Mechanics Lecture Notes

1 Lecture 2: Equilibrium of a solid body

1.1 Introduction

This lecture deals with forces acting on a body at rest. The difference between the particle of the last lecture and the body in this lecture is that all the forces on the particle act through the same point, which is not the case for forces on an extended body. The important concept, again, is the resolution of forces to obtain the equations determining equilibrium.

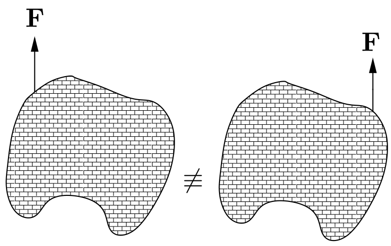
The simplest examples involve essentially one-dimensional bodies such as ladders. Again, it is essential start with a good diagram showing all the forces.

1.2 Key concepts

- Resolution of forces into a single *resultant force* or a *couple*.
- The *moment* of a force about a fixed point.
- Condition for equilibrium: zero resultant force and zero total couple.

1.3 Resolving forces

The difference between forces acting on a particle and forces acting on an extended body is immediately obvious from the intuitive inequivalence of the two situations below: for an extended body, it matters through which points the forces act — i.e. on the position of the *line of action* of the force.



In general, each force acting on a body can be thought of as having two effects: a tendency to translate the body in the direction parallel to the line of action of the force; and a tendency to rotate the body.[1] Clearly, for the body to be in equilibrium these effects must separately balance.

For the translational effects to balance, we need (as in the case of a particle) the vector sum of the forces to be zero:

$$\boxed{\sum_i \mathbf{F}_i = 0.} \tag{1}$$

For the rotational effects about a point P to balance, we need the sum of the effects to be zero, but what does this mean? Intuitively, we expect that a force whose line of action is a long way from P to have more rotating effect than a force of the same magnitude that is nearer and it turns out (see below) that the effect is linear in distance. The rotation effect of a force is called the *moment* of the force.

1.4 The moment of a force

In two dimensions, or in three dimensions in the case of a planar body and forces acting in the same plane as the body, any force tends to rotate the body within the plane or, in other words, about an axis perpendicular to the plane. In this case, we define:

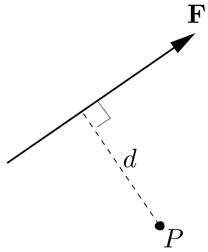
Moment of a force about a point P
=
magnitude of the force
×
the shortest distance between the line of action of the force and P.

[1] Imagine that one point, not on the line of action of the force, is fixed.

with account taken of the direction of the effect: either clockwise or anticlockwise.[2]

In general (in three dimensions when the body and forces are not coplanar), the effect of different forces will be to tend to rotate the body about different axes. In this case, the 'force times distance from line of action to the point' definition of the moment of a force is not adequate. We have to represent the moment as a vector. The important thing to understand is that the direction of vector representing the moment of the force **not** in the direction in which the body might move; it is along an axis about which the body might rotate.

We can obtain the required vector expression for the moment of a force from a diagram. In the diagram below, the magnitude of the moment of the force $\mathbf{F}$ about the point P is $|\mathbf{F}| \times d$.



As mentioned above, the moment of a force is a vector quantity, the direction of the vector being parallel to the axis through P about which the body would rotate under the action of the force. This can be very conveniently expressed using the vector cross product:

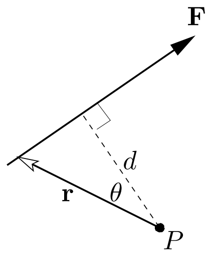
$$\boxed{\text{moment of } \mathbf{F} \text{ about } P = \mathbf{r} \times \mathbf{F}} \qquad (2)$$

where $\mathbf{r}$ is the position vector from P to any point on the line of action of $\mathbf{F}$.

