Mechanics Lecture Notes

1 Lectures 10 and 11: Motion in a circle

1.1 Introduction

The important result in this lecture concerns the force required to keep a particle moving on a circular path: if the radius of the circle is l, the speed of the particle is v and the mass of the particle is m, this force is mv^2/l, directed *towards* the centre of the circle.

This force is sometimes misleadingly described as 'centrifugal' or 'centripetal' and there is much confusion about whether it is directed towards the centre or away from it. But, clearly, a force is needed to prevent the particle moving in a straight line (according to Newton's first law), and this force must be in the direction of deviation from a straight line path; i.e. towards the centre. The force may be provided by the tension in a string, or by the normal reaction if the particle is constrained to move on a circular hoop, or by gravity in the case of a planet orbiting the Sun, or by magnetic fields in the case of the Large Hadron Collider.

The confusion in the direction of the force arises because if you imagine yourself moving freely *in the rotating frame*[1] (in a car, say) you would feel yourself being pushed *outwards* relative to the rotating frame; but this is just because you want to move in a straight line, and the car isn't moving in a straight line. Thus the 'force' you feel in a car as it goes round a bend is merely due to the tendency you have to move in a straight line: it is a *fictitious force*. It is this fictitious force, measured only in the rotating frame, that is the centrifugal ('fleeing from the centre') force. Such fictitious forces are artifacts of working in rotating or other accelerating frames and will not concern us at all here.

1.2 Key concepts

- Use of Cartesian coordinates and angular coordinates to describe motion in a circle.
- The force towards the centre required for circular motion.

1.3 Motion in a circle

For a particle moving on the surface of a sphere centred on the origin and of radius l, we have

$$\mathbf{r}.\mathbf{r} = l^2$$

and differentiating[2] gives

$$2\mathbf{r}.\dot{\mathbf{r}} = 0, \qquad (*)$$

which shows, not very surprisingly, that the velocity is perpendicular to the radius — i.e., it is tangent to the sphere. Similarly, if the speed v is constant (where $\dot{\mathbf{r}}.\dot{\mathbf{r}} = v^2$), we find that the acceleration is perpendicular to the velocity ($\dot{\mathbf{r}}.\ddot{\mathbf{r}} = 0$), which is an important (but not again not surprising, if you think about it) result. In particular, for a particle moving at constant speed in a circle, the acceleration is radial. That doesn't of course mean that the particle is moving towards the centre: only that the change in the velocity vector is radial.

We can go further. Differentiating equation $(*)$ we find

$$\mathbf{r}.\ddot{\mathbf{r}} + \dot{\mathbf{r}}.\dot{\mathbf{r}} = 0, \qquad \text{i.e.} \qquad \hat{\mathbf{r}}.\ddot{\mathbf{r}} = -v^2/r$$

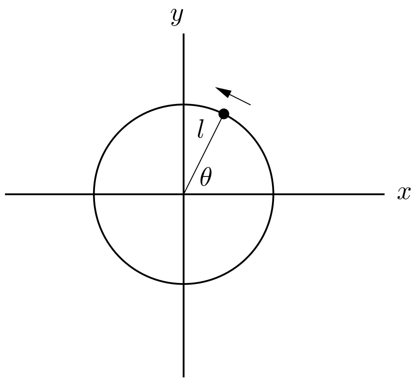
where $\hat{\mathbf{r}}$ is the unit vector in the radial, $\mathbf{r}$, direction. This is the result we are looking for: it gives the formula for the component of acceleration in the radial direction ('towards the centre') and hence the radial force required to maintain the motion.

[1] The rotating frame is accelerating (non-inertial), so Newton's laws do not apply. This point is discussed at length in the Dynamics and Relativity course.

[2] Using the Leibniz rule for vectors:

$$\frac{d}{dt}(\mathbf{a}.\mathbf{b}) = \mathbf{a}.\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt}.\mathbf{b}$$

which can easily be demonstrated using the definition of the dot product in components.



The diagram shows a particle moving on a smooth horizontal ring of radius l. Although the motion is most conveniently described by the angular coordinate θ, it is often easier to work in Cartesian coordinates. Choosing the obvious axes, we have for the coordinates of the particle

$$x = l\cos\theta, \qquad y = l\sin\theta\,.$$

We can find the velocity $(\dot{x}, \dot{y})$ by differentiation:

$$\mathbf{v} = (-l\dot{\theta}\sin\theta, l\dot{\theta}\cos\theta)$$

which satisfies $\mathbf{r}.\mathbf{v} = 0$ as expected. The speed of the particle is given by

$$v = |\mathbf{v}| = \sqrt{(-l\dot{\theta}\sin\theta)^2 + (l\dot{\theta}\cos\theta))^2} = l|\dot{\theta}|.$$

Differentiating $\mathbf{v}$ gives the acceleration:

$$\mathbf{a} = (-l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta\,,\, l\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta)$$

which we can write in the form

$$l\ddot{\theta}\underbrace{(-\sin\theta, \cos\theta)} - l\dot{\theta}^2\underbrace{(\cos\theta, \sin\theta)}\,.$$

