Mechanics Lecture Notes

1 Lecture 3: Centre of mass

1.1 Introduction

This lecture deals with

1.2 Key concepts

- Definition of *centre of mass* of a body as the point through which the resultant of the gravitational forces may be considered to act; the point about which the total moment of the gravitational forces is zero.
- Calculation of centre of mass using moments of the gravitational forces.

1.3 Centre of mass

The centre of mass (or centre of gravity) of a body is the point through which gravity may be considered to act. For the purposes of any calculation involving gravity, we can replace the solid body with a single massive particle at the centre of gravity; it is as if the mass of the body were concentrated at the centre of mass. [1] It can be thought of as the average location of the mass.

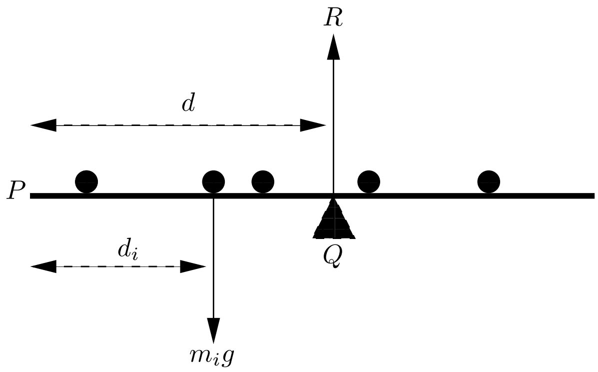
For a one dimensional body (a stick or ladder), the centre of mass is just the point at which the body balances, and the same is true of a two-dimensional body (think of a lamina balancing on a pencil point). It is harder to imagine this for a three dimensional body, but you could think of the centre of mass as being the point such that, for any axis passing through it, there is no tendency for gravity to rotate the body about that axis.

The following example illustrates the process.

Example: the centre of mass of a set of particles

Suppose n particles are attached to a light straight rod which rests on a smooth pivot, as shown in the diagram ($n = 5$ in the diagram). The ith particle has mass m_i and is distance d_i from P, the left end of the rod. The pivot is distance d from one end of the rod. The reaction of the pivot on the rod is R. The system is in equilibrium with the rod horizontal.

As always, the first move is to draw a good diagram.



The blobs in the diagram are the (point!) particles and the ith blob from P, the left end, of the rod has mass m_i and is a distance d_i from P.

For equilibrium we need the both the forces and their moments (about any point) to balance. Summing the forces gives

$$\sum_{i=1}^{n} m_i g - R = 0$$

so the reaction of the pivot on the rod is equal to the total weight of the particles — not entirely unexpected!

[1]This approach works because the forces of gravity acting on the individual particles of the body are all parallel. A formal proof will be given in Relativity and Dynamics course.

Summing the moments of the forces about P gives

$$Rd = \sum_{i=1}^{n} d_i m_i g.$$

So, provided we choose the position of the pivot according to

$$d = \frac{\sum_{i=1}^{n} d_i m_i}{\sum_{i=1}^{n} m_i},$$

the rod and the masses will balance on the pivot just as if it were a single particle of mass equal to the total mass situated on the pivot, which is at the centre of gravity.

We could of course have taken moments about the pivot, Q, to obtain the same result:

$$0 = \sum_{i=1}^{n} (d - d_i) m_i g,$$

and had we started from this equation for d we would not have had to involve R at all.

1.4 Centre of mass: general definition

Extending the above idea to a general system of particles, we define the centre of mass of a set of particles with masses m_i and position vectors (with respect to an arbitrary origin) $\mathbf{r}_i$ to be at the point with position vector

$$\boxed{\frac{\sum m_i \mathbf{r}_i}{\sum m_i}.}$$

Check for yourself that the definition is independent of the origin of the position vectors.

There is an analogous formula for a continuous mass distribution along a one dimensional body (a thin straight rod, say) which is obtained by replacing the sum with an integral and the mass at the point with coordinate x by the $\rho(x)dx$, where $\rho(x)$ is the mass density per unit length at that point. Thus the distance of the centre of mass from the origin (somewhere on the rod, not necessarily at the end) $\bar{x}$, is given by

$$\bar{x} = \frac{\int x \rho dx}{\int \rho dx}.$$

The numerator is the moment of the element of mass at x and the denominator is the total mass.

We can extend this to three dimensions. The position vector of the centre of mass is

$$\frac{\int \mathbf{r} \rho(\mathbf{r}) dV}{\int \rho(\mathbf{r}) dV}$$

though you may not properly understand these volume integrals until the Vector Calculus course in the Lent term. (The position vector defined above has x-component

$$\frac{\int x \rho(\mathbf{r}) dV}{\int \rho(\mathbf{r}) dV}$$

with y and z components similarly defined.)