Why is this cross product the right expression for the moment? Again, we can see from a diagram. The vector $\mathbf{r}$ in the diagram below goes from the point P to an arbitrary point on the line of action of the force. Clearly, $\mathbf{r} \times \mathbf{F}$ is in the correction direction (into the paper). And we have

$$|\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\ |\mathbf{F}|\ \sin\theta = |\mathbf{F}|\ d$$

which agrees with the 2-dimensional case.



To summarise: the magnitude of the moment of a force about a given point is given by the rule 'magnitude of moment equals magnitude of force times shortest distance between line of action of force and the point'. The vector moment has direction normal to the plane containing the point and the line of action of the force.

For the body to be in equilibrium, we require that there is no tendency to turn about any axis. The condition for equilibrium, in addition to (1), is therefore, (in the obvious notation),

$$\boxed{\sum(\text{vector moments of the forces about any point}) \equiv \sum_i \mathbf{r}_i \times \mathbf{F}_i = 0} \qquad (3)$$

There will still perhaps be a question mark in your mind about this result: why doesn't it matter what point we choose to take moments about? This is easily addressed. Suppose we change the point from P to P', where the position vector of P' with respect to P is a fixed vector $\mathbf{a}$. The position vectors in the condition (3) change from $\mathbf{r}_i$ to $\mathbf{r}'_i$, where $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{a}$. If we consider moments about P' instead of moments about P, we have

$$\sum_i \mathbf{r}'_i \times \mathbf{F}_i = \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i - \sum_i \mathbf{a} \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i - \mathbf{a} \times \sum_i \mathbf{F}_i$$

the vanishing of which is equivalent to the condition (3) provided the equilibrium condition (1) holds.

[2] In section 1.6, I will explain why the moment of the force, defined like this, is the correct measure of the rotational effect of the force; for the time being (and for ever if you perfectly sensibly don't want to wade through section 1.6) you should just accept that it is what we need.

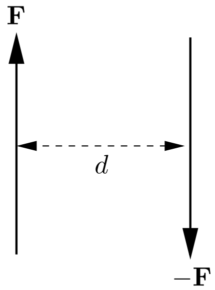
1.5 Couple

A *couple* is a pair of equal and opposite forces.[3] We define the moment of a couple about any point in the obvious way, as the sum of the moments of the two forces about that point. The sum of the moments of two forces will in general depend on the point about which the moment of the individual forces is taken; but this is not the case for a couple. Let the two forces be $\mathbf{F}$ and $-\mathbf{F}$, and let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of any fixed points on their respective lines of action, with respect to a point P. Then

$$\text{moment about } P = \mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times (-\mathbf{F}) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}$$

and this does not depend on the choice of P.

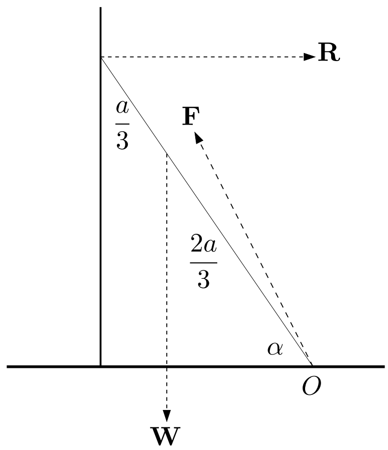
Choosing P on the line of action of one of the forces shows that the magnitude of the couple is just $|\mathbf{F}| \times d$, where d is the distance between the lines of action of the forces.



Example

A light rod[4] of length a stands on rough ground leaning against a smooth wall, and inclined at an angle α to the horizontal. A particle of weight W is placed a distance $\frac{2}{3}a$ up the rod. What is the magnitude of the normal reaction of the wall on the end of the rod?

First, as always, a good picture showing all the forces:



The normal reaction of the ground on the foot of the ladder and the frictional force acting on the foot of the ladder are combined into a single (unknown) force $\mathbf{F}$, acting in an unknown direction. There is no friction at the upper end of the ladder because the wall is smooth.