The two vectors with underbraces are unit vectors pointing tangentially to the circle and radially outwards, respectively. The magnitude of the acceleration is

$$|\mathbf{a}| = \sqrt{l^2(\dot{\theta}^2)^2 + l^2\ddot{\theta}^2}\,.$$

Thus, in order to move in a circle, the particle must experience a force directed towards the centre of the circle, of magnitude

$$F = ml\dot{\theta}^2 \equiv \frac{mv^2}{l}\,.$$

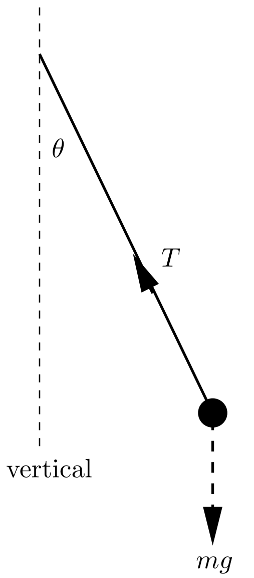
This force is provided by, for example, the normal reaction of the ring on the particle, the tension in a string, or external forces such as gravity (under which an otherwise free particle moves not in a circle but in a parabola), or combinations of these forces.

If $\ddot{\theta} = 0$, the particle moves with constant speed and no net force other than the central force acts on the particle.

1.4 Examples

(i) The simple pendulum

A particle of mass m is attached to one end of a light inextensible rod of length ℓ which is smoothly pivoted at the other end so that it can swing freely in a fixed vertical plane. Find the motion.



Let θ be the angle that the rod makes with the vertical. The easiest way of finding the motion is to use the energy integral: the reason that it is easier than writing down the (second order) equations of motion is that the tension T never needs to be considered.[3] The speed of the mass is $\ell\dot{\theta}$ and its potential energy, relative to its lowest point, is $mg\ell(1-\cos\theta)$. Thus

$$\tfrac{1}{2}m(\ell\dot{\theta})^2 + mg\ell(1-\cos\theta) = E.$$

Here E is a constant that would be determined by the initial conditions. One way forward would be to solve this to obtain $\dot{\theta}$ then integrate[4] but the solution cannot be expressed in terms of elementary functions. However, if the oscillations are small ($|\theta| << 1$), the motion turns out to be SHM. Setting $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ gives

$$\tfrac{1}{2}m(\ell\dot{\theta})^2 + \tfrac{1}{2}mg\ell\theta^2 = E.$$

We could solve this directly, but differentiating with respect to time and cancelling an overall factor of $m\ell\dot{\theta}$ gives immediately

$$\ell\ddot{\theta} + g\theta = 0,$$

which is SHM with period $2\pi\sqrt{\ell/g}$

We can also, as mentioned above, derive the above second order equation of motion directly. Newton's laws of motion are:

$$m\ell\ddot{\theta} = -mg\sin\theta \qquad \text{(component perpendicular to the rod)}$$

$$m\ell\dot{\theta}^2 = T - mg\cos\theta \qquad \text{(component parallel to the rod)}$$

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[3]The tension never needs to be considered because it never contributes to the energy; it never contributes to the energy because the rod has fixed length so no work is done by the tension force. This is a common situation: whenever there is a constraint, there must be a force that ensures the constraint holds; but no work is done against this force. Another example is a particle constrained to move on a surface or a wire, for which the relevant force is the normal reaction which does no work but stops the body falling through the surface.

[4]We can obtain θ as a function of t by doing the following integral and inverting the result:

$$t = \sqrt{\frac{m\ell^2}{2}} \int \frac{d\theta}{\sqrt{E - mg\ell(1-\cos\theta)}}$$

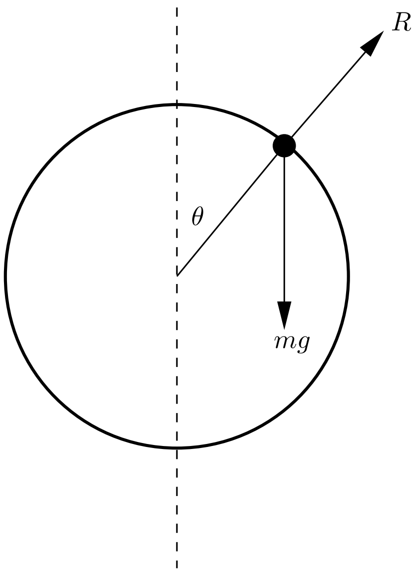
This is an *elliptic* integral and can be evaluated in terms of rather unfriendly functions called *elliptic* functions — elliptic because one way such integrals arise is in calculating the length of an arc of an ellipse.

where, in the second equation, we have used the standard motion in a circle formula for the central acceleration. The first equation is exactly the one we derived previously by differentiating the total energy and once this is solved we can substitute for θ in the second equation to find the tension.

