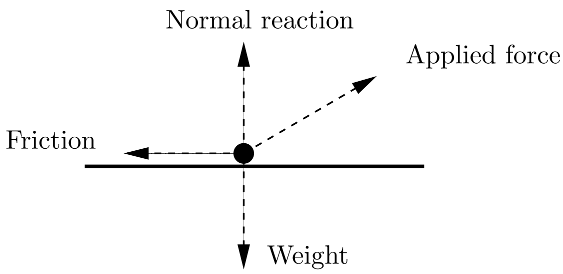
1.3 Forces

We consider here the situation of a stationary particle acted on by a number of forces. It is not very useful to attempt to define exactly what we mean by a *force*: examples of forces will suffice.[1] But we can think of a force as something that tends to produce motion. A force is therefore obviously a vector quantity. Any situation in theoretical physics is described by a mathematical model and the force is part of the model.

There are (as far as is known at present) four fundamental forces: gravity, electromagnetism, weak nuclear force and strong nuclear force. Each force is accompanied by a mathematical model and a set of equations governing the behaviour of the force and objects affected by the force.[2]

All other forces (as far as we know) are derived from these forces. Examples in no particular order are friction, tension in strings, normal reaction forces, air resistance, viscosity, magnetism, gravity, van der Waals forces between molecules, etc. I'm sure that you can think of many more examples.

For example, for a particle on a *rough*[3] horizontal table being pulled by a string (though not hard enough to make the particle move), the forces are as shown in the diagram. There are two external forces, namely the applied (pulling) force acting along the string, and the weight acting downwards. The table exerts two forces on the particle: one is the force of *friction*, which tends to oppose motion; the other is the reaction of the table on the particle that stops the particle falling through the table. This latter force is normal to the surface of the table and is called the *normal reaction.*



[1] It is tempting to use Newton's second law to define force (a force is something that makes a body accelerate), but there is a danger of a circular argument.

[2] Currently accepted mathematical models are General Relativity, Quantum electrodynamics, Electroweak, and Quantum Chromodynamics, respectively. Other theories, such as string theory, attempt to combine these forces.

[3] If the relative surface movement of two objects is subject to resistance the contact between them is called *rough.* Conversely, if there is no resistance the contact is called *frictionless* or *smooth.*

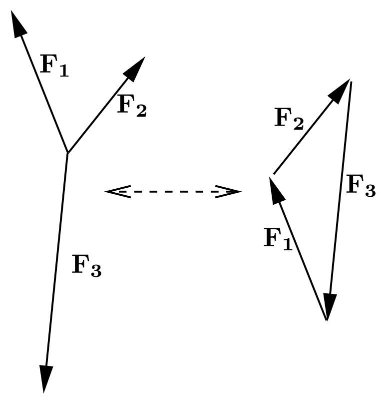
1.4 Equilibrium

A particle or body is said to be *in equilibrium* when all the forces acting on it balance and it is not in motion. Algebraically, this just means that the vector sum of the forces is zero:

$$\sum_i \mathbf{F}_i = \mathbf{0}$$

or, equivalently, the components of the vectors in three directions (which must be linearly independent, of course, but not necessarily orthogonal) sum to zero.

Geometrically, this means that the vectors representing the forces (in both direction and magnitude) can be joined to form a closed polygon.



In order to determine whether a particle would be in equilibrium when acted on by given forces, or in order to determine an unknown force given that the particle is in equilibrium, we have to check that the vector sum of the forces, i.e. the *resultant force*, is zero. That means that the resultant force should have no non-zero component in any direction. Normally, the way to check this is to find the components of the resultant force in three independent directions, which need not be orthogonal but are usually, for convenience, chosen to be orthogonal. This process is called *resolving forces*. It can best be understood in a concrete example.

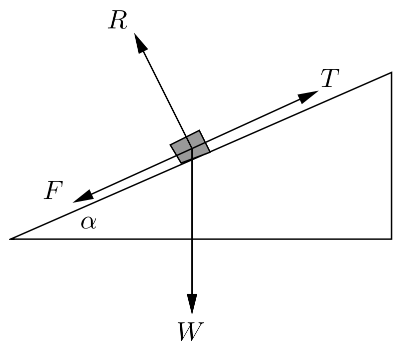
Example

A particle of weight[4] *W lies on a fixed rough plane inclined at angle α to the horizontal. It is held in position by a force of magnitude T acting up the line of greatest slope of the plane. We are going to find the range of values of T for which the particle can be in equilibrium. We will need to find and expression for the frictional force, F, in terms of W, α and T.*

Before anything else, we must draw a good diagram showing all the forces. The importance of a diagram is seen immediately: as soon as we try to draw in the frictional force F we realise that we don't know which way it acts — up or down the plane.

As was stated earlier, *the frictional force opposes the motion*, so if, in the absence of friction, the force T is large enough to pull the particle up the plane, friction acts down the plane. If, in the absence of friction, the weight is enough to pull the particle down the plane, then friction acts up the plane. We first assume the former: friction acts down the plane because without friction the particle would be pulled up the plane.

[4]The weight of the particle is the magnitude of the force it experiences due to gravity; for a particle of mass m, $W = mg$, where g is the (constant) acceleration due to gravity.



The strategy for all similar problems is to determine the equations of equilibrium by *resolving* (i.e. taking components of the vectors) forces in two directions and equating to zero. It helps to choose the directions carefully in order to reduce the number of terms in each equation.

Clearly, for our problem, it is a good plan to resolve parallel and perpendicular to the plane. We have, respectively:

$$T = F + W\sin\alpha \tag{1}$$
$$R = W\cos\alpha \tag{2}$$

Thus $F = T - W\sin\alpha$, using only the first equation.

Normally, we are interested in finding the value of T that will support the particle on the plane. To accomplish this, we have to know something about the frictional force. The *experimental result* relating the frictional force to the normal reaction

$$\boxed{F \le \mu R} \tag{3}$$

is generally used. Here μ is the *coefficient of friction*, the value of which depends on the surfaces involved[5]. When the equality holds, the friction is said to be *limiting*.

In our example, combining equations (1) and (2) with the experimental law (3) gives

$$T \le W(\sin\alpha + \mu\cos\alpha).$$

Note that in the case of limiting friction, T is determined by this equation.

If, instead of assuming that the particle is tending to slip up the plane, we assume that it is tending to slip down the plane (i.e. $F \to -F$), then the frictional force would act up the plane. In this case (check this!) we find

$$T \ge W(\sin\alpha - \mu\cos\alpha)$$

and combining the two results gives the range of values of T for which equilibrium is possible, for a given value of μ:

$$W(\sin\alpha - \mu\cos\alpha) \le T \le W(\sin\alpha + \mu\cos\alpha).$$

Not surprisingly, in order for the particle to remain in equilibrium — i.e. not move — T cannot be too big or too small.

Note that if T were given (for example, if it were the tension in a string that passes over a pulley and has a weight dangling on the other end) the above equation would give bounds on the values of α allowed for equilibrium.

[5]For most common materials, the coefficient of friction lies between 0.3 and 0.6, though it can be considerably higher: for silicone rubber on tarmac it is over 1 (which is a good thing).