Before we go any further, we need to sit back and think. We have only three weapons in our armoury: resolving forces in each of two directions of our choosing; and taking moments about a point of our choosing. If we choose the directions and the point well, we can simplify our task enormously. In this rather straightforward case, we can eliminate the force that we are not interested in (the force on the foot of the ladder) in one go by taking moments about the foot of the ladder.

Taking moments ('force times shortest distance from the line of action of the forces to the point') anticlockwise gives

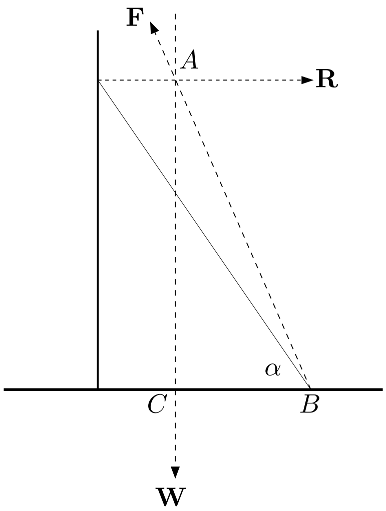
$$\circlearrowleft: \; W \times \frac{2}{3} a \cos\alpha - R \times a \sin\alpha = 0 \quad \Rightarrow \; R = \frac{3}{2} \cot\alpha.$$

[3]Think of turning on an old-fashioned tap.

[4]In the idealisation of the elementary mechanics, rods are straight, rigid and one-dimensional; this one is massless ('light').

We could now, by resolving forces horizontally and vertically, find the horizontal and vertical components of the force on the foot of the ladder, and then the whole problem would be solved.

Sometimes it is possible to solve such problems elegantly by geometry (but not necessarily more easily; you have to be good at geometry). Let us draw the diagram again, this time paying attention to the point at which the lines of actions of the forces intersect.



Note first that the three lines of action must intersect. Otherwise, we could take moments about the point of intersection of any pair (if not parallel); the two corresponding forces would have no moment about this point, since their lines of action passes through the point, leaving a non-zero moment from the third force. The total moment would thus be non-zero and the rod could not be in equilibrium.

The triangle ABC can be thought of completely geometrically, in which case it is soluble since we know two sides (horizontal and vertical) and the included angle (a right-angle).

But the the three forces $\mathbf{R}$, $\mathbf{F}$ and $\mathbf{W}$ are parallel to the sides of the triangle and, since they sum to zero (equilibrium condition) they can be represented as the sides of a triangle. This triangle must be similar to ABC, so we can find the relationships between the forces. For example, $AC/CB = |\mathbf{W}|/|\mathbf{R}|$ and $CB = \frac{2}{3}a\cos\alpha$ and $AC = a\sin\alpha$ so we obtain W in terms of R as before.

1.6 Moment of a force: justification of definition

NB this section is an optional extra: a gigantic footnote[5]

To emphasise the point in the heading: you do not need to know the material in this section; in fact, hardly anyone knows it.[6]

However, if you ever ask yourself why the moment of a force is defined as above (i.e. why is this the appropriate tool for investigating equilibrium), you will find this interesting.

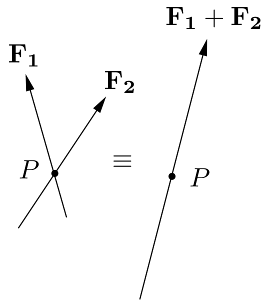
We will investigate the *resultant* of a number of forces acting on a body, which means, as in the case of a single particle, a reduced system of forces that has exactly the same effect on the body as the original system of forces. In the case of a single particle the reduced system is just one force; in the case of a system of forces acting on a body, the reduced system turns out to be a single force or, in very special cases, a couple.

We consider the case of just two forces, $\mathbf{F_1}$ and $\mathbf{F_2}$, in two dimensions; the generalisation to more forces in two dimensions is obvious (you just reduce the forces in pairs) and the three-dimensional case can be reduced to three two-dimensional cases by looking at the components of the forces in, for example, the x-y plane.