In these equations, we were lucky in that the tension plays no role in obtaining the motion, which was determined by the first equation alone, so it need not have been considered at all. Of course, had we resolved horizontally and vertically instead of perpendicular and parallel to the rod, the tension would have played a bigger part.

(ii) *A particle of mass m slides on the surface of a smooth cylinder under the action of gravity. The axis of the cylinder is horizontal and the motion is in a vertical plane. The particle was released from rest from a point very close to the top of the cylinder. Find the position of the particle when it leaves the surface of cylinder.*

How can we determine the point at which the particle leaves the cylinder? While in contact with the cylinder, the particle experiences a normal reaction from the surface and the cylinder experiences an equal and opposite normal reaction, by Newton's third law. When this force is zero, contact is broken and the particle then falls freely under gravity in a parabola.[5]



We can use conservation of energy to obtain the speed of the particle directly in terms of its angular position (comparing KE+PE with initial KE+PE):

$$\tfrac{1}{2}mv^2 - mga(1-\cos\theta) = 0 + 0. \quad \text{i.e.} \quad v^2 = 2ga(1-\cos\theta).$$

To find the normal reaction of the cylinder on the particle, we use the radial component of the equation of motion:

$$mg\cos\theta - R = mv^2/a$$

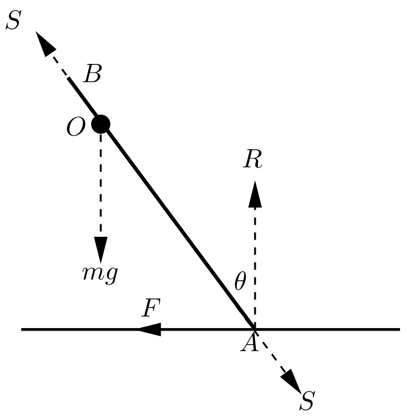
so

$$R/m = g\cos\theta - v^2/a = g(3\cos\theta - 2).$$

This is positive until $\cos\theta = 2/3$ at which point the particle leaves the surface of the cylinder.

[5] We can think of it another way. If the cylinder were taken away, the particle would fall in a parabola under gravity. If the radius of curvature of this parabola were less than a (i.e. of the path were more curved than the surface of the cylinder) , the trajectory would bend into the region previously occupied by the cylinder. In this case, if the cylinder were replaced, the particle would have to move on its surface. The condition that the particle loses contact with the cylinder is therefore that the radius of curvature of the parabolic trajectory exceeds a. For more details about radius of curvature, wait for the Vector Calculus course.

(iii) *A particle of mass m is riding a massless bicycle at constant speed v round a rough circular path of radius a at speed v. At what angle to the vertical must it lean?*



The diagram shows a bicycle AB, the point A representing the point of contact between a tyre and the path. The particle is sitting at O. To keep it simple, we assume that the particle is going round a circle of radius a (rather than the more natural assumption that the point A is going round a circle of given radius). The point O is a distance d from A.

The forces on the bicycle are: the normal reaction R of the path on the tyre; gravity, acting through O; and the friction between the tyre and the path which prevents slipping. Since there must be a horizontal force on the bicycle making it go round the corner, and the only horizontal force is friction, it is clear that the direction of the frictional force must be as shown in the diagram (inwards).

There is also an internal force of stress, S, in the frame of the bicycle. If we take this into account, we can use Newton's second law for any point of the bicycle. For the particle at O, we have (taking vertical and horizontal components)

$$0 = mg - S\cos\theta \quad \text{and} \quad \frac{mv^2}{a} = S\sin\theta.$$

Dividing gives

$$\tan\theta = \frac{v^2}{ag}.$$

For example, if $v = 5$ metres/sec (which is about 12mph) and $a = 10$ metres, and taking $g = 10$ metres per second per second results in $\tan\theta = 1/4$, i.e. $\theta \approx 15°$, which is not absurd despite the simplicity of the model.

Note that we could find R and F in terms of S and θ, and hence in terms of mg and v, by using Newton's second law on the point A instantaneously in contact with the path. We don't have to worry about the acceleration of this point, since it is massless by assumption:

$$0 = S - R\cos\theta - F\sin\theta \quad \text{and} \quad 0 = R\sin\theta - F\cos\theta.$$

Solving these equations and using $F = \mu R$ at limiting friction, we would find the maximum speed that the bicycle could go round the corner without slipping. (It is easy to see that $\mu = \tan\theta$; setting $\mu \approx 0.6$ for dry rubber on asphalt corresponds to 31°, which is again not completely ridiculous).

The above equations will yield $R = mg$, which we could also have obtained directly by considering the vertical forces on the whole bicycle. The reason I didn't do this is because we have not discussed the motion of a rigid body (even though we did discuss the equilibrium of a rigid body). In this case, since no part of the bicycle is accelerating in the vertical direction, we can simply apply the equilibrium condition that the net force in the vertical direction (i.e. $R - mg$ is zero).