There are three cases to consider.

Case (i) $\mathbf{F_1}$ and $\mathbf{F_2}$ are not parallel

In this case, the lines of actions of $\mathbf{F_1}$ and $\mathbf{F_2}$ intersect, at P, say. The resultant, $\mathbf{F}$, of the two forces is just $\mathbf{F_1} + \mathbf{F_2}$ and it acts through P.



It is a simple exercise to check that the moment of $\mathbf{F}$ about any point Q is the same as the sum of the moments of $\mathbf{F}_1$ and $\mathbf{F}_2$ about Q.

This means that $\mathbf{F}$ has the same translational and rotational effect as $\mathbf{F}_1$ and $\mathbf{F}_2$ combined, provided we use the definition of the moment of a force given in section (1.4).

Case (ii) $\mathbf{F_1}$ and $\mathbf{F_2}$ are parallel, but $\mathbf{F_1} + \mathbf{F_2} \neq \mathbf{0}$.

In this case, the lines of action of the forces do not act through a common point, so we cannot immediately use the method above. Instead, the method can be used indirectly through a lovely construction. All we do is to add a pair of equal and opposite forces[7] as shown in the diagrams, to give two new forces that are no longer parallel. The diagrams on the next page illustrate the construction.

The diagrams show:

(i) Two parallel forces, $\mathbf{F_1}$ and $\mathbf{F_2}$, with $\mathbf{F_1}+\mathbf{F_2} \neq \mathbf{0}$ (i.e. exactly the situation we are considering).

(ii) In this diagram, two equal and opposite forces, $\mathbf{F}$ and $-\mathbf{F}$, have been added to the previous diagram. Adding these forces clearly has no effect: they just cancel each other out.

(iii) But if instead of cancelling them out we add them to $\mathbf{F_1}$ and $\mathbf{F_2}$, respectively, we obtain two resultant forces (by Case (i) above) which are the diagonal forces in the diagram.

(iv) The forces in **(i)** are therefore equivalent to the two forces in this diagram.

(v) Since the forces in **(iv)** are not parallel, they can be resolved (by Case (i) above) into a single force of magnitude (unsurprisingly) $\mathbf{F_1} + \mathbf{F_2}$ as shown in this diagram. But where does the line of action of this resultant lie?

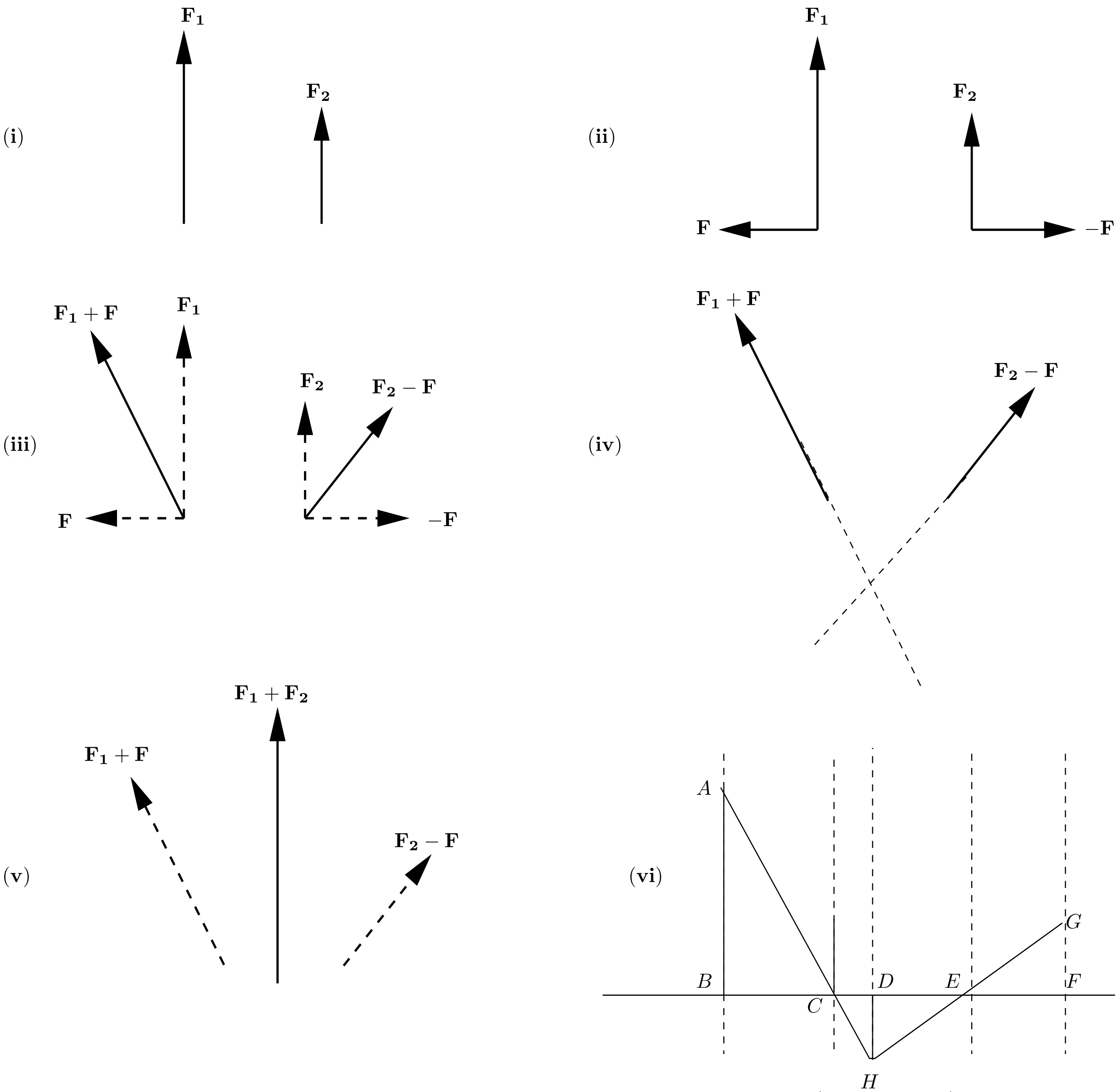
(vi) A bit of geometry does the trick. In this diagram, the AB and FG represent in direction and magnitude the original forces $\mathbf{F_1}$ and $\mathbf{F_2}$. CB and EF represent the equal and opposite forces we added in, and the two resultants are represented by CA and EG. Now using similar triangles gives

[5] which I put in because I found it interesting.

[6] It is not covered in any recent A-level text book that I could find, though it was covered in the book I had at school, by Humphrey and Topping.

[7] So simple!

the magic result $DC \times |\mathbf{F_1}| = DE \times |\mathbf{F_2}|$ — i.e. the resultant acts through the point where the moments of the original forces balance.



We thus find that the resultant of the two forces is a single force $\mathbf{F}_1 + \mathbf{F_2}$ (unsurprisingly), the line of action of which — and this is the important result — lies at distances d_1 and d_2 from the lines of action of $\mathbf{F_1}$ and $\mathbf{F_2}$, respectively, such that $d_1|\mathbf{F_1}| = d_2|\mathbf{F_2}|$; i.e. so that the *moments of the two forces* as defined above are equal.

Case (iii) $\mathbf{F_1}$ and $\mathbf{F_2}$ are parallel, and $\mathbf{F_1} + \mathbf{F_2} = \mathbf{0}$.

This is the exceptional case, when the system of forces cannot be reduced to a single force. No further reduction is possible and we are left with a couple. It is easy to see that the construction of Case (ii) breaks down: the construction gives another pair of equal and opposite forces.